Robotics: Basics and Selected Advanced Concepts
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Lecture – 08
Inverse Kinematics of Serial Robots

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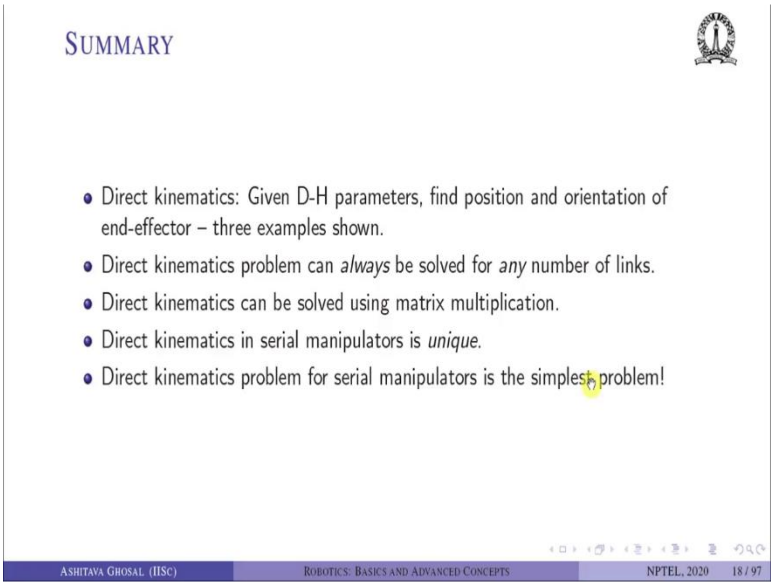

Welcome to this NPTEL course on Robotics: Basics and Advanced Concepts. In the last lecture we had looked at the direct kinematics of serial robots. In this lecture, we will look at the Inverse kinematics of serial robots. To recapitulate in the last lecture the direct kinematics problem was stated and the direct kinematics problem is given the DH parameters find the position and orientation of the end effector.

I had shown you three examples a planar example the PUMA 560 example and the SCARA robot. The direct kinematics can always be solved for any number of links. It just simply involves multiplication of matrices ok. So, there is no reason why we cannot multiply matrices and hence for any number of links you give me I need to obtain the link transformation matrices and then just multiply in some appropriate order.

The direct kinematics in serial manipulator is unique ok. So, there is only one possible solution to the direct kinematics problem and the direct kinematics problem for serial manipulator is the simplest problem. In this lecture we look at the inverse kinematics of a serial robot.

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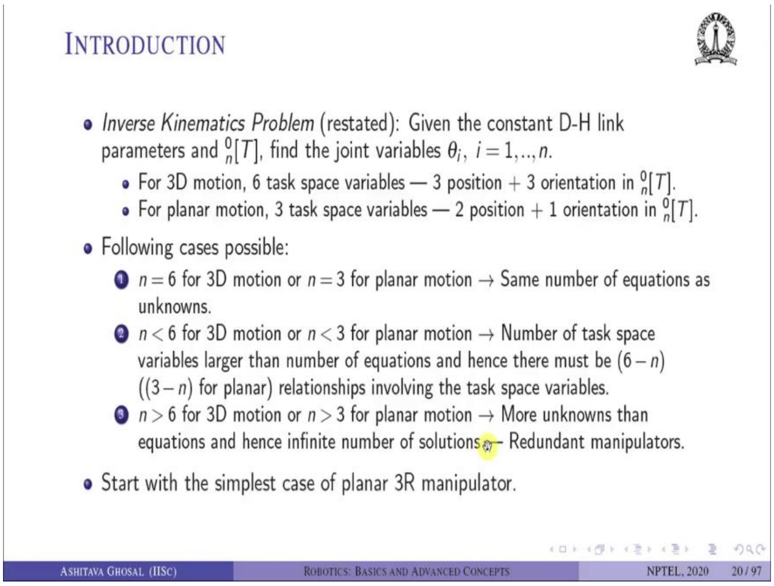

So, what is the inverse kinematics problem? To restate it once more, given the constant D-H link parameters and the position and orientation of the last link or the end effector ${}^0_n[T]$, find the joint variables θ_i, i equals 1 through n ok. So, what are the constant D-H link parameters? Just to recapitulate we have a twist angle, we have a link length and either a link offset or a link rotation depending on what kind of joint it is.

To continue for 3D motion, we have 6 task space variables. So, 3 position and 3 orientation in this transformation matrix ${}^0_n[T]$. And for planar motion there are 3 task space variables, 2 position let us say x and y and one orientation in this given transformation matrix.

So, there are following cases which are possible now. So, if n is equal to 3 for planar motion or n is equal to 6 for 3D motion, we have the same number of equations or same number of unknowns. Just or what do you mean by same number of equations? In ${}^0_n[T]$ we know that there are independent equations coming from the rotation matrix and from the last column which is the position vector.

If n is 6 then in the ${}^0_n[T]$, we can get 3 equations from the rotation part and 3 from the translation part and hence we have 6 task space variables and we have 6 joined variables - n is equal to 6. For n less than 6 for 3D motion or n less than 3 for planar motion, the number of task space variables are larger than the number of equations and hence there must be 6 - n for spatial and 3 - n for planar relationships involving the task space variables.

So, if you recollect in the SCARA robot the motion which is the 4 degree of freedom robot and the end effector can be determined by position can be determined by x, y, z and 1 orientation. So, although the end effector is moving in 3D space there are the 2 other angles orientation angles are not there. So, hence there are some constraints between the task space variables.

If n is greater than 6 for 3D motion or n is greater than 3 for planar motion, we have more unknowns than equations and hence infinite number of solutions. These are a special class of manipulators which are called redundant manipulators and we will be looking at them in detail later. So, let us start the inverse kinematics problem for the simplest case of a planar 3R manipulator.

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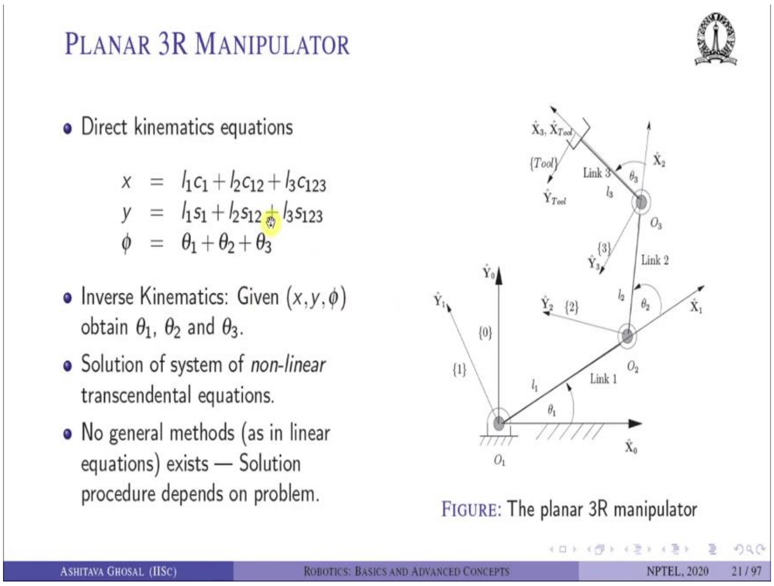

So, the figure in this slide show a planar 3R manipulator, basically there are 3 rotary joints at O_1, O_2 and O_3. The direct kinematics equation are known. So, we can obtain the x, y and orientation of the $\{Tool\}$ coordinate system or the third coordinate system as given by $l_1 c_1 + l_2 c_{12} + l_3 c_{123}$ and y is the $l_1 s_1 + l_2 s_{12} + l_3 s_{123}$. And the orientation of the tool coordinate system or the n th coordinate system, in this case n is 3, is given by $\theta_1 + \theta_2 + \theta_3$. So, what is the inverse kinematics problem for this planar 3R manipulator? We are given x, y and ϕ, the left-hand side of these equations and we want to obtain by θ_1, θ_2 and θ_3.

So, what you can see is these are 3 non-linear equations. So, the first 2 involves sin and cosine of angles ok. So, they are non-linear transcendental equations, the third one is linear. In order to solve these 3 non-linear equations or 2 non-linear and 1 linear equation we do not have any general method to solve non-linear transcendental equations.

The solution of non-linear transcendental equations depends on the problem. This is unlike a linear set of equations - a set of linear equations where we can easily solve if you have y equals [A]x, we can always find solution x for a given y provided certain conditions are met on that matrix [A].

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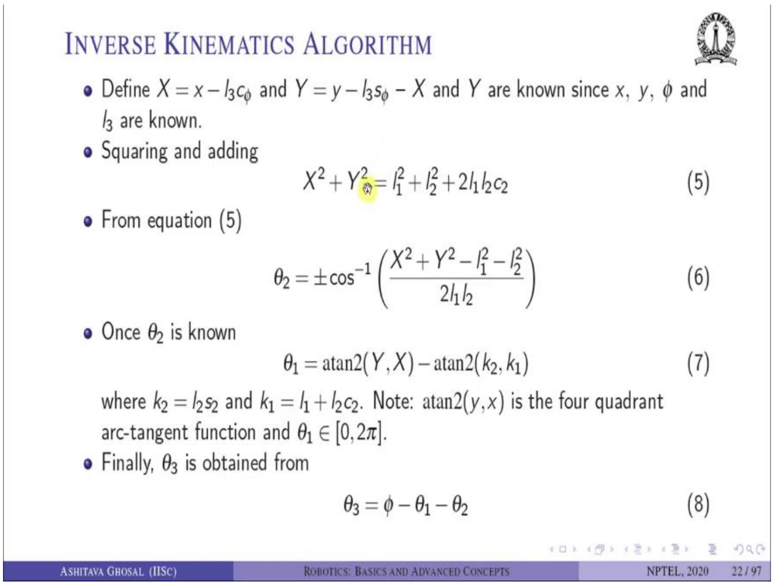

So, what do we do? So, what we can do is we can see that we can define a new variable capital X which is x - l_3 cos ϕ and a capital Y which is y - l_3 sin ϕ ok. So, X and Y, capital X and capital Y, are known, because since small x small y and ϕ are given - this is the inverse kinematics problem.

So, if you now add and square and add $X^2 + Y^2$ square we can see that you will get $l_1{}^2 + l_2{}^2 + 2\, l_1\, l_2 \cos \theta_2$. Hence, from equation (5) we can show that θ_2 is $\cos^{-1}$ $(X^2 + Y^2 - l_1^2 - l_2^2)$ divided by $2\, l_1\, l_2$ ok.

So, what have we done? We have started from these 3 equations we have defined capital X as x -l_3 cos ϕ, so this has gone there. So, X is now $l_1\, c_1 + l_2\, c_{12}$, Y is also $l_1\, s_1 + l_2\, s_{12}$; so, when you square and add X and Y, so $(l_1\, c_1)^2 + (l_1\, s_1)^2$, θ_1 will go away.

Likewise, θ_{12} will go away and we will be left with only this equation - $X^2 + Y^2 = l_1^2 + l_2^2 + 2\, l_1\, l_2\, c_2$ and we can find θ_2 ok. So, I am going over this little slowly because we can easily see what we are trying to do ok.

Once θ_2 is known we can find out what is θ_1 by doing this atan2(Y, X) - atan2(k_2, k_1) ok, where k_2 is $l_2\, s_2$ and k_1 is $l_1 + l_2\, c_2$ ok. So, basically what are we doing? We are saying X is some $l_1\, c_1 + l_2\, c_{12}$. So, we can expand c_{12} as cos a cos b - sin a sin b and so on, and then take terms which are in θ_2 which is now known and solve for θ_1.

So, we are going to use atan2 again such that we can get this θ_1 in the correct quadrant. And then finally θ_3 is obtained from $\phi - \theta_1 - \theta_2$. So, as you can see, we have used some knowledge of trigonometry to first find θ_2, then we taken terms which are now known and solved for θ_1 and then finally θ_3 is simply $\phi - \theta_1 - \theta_2$ ok.

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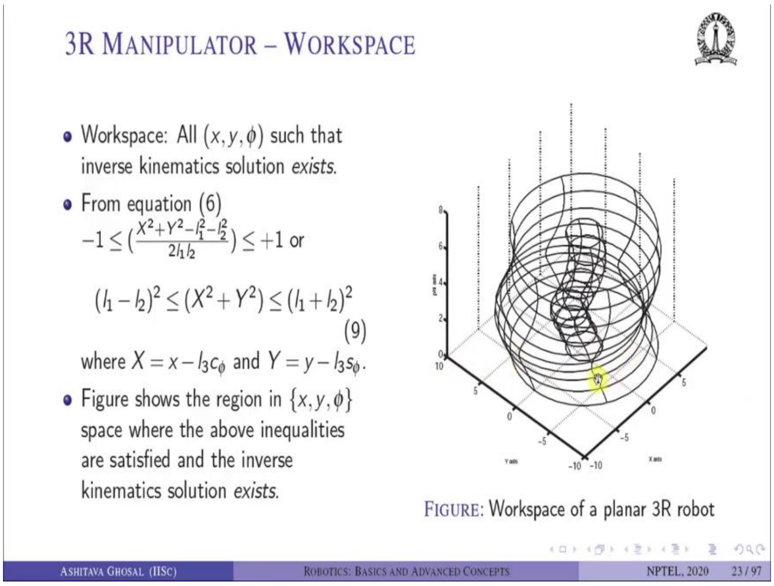

So, let us continue. So can we solve this inverse kinematics problem everywhere? The answer is no. Why? Because if you look at this equation θ_2 is $\pm \cos^{-1}$ of something; so, whatever is inside this bracket which is $(X^2 + Y^2 - l_1^2 - l_2^2)$ divided by $2\, l_1\, l_2$ must be within $\pm$ 1, because only $\cos^{-1}$ of a number which is within $\pm$ 1 is defined ok.

So, hence we can say that this condition must always be true - which is this quantity ($(X^2 + Y^2 - l_1^2 - l_2^2) / 2\, l_1\, l_2$) lie between $\pm$ 1. And we can now simplify this and write it as $(X^2 + Y^2)$ should be greater than $(l_1 - l_2)^2$ and less than $(l_1 + l_2)^2$.

And finally, we can substitute back X as $x - l_3 \cos \phi$ and Y as $y - l_3 \sin \phi$ ok. The figure in this right-hand side shows a plot of x, y and angle ϕ phi ok. So, every section represents a circle of radius between $(l_1 + l_2)^2$ and $(l_1 - l_2)^2$ and then if you substitute back this $x - l_3 \cos \phi$ and $y - l_3 \sin \phi$, you will get this complicated looking figure ok.

So, hence only points which are lying between these big circles and small circles, x, y and ϕ - and ϕ is along the vertical direction, you can only find cosine inverse of that number ok. So, this is called the workspace of the planar robot. So, all x, y and ϕ which is inside this region, which is shown here, I can solve the inverse kinematics.

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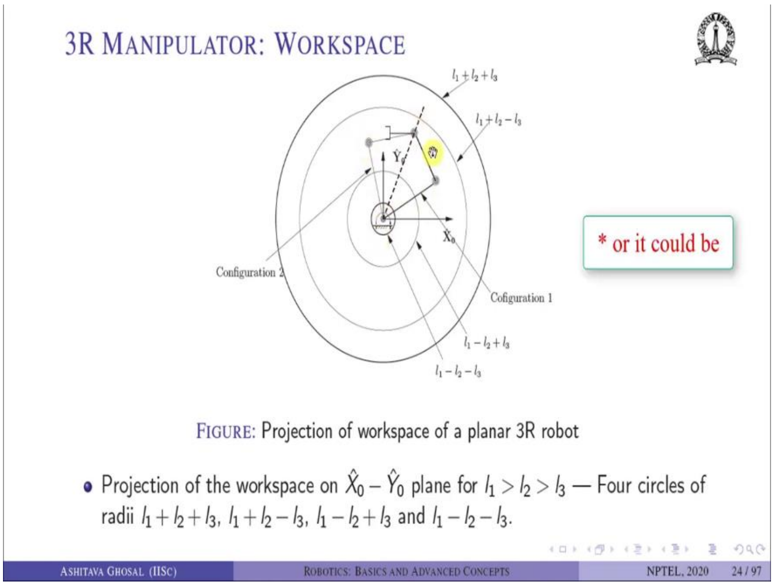

We can also try and see what is exactly happening a little bit more detail. So, I can project all these circles back to the X-Y plane. So, I will get the following circles. So, I will get one completely outside circle which is $l_1 + l_2 + l_3$ there will be also one small inside circle which is $l_1 - l_2 - l_3$ and then there are these 2 other circles - one is $l_1 + l_2 - l_3$ and one is $l_1 - l_2 + l_3$ ok. So, this is nothing but the projection of those that figure onto the X-Y plane ok.

So we have these 4 circles, so now if you pick a point on this region between $l_1 - l_2 + l_3$ and $l_1 + l_2 - l_3$ so this is the point. As you can see so this point is nothing but X, Y ok, it is $x - l_3 \cos \phi$ and $y - l_3 \sin \phi$. So, this X, Y this is the point and I can reach this point in 2 ways. So, the planar robot can be like this and the (Refer Time: 14:00) and it could be like this, that is what is meant by there are 2 possible solutions for theta 2.

So, if you go back and see there are 2 possible solutions of θ_2 is ± cos $^{-1}$ of this ok. And geometrically what is happening? One is like this. So, this is the angle θ_2 and the other one is - θ_2 to reach that same X and Y.

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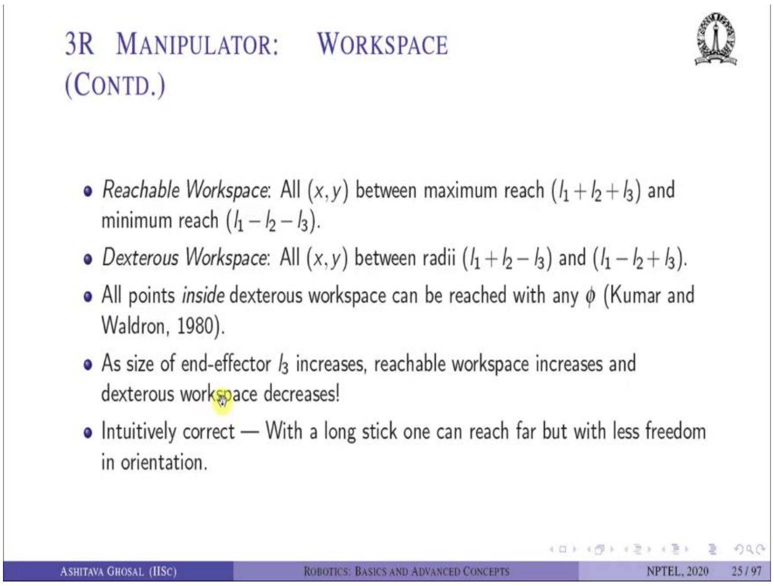

So, the reachable workspace is defined as all (x, y) between the maximum reach which is ($l_1 + l_2 + l_3$) and minimum reach which is ($l_1 - l_2 - l_3$). We can also define something called as a dexterous workspace and this is all (x, y) between the radius ($l_1 + l_2 - l_3$) and ($l_1 - l_2 + l_3$) ok.

So, all points inside the dexterous workspace can be reached with any ϕ. So, as the size l_3 increases what you can see that the dexterous workspace will become smaller ok. The reachable workspace will increase because it is ($l_1 + l_2 + l_3$) ok, whereas the dexterous workspace is ($l_1 + l_2 - l_3$) ok. So, this is the well-known result which was obtained long time back and it is very nice result which says that as the size of the end effector l_3 increases reachable workspace increases and dexterous workspace decreases. And this is also sort of intuitively correct. So, if you are holding a long stick you can reach far away you can reach the roof of the room, but if you use a long stick at the end you have very little freedom in orienting the tip. So, that is the whole idea of a dexterous workspace I can reach any point with whatever orientation I feel I want.

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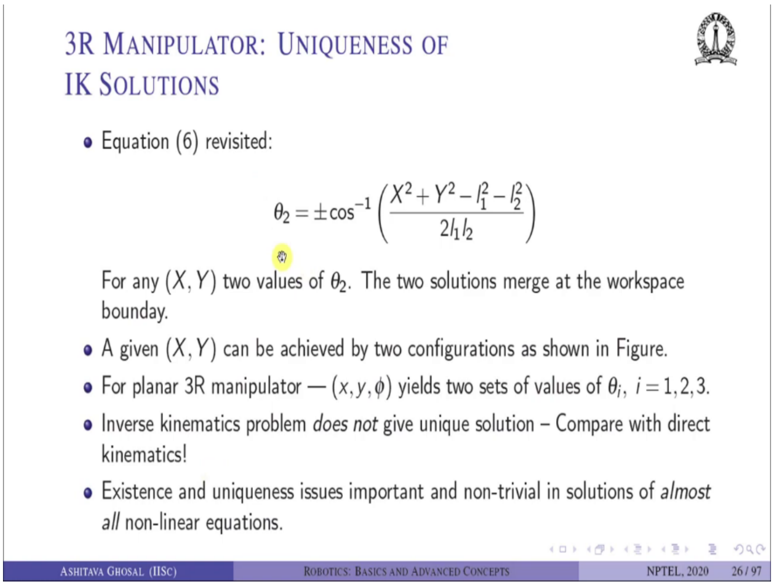

Let us go back to again that equation of θ_2 ok. So, I showed you θ_2 is ± cos inverse this. So, hence for any (X, Y), there are two possible values of θ_2 and these 2 values merge at the workspace boundary. So, at a given (X, Y) can be achieved by 2 configurations as shown in the figure, I showed you this figure, that if I take any point (X, Y) here I can reach either like this or I can reach like this. So, one of them is + θ_2 and the other one is - θ_2.

So, in general what we can say is for a planar 3R manipulator, given x, y and ϕ we can obtain this x, y and ϕ by 2 sets of values of θ, 1, 2 and 3 ok. So, in conclusion the inverse kinematics problem does not give unique solutions ok. Remember in direct kinematics we always could get a unique solution ok. Multiplying matrices given θs I could find the position and orientation of the end effector uniquely. In the inverse kinematics problem given the position and orientation of the end effector, I get in this case of a planar 3R, 2 possible sets of values of θ_1 θ_2 and θ_3.

So, in general or more abstractly what we have is something called as a existence and uniqueness issues in solution of non-linear equation ok. So, it is quite hard and non- trivial to obtain the existing and uniqueness conditions for solution of non-linear equation. In this case uniqueness means what? We have 2 possible sets the solutions are not unique and existence means what we can obtain the values of x, y and ϕ such that we can get a inverse kinematic solution ok. So, all x, y, ϕ for which inverse kinematic solution exists, that gives us this existence criteria and uniqueness criteria is how many possible joint angles satisfies the given x, y and ϕ.

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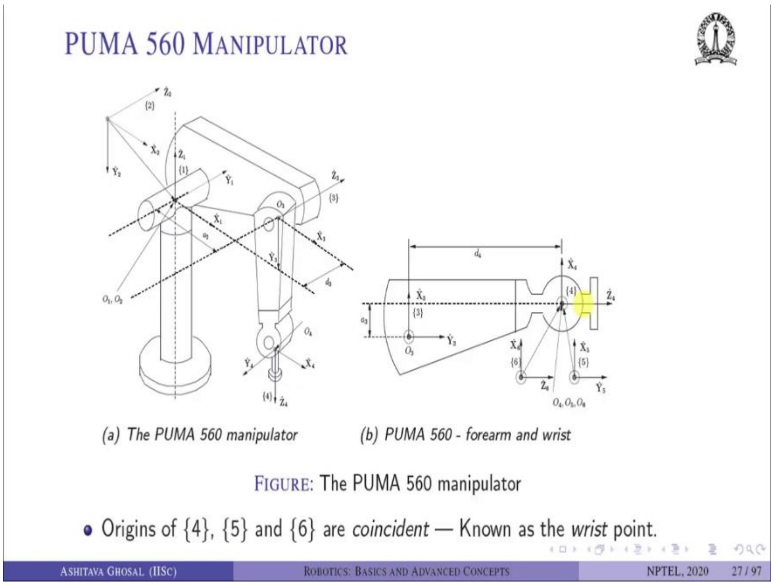

Now, let us continue we look at a 6 degree of freedom PUMA 560 robot. This has been shown earlier also there are 6 joints 1 is along Z_1, one is along Z_2 then Z_3 and then there are 3 more which are intersecting at this point which is the point of intersection of axis 4, 5 and 6. So, this origins of {4}, {5} and {6} are coincident and this is called the wrist point.

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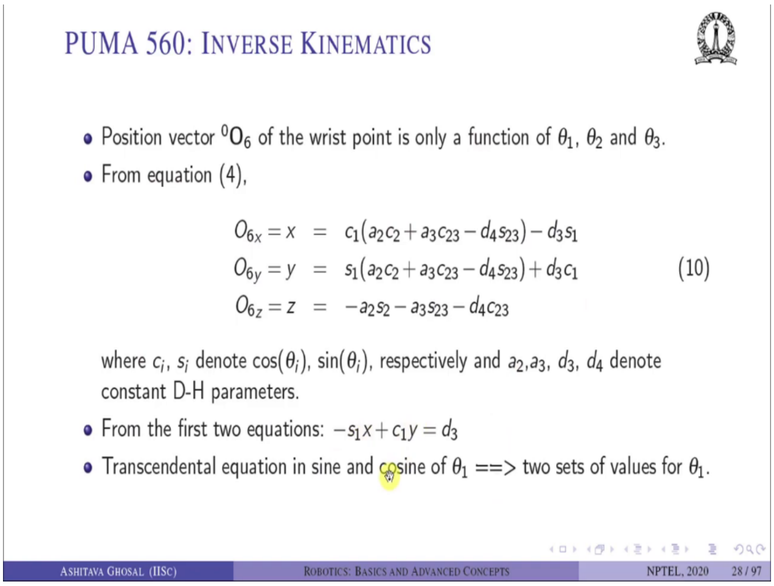

So, it turns out that the position vector of this wrist point is only a function of θ_1, θ_2 and θ_3 ok. So, these are the 3 equations O_{6x} which is shortened as x is given by c_1 (a_2c_2 +a_3c_{23} –

d_4s_{23}) – d_3s_1 and so on ok. So, does this make sense? Yes if you think about it little bit that the links 4, 5 and 6 are after the joint axis 4, 5 and 6 ok. So, hence the position vector which is lying on the joint 4, axis 5, or 6 axis because all of them are at the same place, can only of be a function of all the angles before this origin. And what are the angles? One is θ_1, one is θ_2 and one is θ_3, the θ_4, θ_5 and θ_6 affect the orientation of the end effector it does not affect the origin of the last link ok.

So, how do we solve the inverse kinematics of the PUMA? So, basically what are we given? We are given x, y and z and fortunately we now have 3 equations in 3 unknowns. What are the unknown's - θ_1, θ_2 and θ_3.

So, from this first 2 equations if you multiply the first equation by - s_1 and the second equation by + c_1 -- s_1 is sin θ_1, c_1 is cos θ_1 -- and add them you can see everything drops out and we get a nice simple equation which is $-s_1 x + c_1 y = d_3$. So, you will get $d_3 s_1^2$, $d_3 c_1^2$, when you add them, they will become 1 ok. So, we have a single transcendental equation in θ_1 ok. So, how do we solve this?

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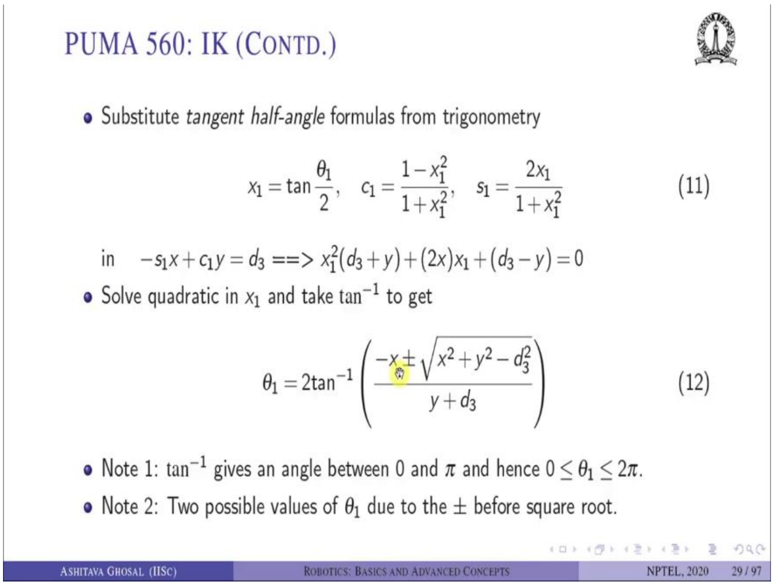

So, the kinematics people have figured out this really nice way of trying to solve a single transcendental equation. So, the idea is that we convert a transcendental equation into a polynomial and how can we do that? Let us define a new variable x_1 which is tan $\theta_1/2$ ok. So, if x_1 is tan $\theta_1/2$, cos θ_1 is $(1 - x_1^2)/(1 + x_1^2)$ and sin θ_1 is $2x_1/(1 + x_1^2)$.

So, hence - $s_1 x + c_1 y = d_3$ can be written as a quadratic in x_1 and remember x_1 is tan $\theta_1/2$. So, what have we done? We have taken a transcendental equation in sin and cos θ_1 and obtained a quadratic in tan $\theta_1/2$ ok. So, this is the well-known tangent half angle formulas from trigonometry.

So, once I have a quadratic of this form, I can easily solve for x_1. Now quadratic equations, we know the roots of a quadratic equation in closed form and then we can find out θ_1 which is $\tan^{-1}$ of that quantity and since we are finding out $\theta_1/2$. So, actual θ_1 is 2 $\tan^{-1}$ of the roots of the quadratic equation ok.

So, a few observations so tan inverse gives an angle between 0 and π normally. So, hence 2 $\tan^{-1}$ gives the value between 0 and 2π, so we are getting the angle in the right quadrant. So, we do not have to use atan2 here -- this idea of a tangent half angle makes ensures or makes us get an angle which is in the right quadrant.

The second observation is we get two possible values of θ_1 due to this ± sign in the square root ok. So, again uniqueness no ok. So given x, y and z in this form of equation of the wrist point, I am getting a value of θ_1 which are 2 of them. So, this is not unique - very similar to the previous case of the planar 3R case, where we could get two possible values of an angle which satisfies a given position and orientation. In this case two possible values of θ_1 which for a given wrist point ok.

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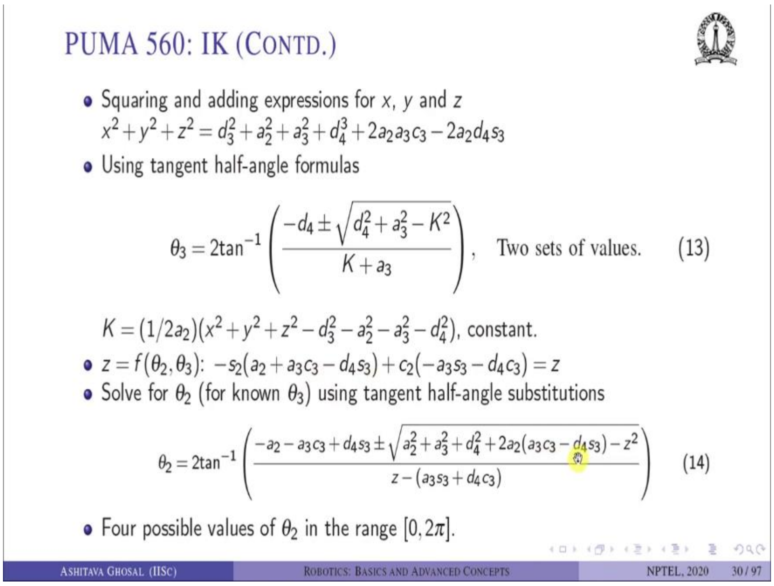

Let us continue. If you square and add those 3 equations, which is $x^2 + y^2 + z^2$, you can see that you will end up with a single equation in θ_3. So, this is correct? Yes. So, $x^2 + y^2 + z^2$, so θ_1 will vanish, θ_2 will also vanish. It turns out, because you can see the pattern - it is - $a_2\, s_2 - a_3\, s_{23}$ - $d_4\, c_{23}$; whereas, here it is, you know, plus and minus, sort of very similar but here it is c_2, here it is s_2 ok.

So, if you expand, the square and adding that expression and do a little bit of simplification you can get a single equation in cosine θ_3 and sin θ_3 ok. Again, this is a transcendental equation in cosine and sin θ_3, we can again use the tangent half angle formulas to obtain θ_3.

So, here also you can see that you get 2 sets of values of θ_3. Again square root solutions of the quadratic equation where K is a constant ok. Finally, we can see that the last equation z is function of only θ_2 and θ_3. So now that we know θ_3, we can collect terms with a θ_3 inside this bracket and again we have a simple transcendental equation in θ_2.

So, - s_2 into something which is now known, c_2 into something which is now known equal to z and we can solve this again using tangent half angle substitution. And we will get θ_2 as 2 $\tan^{-1}$ of really complicated long expression ok. So, how many values of θ_2 we get? We get four possible values of θ_2. Why? Because θ_3 already had 2 possible values and θ_2 depends on θ_3, so you can see it is $a_3\, c_3$ is there $a_3\, c_3$, $d_4\, s_3$ all these terms are here. So, we get four possible values of θ_2 in the range 0 to 2π.

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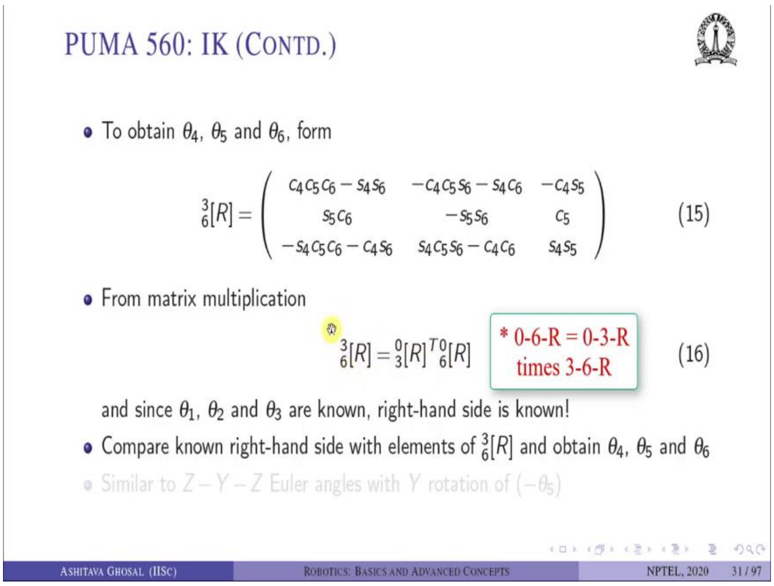

To obtain θ_4, θ_5 and θ_6 which is basically the last 3 angles after the wrist, you can see that the rotation matrix ${}^3_6[R]$ is of this form -- it is ($c_4\, c_5\, c_6$ - $s_4\, s_6$) r_{11} is this. r_{23} is c_5, r_{21} is $s_5\, c_6$ and so on ok. This matrix ${}^3_6[R]$ can be written as ${}^0_3[R]^T\, {}^0_6[R]$ right, because ${}^0_6[R] = {}^0_3[R]\,{}^3_6[R]$. We multiply by the inverse which is the same as the transpose and we get this matrix equation.

The right-hand side is known because ${}^0_6[R]$ is given to you for the inverse kinematics problem and ${}^0_3[R]$ contains only θ_1, θ_2, and θ_3. So, basically right-hand side is known and left-hand side has 3 variables θ_4, θ_5 and θ_6 ok.

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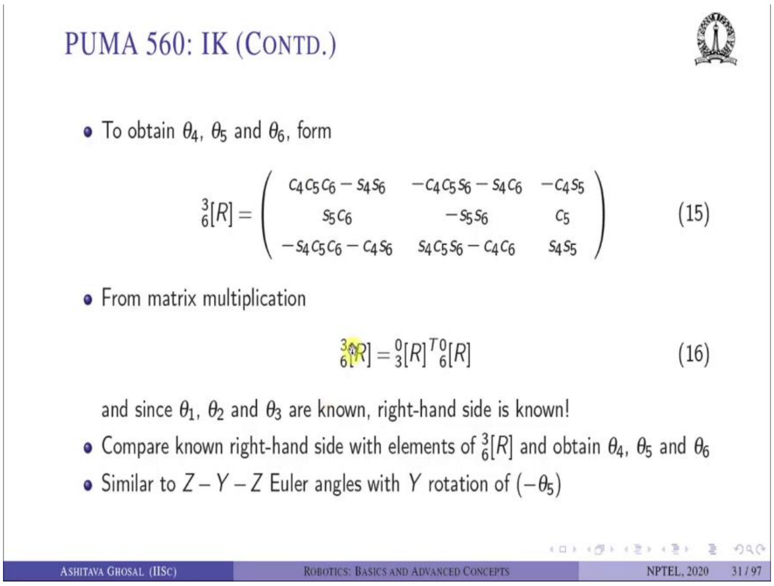

So, we can just compare term by term and find out what is θ_4, θ_5 and θ_6. So, for example, c_5 will be equal to some known number. So, θ_5 will be $\cos^{-1}$ of that known number. It turns out that in this case it is even simpler. Why? Because this matrix is very similar to what is called as the *Z*-*Y*-*Z* Euler angles with the *Y* rotation of – θ_5 ok.

So, last week we had looked at Euler angles representation of orientation of a rigid body using simple rotations. So, we have, in this example, a simple rotation about *Z* a simple rotation followed by a simple rotation about *Y* and another simple rotation about *Z*. However, unlike what we had done earlier the second rotation is by a minus angle θ_5.

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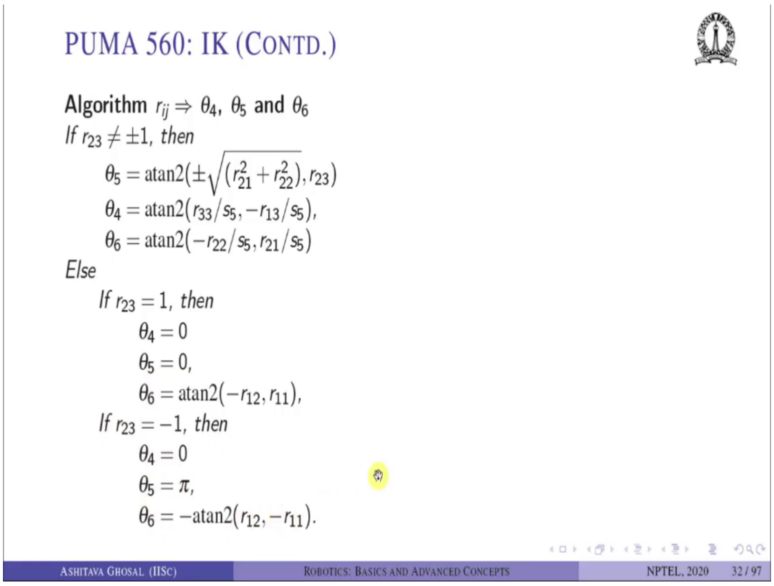

So, we can write the inverse Euler angle transformation just like we had done last week. So, given r_{ij} how do I find out θ_4, θ_5 and θ_6? Again, there is a singular configuration when r_{23} is ± 1. So, if r_{23} is not equal to ± 1, θ_5 can be first found out by atan2 (± $\sqrt{(r_{21}^2 + r_{22}^2)}$, r_{23}). Is it true? Yes, ($r_{21}^2 + r_{22}^2$) and then this will be a function of only sin θ_5 and cos θ_5 and we can use atan2 to find θ_5.

Then we can divide by sin θ_5 and find θ_4 again using an atan2. And θ_6 - divided by sin θ_5 these 2 - r_{22} and r_{21} and find θ_4, θ_5 and θ_6. If r_{23} is + 1, this is a special or a singular configuration, we set θ_4 as 0, θ_5 as 0 and θ_6 this. If r_{23} is – 1, then we set θ_4 as 0 θ_5 as π and θ_6 this ok.

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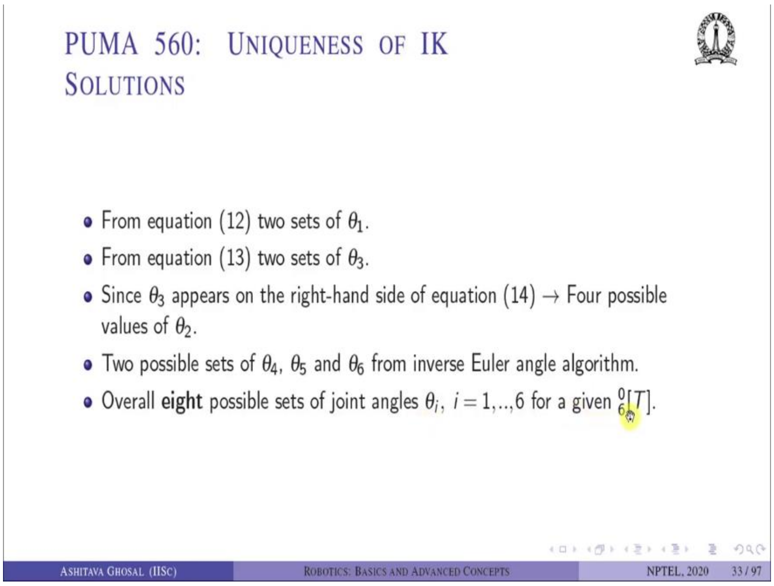

So, in summary what have we done? We have obtained two sets of θ_1, two sets of θ_3, since θ_3 appears on the right-hand side of equation (14). So, four possible values of θ_2 and then we also have two possible sets of θ_4, θ_5 and θ_6 - because you can see θ_5 has ± sign here. So, I can get two possible θ_5 and then when we divide it by θ_5 you will get two possible values of θ_4 and θ_6 ok. So, overall, what have we obtained? We have obtained eight possible sets of joint angles θ_i for a given position and orientation of the sixth link with respect to the 0th link - given ${}^0_6[T]$ ok.

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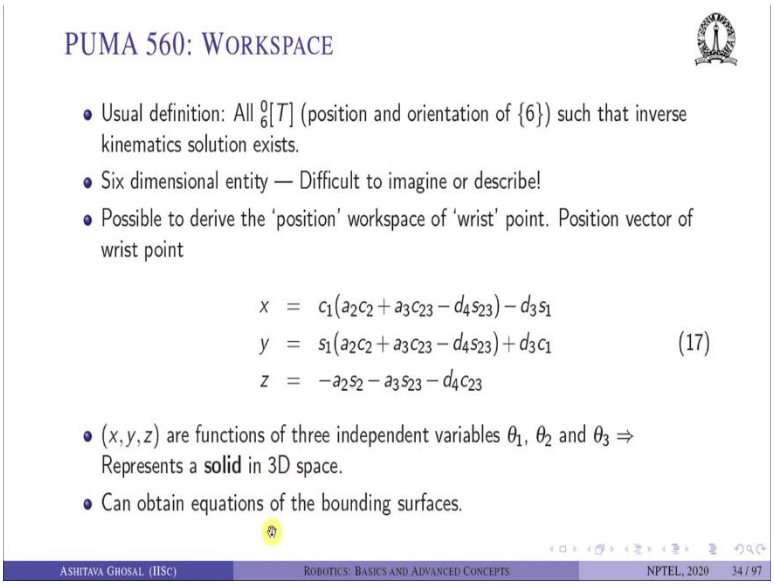

Now, let us see whether you can discuss a little bit about the workspace of this robot. So, basically under what condition the inverse kinematic solution exist. So, usual definition that all position and orientation of this 6th coordinate system such that the inverse kinematic solution exists. In this case it is very difficult to imagine or even visualize or describe, because it is a six-dimensional entity -- we have 3 positions *x*, *y*, *z* and 3 orientations.

It is possible to derive the position workspace of the wrist point. Why? Because we know that the position vector of the wrist point is *x*, *y* and *z* and they are functions of only θ_1, θ_2 and θ_3 and the constant D-H parameters. So, *x*, *y*, *z* are functions of three independent variables θ_1, θ_2 and θ_3, so it represents a solid in 3D space. So, it is like a solid region in 3D space where the inverse kinematics exist solution exist. We can find the bounding surfaces of the solid region ok.

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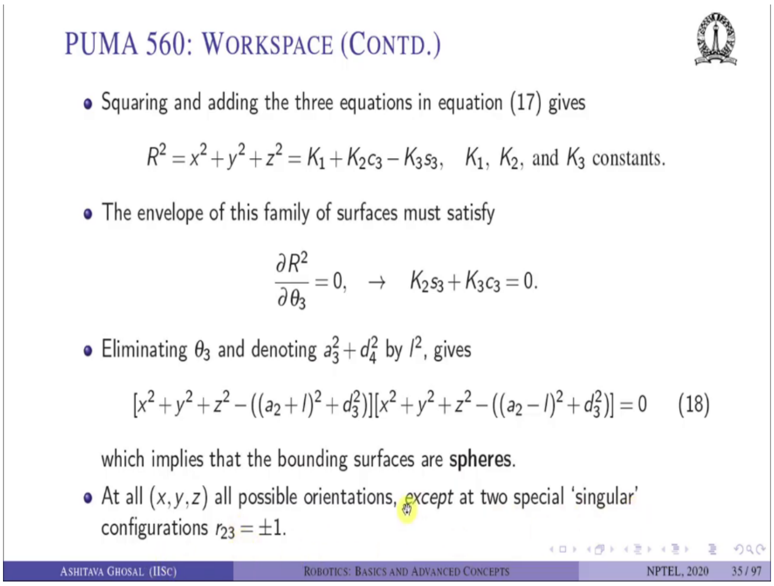

And how do we find out that? So here are the steps, so if we square and add these three equations, let us call that as R^2 which is $x^2 + y^2 + z^2$ we can see it is a function of only θ_3, K_1, K_2, K_3 are constant. So, this is a family of surfaces. So, if (x y z) $x^2 + y^2 + z^2$ was equal to constant; that is a sphere ok. But then as θ_3 changes we have a family of surfaces.

The envelope of this family of surfaces can be obtained by taking the partial derivative of this equation with respect to this variable θ_3 which is on the right-hand side. If you take this we will get one single equation $K_2 s_3 + K_3 c_3$ equal to 0 and then we eliminate θ_3 from these 2 equations ok.

So, we can eliminate in formally using Sylvester's method, but we can just simply see that we can eliminate θ_3 and if you denote $a_3^2 + d_4^2$ by l^2 we will get this expression, which is $x^2 + y^2 + z^2$. So, basically this is the radius vector, the distance from the origin is minus some number and this is also minus, so bounding surfaces at 2 spheres. So, there is a sphere which is at a distance of surface is $(a_2 + l)^2 + d_3^2$ and the other is $(a_2 - l)^2 + d_3^2$ ok. So, we have 2 spheres which are the bounding surfaces of this solid region, where the inverse kinematic solution exists. And at all (x, y, z) where this inverse kinematic solution exist, we can find the orientation except two special singular configurations when r_{23} is ± 1 ok.

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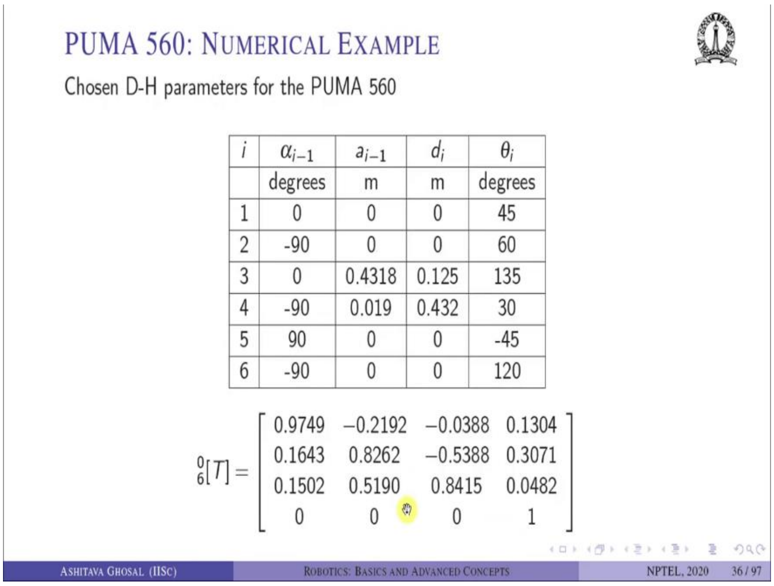

So, let us look at a numerical example ok, so this is taken from literature the DH parameter of a PUMA 560 the constant values are given in this table. We have chosen θ_i arbitrarily as 45 degrees, 60 degrees, 135, 30, - 45, and 120 and these numbers 0.4318 for a_{i-1} for link 3 and d_i is 0.125 and so on this is from literature.

So, PUMA robot comes with these D-H tables. So, once we have this D-H table and once I give you this θ then I can find out ${}^0_6[T]$. So, this is 4 by 4 homogeneous transformation matrix the last row is 0 0 0 , the rotation matrix is this top 3 by 3 and the position vector of the 6 th or the last coordinate system is given by 0.1304, 3071 and 0.0482. This is just multiplication of 6 matrices derived from taking each row of the D-H table.

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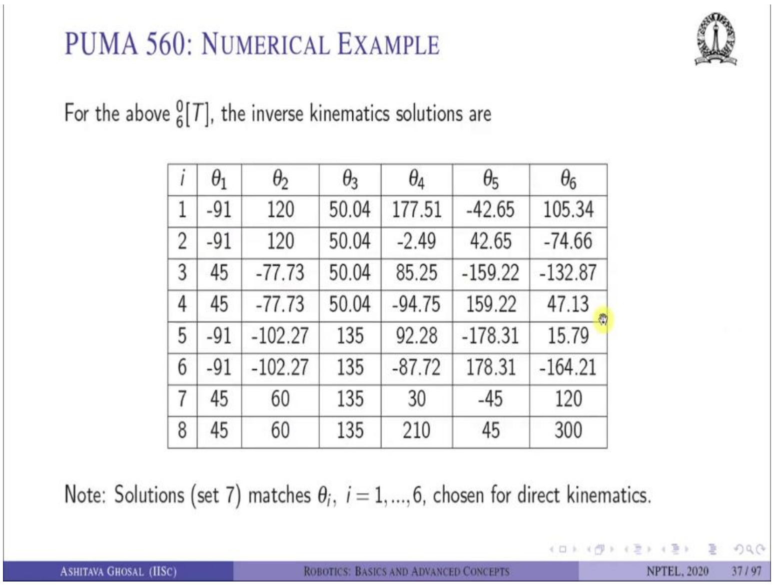

We can now take this ${}^0_6[T]$ and run our inverse kinematic steps whatever we have discussed. What are the inverse kinematic steps? We first find out θ_1 then find out θ_3 then find out θ_2 and then find out θ_4, θ_5 and θ_6 using those steps which I have discussed few minutes back. So, it turns out that I will get 8 possible solution sets - we expect 8 possible solution sets so ok.

So, θ_1, θ_2, θ_3, θ_4, θ_5, and θ_6 and the solution sets - each row corresponding to a single solution set. So, as you can see the set 7, 45, 60, 135, 30, - 45, 120 is what we started with here in the D-H table. So, that makes sense right because I took a set of constant D-H parameters and a set of values of θs, I obtained the ${}^0_6[T]$ transformation matrix and then I would run this took that same data set and run the inverse kinematics. So, one of the solution sets better be what we started with and that is indeed true.

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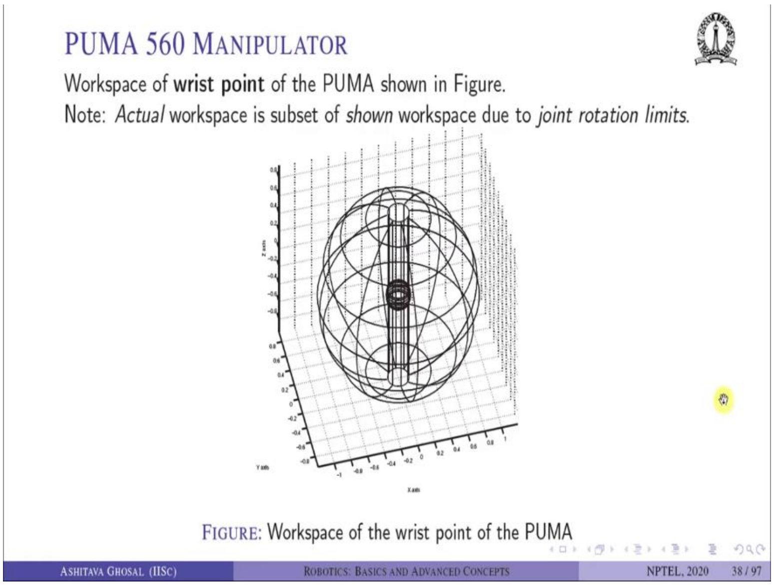

What is the workspace looks like? We know it is bounded by 2 spheres. So, there is a outside sphere and there is an inside sphere, interestingly there is also some kind of a small hollow region where it is not reachable ok. So, it is like a cylinder with some spheres and so the workspace of a PUMA robot the wrist point looks like this 2 spheres bounded by 2 spheres.

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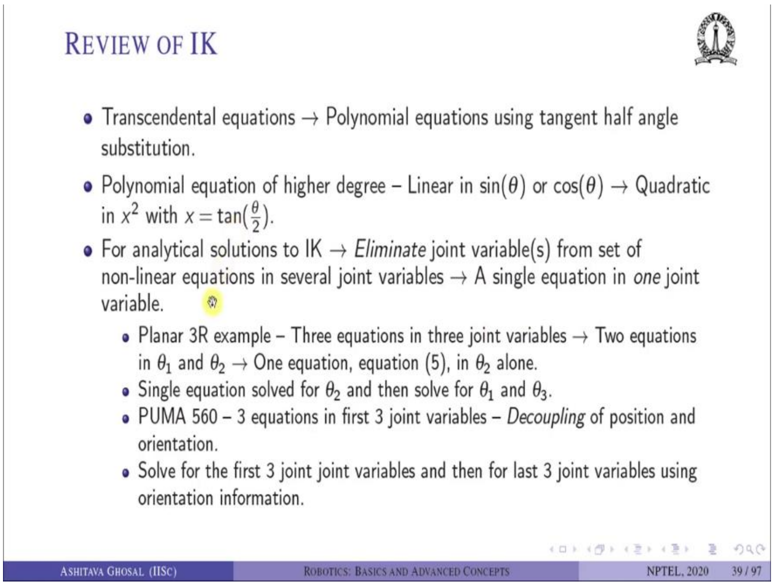

So, let us just quickly review the inverse kinematics whatever we have done till now. So, first important observation is to solve the inverse kinematics problems we have to deal with transcendental equations. So, first step is we can obtain polynomial equations using tangent half angle substitution. So, the polynomial equation is always of a higher degree. So, if you have sin θ and cos θ it will become quadratic in x^2, where x is tan $\theta/2$ ok.

For analytical solutions to the inverse kinematics problem, we have to eliminate joint variables from a set of non-linear equation in several joint variables ok. So, what do we want? We want a single equation in one joint variable that is very useful or important. So, in the case of the planar 3R example, we started with three equations in three joint variables, we obtained two equations in θ_1 and θ_2 - remember capital X and capital Y - and then we obtained one equation in θ_2 alone ok.

So, this single equation was solved for θ_2 and then we solve for θ_1 and θ_3. For the PUMA 560 we have 3 equations in 3 joint variables - the wrist point. So, the position and orientation could be decoupled, we could just take the position equations and solve for 3 angles and then using those 3 angles we could solve for the last 3 joint angles, which is which uses the orientation information.

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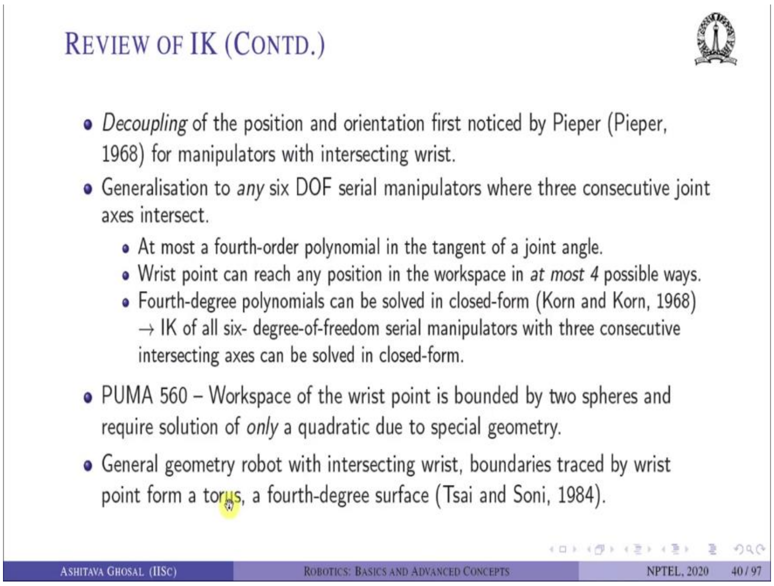

So, this decoupling of position and orientation was first noticed by Pieper in 1968 for manipulators with intersecting wrist - intersecting wrist means the last 3 joint angles joint axis intersect at a point. This was eventually generalized to any six degree of freedom serial manipulator, where three consecutive joint axes intersect. It was shown that at most a fourth order polynomial in the tangent of a joint angle is what we will get.

So, the wrist point can reach any position in the workspace in at most 4 possible ways, because we have 4 possible solutions of the tangent of the joint angle and this fourth degree polynomial which we get can be solved in closed form - this is very important. You know we can solve a quadratic equation in closed form, we can solve a cubic equation in closed form and we can also solve a quadratic equation in closed form. Any polynomial higher than 4 we cannot solve in closed form.

So, it turns out that the inverse kinematics of all six degree of freedom serial manipulators with three consecutive intersecting axis can be solved in closed form. It is a very useful and neat result. So, for the PUMA 560 the workspace of the wrist point is bounded by two spheres and requires the solution of only a quadratic. We do not have to solve a quartic equation for PUMA because of the special geometry. So, you know some axis are intersecting some axis are parallel and so on.

It was shown eventually that for general geometry robot with intersecting wrist the boundary is traced by the wrist point form a torus. So, they are not spheres, they are this solids or surfaces called torus, which is the fourth degree surface ok. A sphere is second order second degree torus is fourth degree.

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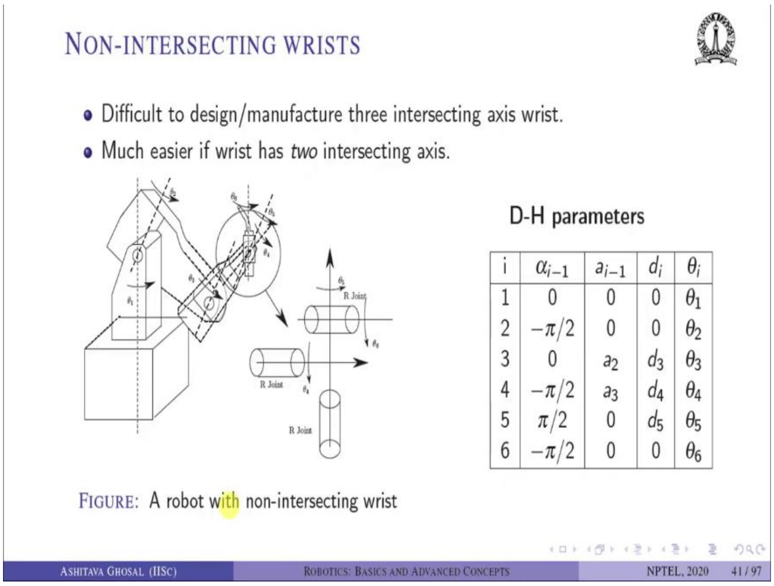

Let us continue. What happens if you have non intersecting wrist? So, intersecting wrist problem is more or less solved - you know we have 4 solutions quartic and then we can do this inverse Euler angle transformation and find the last 3 angles. However, it is very very difficult to manufacture three intersecting axis wrist.

Why? Because it is sort of - you can - imagine that you have 3 lines which are meeting at a point and then we are going to manufacture this or locate these 3 motors such that their axis intersects at a point. It will never happen manufacturing wise ok. It is much easier if the wrist has two intersecting axis ok.

So, here is an example of a robot with the last two axis intersecting and then again previous two axis intersecting; but not all three intersecting at the same place. So, schematically it is shown here θ_4 and θ_5 intersects at the same place and θ_5 and θ_6 intersect at the same place, but all of them do not intersect at the same place.

So, the D-H table for this robot, this is well known welding robot, which was built long time back, it is very similar to the PUMA except there is a d_5 here and this d_5 is non zero. So, the d_5 is the last this joint link offset for the 5 th link ok.

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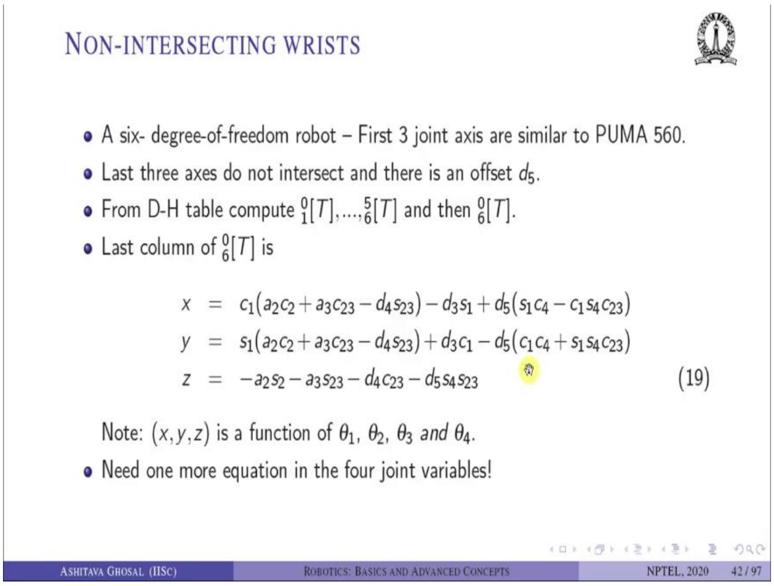

So, it is a six degree of freedom robot - first 3 joints are very similar to the PUMA 560, the last three axes do not intersect, there is an offset d_5. So, from the D-H table we can compute ${}^0_1[T]$, ${}^1_2[T]$ and all the way till ${}^0_6[T]$. So, if you compute ${}^0_6[T]$ then last column of ${}^0_6[T]$, which is the position of the last link with respect to the fixed link can be shown to be a function of now 4 joint angles. It is θ_1, θ_2, θ_3 and also θ_4.

So, if d_5 were to be 0 we will get back the equations of the PUMA, but however there are these additional terms $d_5 s_4 s_{23}$ d_5 ($s_1 c_4 - c_1 s_4 c_{23}$) and so on ok. So, what can we notice? We can see that the *x, y* and *z* the n^{th} origin of the 6^{th} coordinate system or the last link is now a function of θ_1, θ_2, θ_3 and θ_4. So, we have 3 equations in 4 unknowns. So we need one more equation in the fourth joint variables and how can we obtain this as follows.

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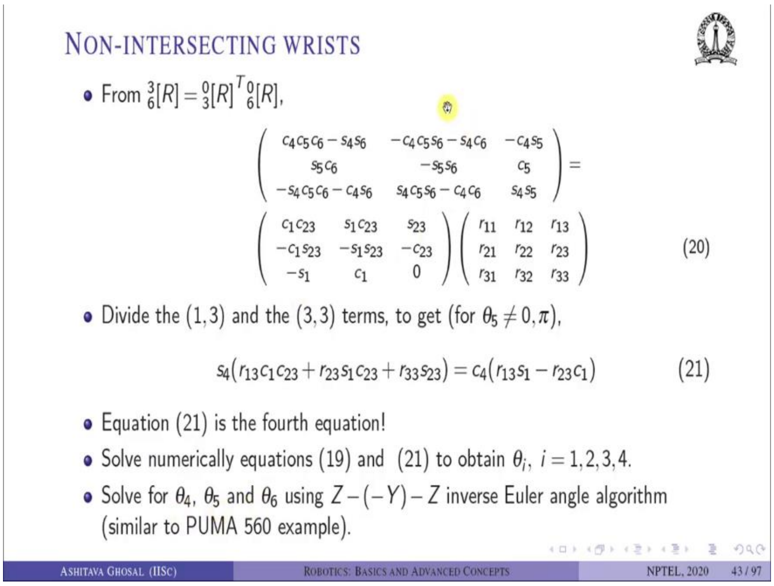

We can rewrite the ${}^3_6[R]$ which is the rotation matrix of the 6 th link with respect to the third link as product of 2 rotation matrices -${}^0_3[R]^T\,{}^0_6[R]$. Symbolically it is shown here. So, this ${}^3_6[R]$ is very similar to what we had for the PUMA, ${}^0_3[R]^T$ will be a function of θ_1, θ_2, θ_3 and ${}^0_6[R]$ is given to us - we are trying to solve the inverse kinematics problem.

So, basically we have one side θ_4, θ_5, and θ_6 and we have another side θ_1, θ_2, and θ_3. So, if you divide the (1, 3) term which is $- c_4 s_5$ and (3, 3) term which is $c_4 s_5$. So, θ_5 not equal to 0 - because we cannot divide 0 by 0- we can get one equation which is s_4 into whatever is there on the right side which is $r_{13}\, c_1 c_{23}$ and so on; will be equal to c_4 into something else where the r_{ij} in this equation are known. But θ_1, θ_2, θ_3 are unknowns, so this is an equation which involves θ_1, θ_2, θ_3 and the given r_{ij}'s and hence this is the fourth equation ok.

We had 3 equations in *x, y* and z and somehow we have managed to derive a fourth equation again in terms of θ_1, θ_2, θ_3 and θ_4. So, we have 4 equations and 4 unknowns, and we can at least solve numerically these equations to obtain θ_1, θ_2, θ_3 and θ_4. And once θ_1,

θ_2, θ_3 and θ_4 is obtained we can find theta θ_4, θ_5, θ_6 by again this inverse Euler angle algorithm similar to the PUMA Z – (-Y) - Z.

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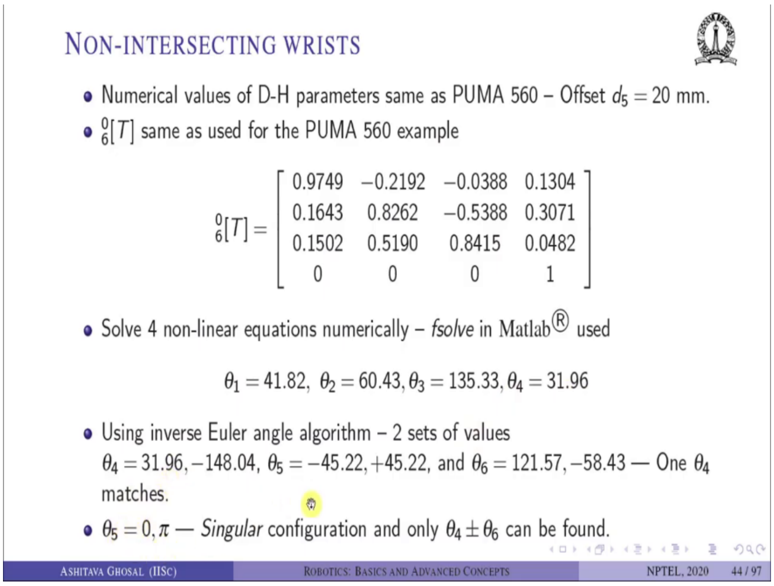

So, let us look at a numerical example the ${}^0_6[T]$ is same as the PUMA example ok, we have taken the same set of angles and the same D-H parameters and we assume d_5 is 20 mm. This is reasonable because the last link offset is small - it is not very large. So, if we solve these 4 equations in Matlab using this solve using the solution program called *fsolve*. How many if you have heard *fsolve*, I do not know but there is a way to solve non-linear equations in Matlab numerically using this routine called *fsolve* and we will get θ_1, θ_2, θ_3 and θ_4 ok. So, it is a numerical procedure, so we will have to have a certain guess and then it will converge to the final solution and it turns out we will get θ_1 as 41.82, θ_2 as 60.43, θ_3 as 135.33, θ_4 as 31.96 ok.

And using inverse Euler angle algorithm we get 2 sets of values of θ_4, θ_5, θ_6, and we can see one θ_4 matches - just to give you some confidence that our numerical solution is ok. So, θ_4 is 31.96, here also one of the θ_4 is 31.96 ok. If θ_5 were 0 or π, this is the singular configuration, and we can only solve $\theta_4 \pm \theta_6$ ok.

So, do we know these numbers are ok? Yes, because, what have we done? We have added at d_5 which is the small number. What was θ_1 before for the PUMA? It was 45, θ_2 was 60, θ_3 was 135, θ_4 was 30. So, although we have added an offset, it is sort of close to what we started with the PUMA example. So, it gives us more or less confidence that this numerical solution is correct.

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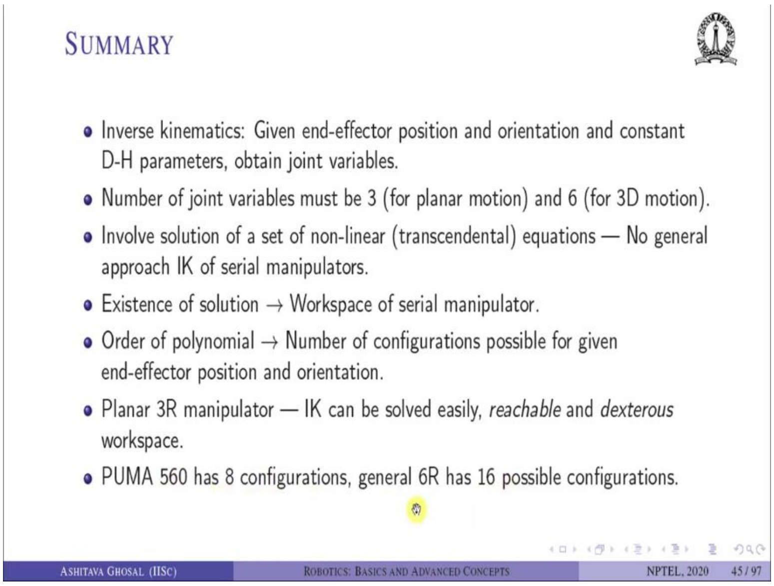

So, in summary the inverse kinematics problem is defined as given end effector position and orientation and all the constant D-H parameters, obtain the joint variables. So, the number of joint variables must be 3 for planar motion and 6 for 3D motion. Only then we have the equal number of equations and equal number of unknowns.

The inverse kinematics involve solutions of a set of non-linear transcendental equations. So, there are no general approach of inverse kinematics of serial robots. The existence of solution leads to the notion of workspace of a serial manipulator. So, we looked at the planar 3R example and stored that $\cos^{-1}$ of something, that something should lie between ±1 and then that gives this whole idea of a workspace of the planar robot.

The order of the polynomial, the single polynomial which you obtain to find one of the joint variables gives you the number of possible configurations for a given end effector position and orientation. So, in the case of planar 3R robot the IK inverse kinematics could be easily solved, we could also give this notion of a reachable workspace and dexterous workspace. So, we could find what is the furthest the robot can reach and what is the region in the workspace where you could achieve arbitrary orientation.

The PUMA 560 has 8 possible configurations and we will see later on in this week that the general 6 degree of freedom robot has 16 possible configurations. So, with this I will stop. In the next lecture we will look at 2 special kinds of serial robots, one in which the number of joint angles is less than 6 and the number of joint angle is greater than 6.