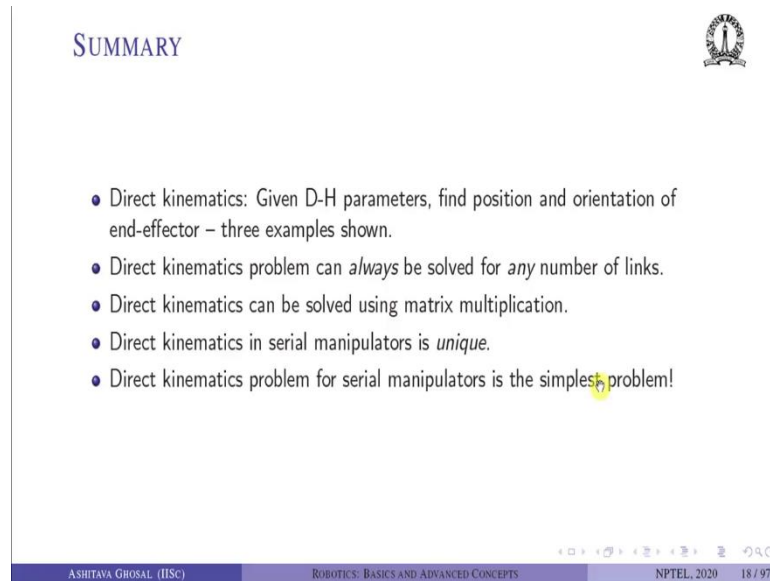


**Robotics: Basics and Selected Advanced Concepts**  
**Prof. Ashitava Ghosal**  
**Department of Mechanical Engineering**  
**Indian Institute of Science, Bengaluru**

**Lecture – 08**  
**Inverse Kinematics of Serial Robots**

(Refer Slide Time: 00:21)



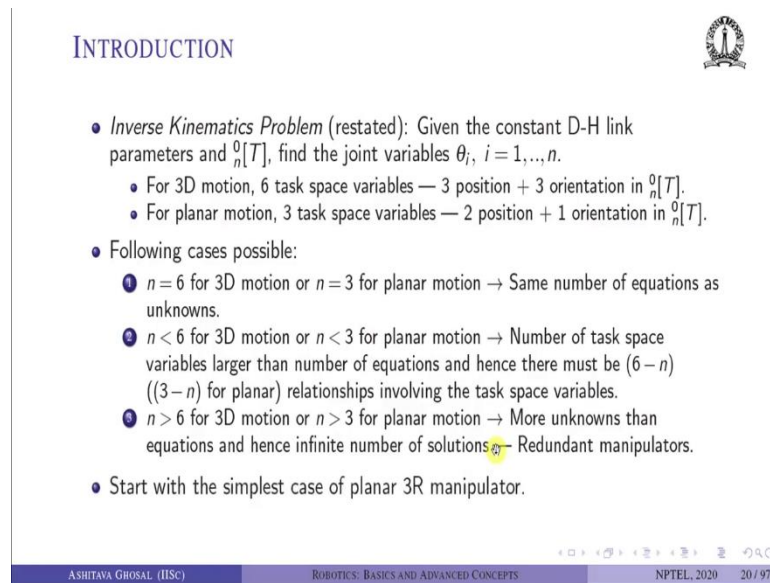
The slide is titled "SUMMARY" in blue text. It features a list of five bullet points. The first bullet point is "Direct kinematics: Given D-H parameters, find position and orientation of end-effector – three examples shown." The second is "Direct kinematics problem can *always* be solved for *any* number of links." The third is "Direct kinematics can be solved using matrix multiplication." The fourth is "Direct kinematics in serial manipulators is *unique*." The fifth is "Direct kinematics problem for serial manipulators is the simplest problem!" The word "simplest" is highlighted in yellow. In the top right corner, there is a small circular logo of the Indian Institute of Science. At the bottom, there is a footer with the text "ASHITAVA GHOSAL (IISc)", "ROBOTICS: BASICS AND ADVANCED CONCEPTS", "NPTEL, 2020", and "18 / 97".

Welcome to this NPTEL course on Robotics: Basics and Advanced Concepts. In the last lecture we had looked at the direct kinematics of serial robots. In this lecture, we will look at the Inverse kinematics of serial robots. To recapitulate in the last lecture the direct kinematics problem was stated and the direct kinematics problem is given the DH parameters find the position and orientation of the end effector.

I had shown you three examples a planar example the PUMA 560 example and the SCARA robot. The direct kinematics can always be solved for any number of links. It just simply involves multiplication of matrices ok. So, there is no reason why we cannot multiply matrices and hence for any number of links you give me I need to obtain the link transformation matrices and then just multiply in some appropriate order.

The direct kinematics in serial manipulator is unique ok. So, there is only one possible solution to the direct kinematics problem and the direct kinematics problem for serial manipulator is the simplest problem. In this lecture we look at the inverse kinematics of a serial robot.

(Refer Slide Time: 01:47)



**INTRODUCTION**

- *Inverse Kinematics Problem* (restated): Given the constant D-H link parameters and  ${}^0_n[T]$ , find the joint variables  $\theta_i$ ,  $i = 1, \dots, n$ .
  - For 3D motion, 6 task space variables — 3 position + 3 orientation in  ${}^0_n[T]$ .
  - For planar motion, 3 task space variables — 2 position + 1 orientation in  ${}^0_n[T]$ .
- Following cases possible:
  - 1  $n = 6$  for 3D motion or  $n = 3$  for planar motion → Same number of equations as unknowns.
  - 2  $n < 6$  for 3D motion or  $n < 3$  for planar motion → Number of task space variables larger than number of equations and hence there must be  $(6 - n)$  ( $(3 - n)$  for planar) relationships involving the task space variables.
  - 3  $n > 6$  for 3D motion or  $n > 3$  for planar motion → More unknowns than equations and hence infinite number of solutions — Redundant manipulators.
- Start with the simplest case of planar 3R manipulator.

ASHITAVA GHOSAL, (IISc)      ROBOTICS: BASICS AND ADVANCED CONCEPTS      NPTEL, 2020      20 / 97

So, what is the inverse kinematics problem? To restate it once more, given the constant D-H link parameters and the position and orientation of the last link or the end effector  ${}^0_n[T]$ , find the joint variables  $\theta_i$ ,  $i$  equals 1 through  $n$  ok. So, what are the constant D-H link parameters? Just to recapitulate we have a twist angle, we have a link length and either a link offset or a link rotation depending on what kind of joint it is.

To continue for 3D motion, we have 6 task space variables. So, 3 position and 3 orientation in this transformation matrix  ${}^0_n[T]$ . And for planar motion there are 3 task space variables, 2 position let us say  $x$  and  $y$  and one orientation in this given transformation matrix.

So, there are following cases which are possible now. So, if  $n$  is equal to 3 for planar motion or  $n$  is equal to 6 for 3D motion, we have the same number of equations or same number of unknowns. Just or what do you mean by same number of equations? In  ${}^0_n[T]$  we know that there are independent equations coming from the rotation matrix and from the last column which is the position vector.

If  $n$  is 6 then in the  ${}^0_n[T]$ , we can get 3 equations from the rotation part and 3 from the translation part and hence we have 6 task space variables and we have 6 joined variables -  $n$  is equal to 6. For  $n$  less than 6 for 3D motion or  $n$  less than 3 for planar motion, the number of task space variables are larger than the number of equations and hence there must be  $6 - n$  for spatial and  $3 - n$  for planar relationships involving the task space variables.

So, if you recollect in the SCARA robot the motion which is the 4 degree of freedom robot and the end effector can be determined by position can be determined by  $x$ ,  $y$ ,  $z$  and 1 orientation. So, although the end effector is moving in 3D space there are the 2 other angles orientation angles are not there. So, hence there are some constraints between the task space variables.

If  $n$  is greater than 6 for 3D motion or  $n$  is greater than 3 for planar motion, we have more unknowns than equations and hence infinite number of solutions. These are a special class of manipulators which are called redundant manipulators and we will be looking at them in detail later. So, let us start the inverse kinematics problem for the simplest case of a planar 3R manipulator.

(Refer Slide Time: 05:03)

### PLANAR 3R MANIPULATOR

- Direct kinematics equations
 
$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$

$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$

$$\phi = \theta_1 + \theta_2 + \theta_3$$
- Inverse Kinematics: Given  $(x, y, \phi)$  obtain  $\theta_1, \theta_2$  and  $\theta_3$ .
- Solution of system of *non-linear* transcendental equations.
- No general methods (as in linear equations) exists — Solution procedure depends on problem.

FIGURE: The planar 3R manipulator

ASHITAVA GHOSAL (IISc)
ROBOTICS: BASICS AND ADVANCED CONCEPTS
NPTEL, 2020 21 / 97

So, the figure in this slide show a planar 3R manipulator, basically there are 3 rotary joints at  $O_1$ ,  $O_2$  and  $O_3$ . The direct kinematics equation are known. So, we can obtain the  $x$ ,  $y$  and orientation of the  $\{Tool\}$  coordinate system or the third coordinate system as given by  $l_1 c_1 + l_2 c_{12} + l_3 c_{123}$  and  $y$  is the  $l_1 s_1 + l_2 s_{12} + l_3 s_{123}$ . And the orientation of the tool coordinate system or the  $n$  th coordinate system, in this case  $n$  is 3, is given by  $\theta_1 + \theta_2 + \theta_3$ . So, what is the inverse kinematics problem for this planar 3R manipulator? We are given  $x$ ,  $y$  and  $\phi$ , the left-hand side of these equations and we want to obtain by  $\theta_1, \theta_2$  and  $\theta_3$ .

So, what you can see is these are 3 non-linear equations. So, the first 2 involves sin and cosine of angles ok. So, they are non-linear transcendental equations, the third one is linear. In order to solve these 3 non-linear equations or 2 non-linear and 1 linear equation we do not have any general method to solve non-linear transcendental equations.

The solution of non-linear transcendental equations depends on the problem. This is unlike a linear set of equations - a set of linear equations where we can easily solve if you have  $y$  equals  $[A]x$ , we can always find solution  $x$  for a given  $y$  provided certain conditions are met on that matrix  $[A]$ .

(Refer Slide Time: 07:03)

### INVERSE KINEMATICS ALGORITHM

- Define  $X = x - l_3 c_\phi$  and  $Y = y - l_3 s_\phi$  -  $X$  and  $Y$  are known since  $x$ ,  $y$ ,  $\phi$  and  $l_3$  are known.
- Squaring and adding
 
$$X^2 + Y^2 = l_1^2 + l_2^2 + 2l_1 l_2 c_2 \quad (5)$$
- From equation (5)
 
$$\theta_2 = \pm \cos^{-1} \left( \frac{X^2 + Y^2 - l_1^2 - l_2^2}{2l_1 l_2} \right) \quad (6)$$
- Once  $\theta_2$  is known
 
$$\theta_1 = \text{atan2}(Y, X) - \text{atan2}(k_2, k_1) \quad (7)$$

where  $k_2 = l_2 s_2$  and  $k_1 = l_1 + l_2 c_2$ . Note:  $\text{atan2}(y, x)$  is the four quadrant arc-tangent function and  $\theta_1 \in [0, 2\pi]$ .
- Finally,  $\theta_3$  is obtained from
 
$$\theta_3 = \phi - \theta_1 - \theta_2 \quad (8)$$

ASHITAVA GHOSAL (IITB)     ROBOTICS: BASICS AND ADVANCED CONCEPTS     NPTEL, 2020 22 / 97

So, what do we do? So, what we can do is we can see that we can define a new variable capital  $X$  which is  $x - l_3 \cos \phi$  and a capital  $Y$  which is  $y - l_3 \sin \phi$  ok. So,  $X$  and  $Y$ , capital  $X$  and capital  $Y$ , are known, because since small  $x$  small  $y$  and  $\phi$  are given - this is the inverse kinematics problem.

So, if you now add and square and add  $X^2 + Y^2$  square we can see that you will get  $l_1^2 + l_2^2 + 2 l_1 l_2 \cos \theta_2$ . Hence, from equation (5) we can show that  $\theta_2$  is  $\cos^{-1} (X^2 + Y^2 - l_1^2 - l_2^2) / (2 l_1 l_2)$  ok.

So, what have we done? We have started from these 3 equations we have defined capital  $X$  as  $x - l_3 \cos \phi$ , so this has gone there. So,  $X$  is now  $l_1 c_1 + l_2 c_{12}$ ,  $Y$  is also  $l_1 s_1 + l_2 s_{12}$ ; so, when you square and add  $X$  and  $Y$ , so  $(l_1 c_1)^2 + (l_1 s_1)^2$ ,  $\theta_1$  will go away.

Likewise,  $\theta_{12}$  will go away and we will be left with only this equation -  $X^2 + Y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 c_2$  and we can find  $\theta_2$  ok. So, I am going over this little slowly because we can easily see what we are trying to do ok.

Once  $\theta_2$  is known we can find out what is  $\theta_1$  by doing this  $\text{atan2}(Y, X) - \text{atan2}(k_2, k_1)$  ok, where  $k_2$  is  $l_2 s_2$  and  $k_1$  is  $l_1 + l_2 c_2$  ok. So, basically what are we doing? We are saying  $X$  is some  $l_1 c_1 + l_2 c_{12}$ . So, we can expand  $c_{12}$  as  $\cos a \cos b - \sin a \sin b$  and so on, and then take terms which are in  $\theta_2$  which is now known and solve for  $\theta_1$ .

So, we are going to use  $\text{atan2}$  again such that we can get this  $\theta_1$  in the correct quadrant. And then finally  $\theta_3$  is obtained from  $\phi - \theta_1 - \theta_2$ . So, as you can see, we have used some knowledge of trigonometry to first find  $\theta_2$ , then we taken terms which are now known and solved for  $\theta_1$  and then finally  $\theta_3$  is simply  $\phi - \theta_1 - \theta_2$  ok.

(Refer Slide Time: 10:13)

### 3R MANIPULATOR – WORKSPACE

- Workspace: All  $(x, y, \phi)$  such that inverse kinematics solution exists.
- From equation (6)
 
$$-1 \leq \left( \frac{X^2 + Y^2 - l_1^2 - l_2^2}{2l_1 l_2} \right) \leq +1$$

$$(l_1 - l_2)^2 \leq (X^2 + Y^2) \leq (l_1 + l_2)^2 \quad (9)$$
 where  $X = x - l_3 c_\phi$  and  $Y = y - l_3 s_\phi$ .
- Figure shows the region in  $\{x, y, \phi\}$  space where the above inequalities are satisfied and the inverse kinematics solution exists.

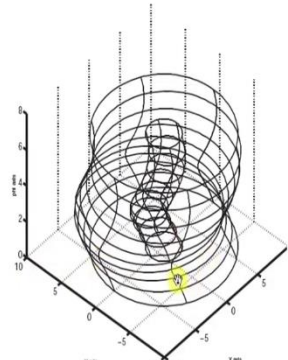


FIGURE: Workspace of a planar 3R robot

ASHITAVA GHOSAL (IISc)
ROBOTICS: BASICS AND ADVANCED CONCEPTS
NPTEL, 2020 23 / 97

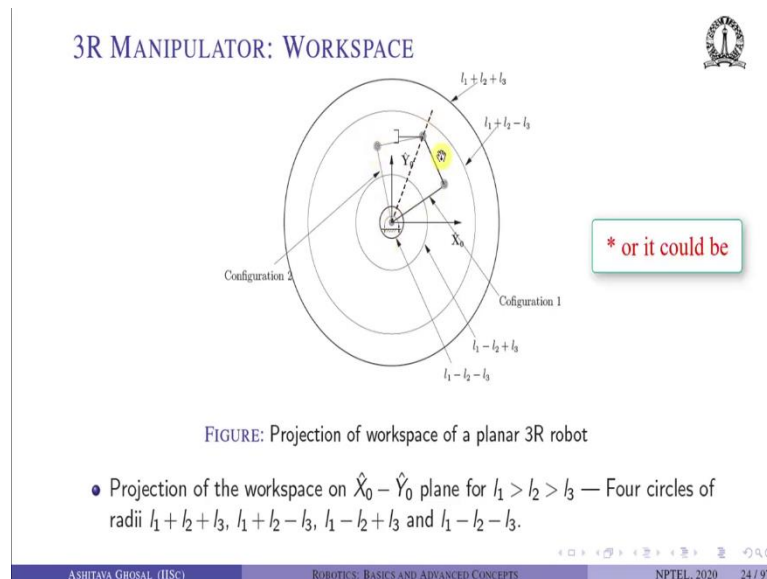
So, let us continue. So can we solve this inverse kinematics problem everywhere? The answer is no. Why? Because if you look at this equation  $\theta_2$  is  $\pm \cos^{-1}$  of something; so, whatever is inside this bracket which is  $(X^2 + Y^2 - l_1^2 - l_2^2) / 2 l_1 l_2$  must be within  $\pm 1$ , because only  $\cos^{-1}$  of a number which is within  $\pm 1$  is defined ok.

So, hence we can say that this condition must always be true - which is this quantity  $((X^2 + Y^2 - l_1^2 - l_2^2) / 2 l_1 l_2)$  lie between  $\pm 1$ . And we can now simplify this and write it as  $(X^2 + Y^2)$  should be greater than  $(l_1 - l_2)^2$  and less than  $(l_1 + l_2)^2$ .

And finally, we can substitute back  $X$  as  $x - l_3 \cos \phi$  and  $Y$  as  $y - l_3 \sin \phi$  ok. The figure in this right-hand side shows a plot of  $x, y$  and angle  $\phi$  phi ok. So, every section represents a circle of radius between  $(l_1 + l_2)^2$  and  $(l_1 - l_2)^2$  and then if you substitute back this  $x - l_3 \cos \phi$  and  $y - l_3 \sin \phi$ , you will get this complicated looking figure ok.

So, hence only points which are lying between these big circles and small circles,  $x, y$  and  $\phi$  - and  $\phi$  is along the vertical direction, you can only find cosine inverse of that number ok. So, this is called the workspace of the planar robot. So, all  $x, y$  and  $\phi$  which is inside this region, which is shown here, I can solve the inverse kinematics.

(Refer Slide Time: 12:37)

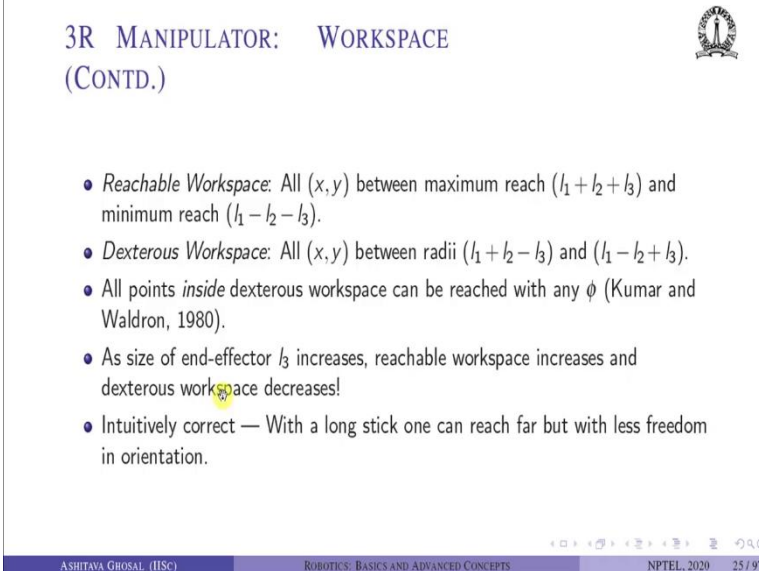


We can also try and see what is exactly happening a little bit more detail. So, I can project all these circles back to the  $X-Y$  plane. So, I will get the following circles. So, I will get one completely outside circle which is  $l_1 + l_2 + l_3$  there will be also one small inside circle which is  $l_1 - l_2 - l_3$  and then there are these 2 other circles - one is  $l_1 + l_2 - l_3$  and one is  $l_1 - l_2 + l_3$  ok. So, this is nothing but the projection of those that figure onto the  $X-Y$  plane ok.

So we have these 4 circles, so now if you pick a point on this region between  $l_1 - l_2 + l_3$  and  $l_1 + l_2 - l_3$  so this is the point. As you can see so this point is nothing but  $X, Y$  ok, it is  $x - l_3 \cos \phi$  and  $y - l_3 \sin \phi$ . So, this  $X, Y$  this is the point and I can reach this point in 2 ways. So, the planar robot can be like this and the (Refer Time: 14:00) and it could be like this, that is what is meant by there are 2 possible solutions for theta 2.

So, if you go back and see there are 2 possible solutions of  $\theta_2$  is  $\pm \cos^{-1}$  of this ok. And geometrically what is happening? One is like this. So, this is the angle  $\theta_2$  and the other one is  $-\theta_2$  to reach that same  $X$  and  $Y$ .

(Refer Slide Time: 14:31)



3R MANIPULATOR: WORKSPACE  
(CONTD.)

- *Reachable Workspace*: All  $(x, y)$  between maximum reach  $(l_1 + l_2 + l_3)$  and minimum reach  $(l_1 - l_2 - l_3)$ .
- *Dexterous Workspace*: All  $(x, y)$  between radii  $(l_1 + l_2 - l_3)$  and  $(l_1 - l_2 + l_3)$ .
- All points *inside* dexterous workspace can be reached with any  $\phi$  (Kumar and Waldron, 1980).
- As size of end-effector  $l_3$  increases, reachable workspace increases and dexterous workspace decreases!
- Intuitively correct — With a long stick one can reach far but with less freedom in orientation.

ASHITAVA GHOSAL (IISc)      ROBOTICS: BASICS AND ADVANCED CONCEPTS      NPTEL, 2020      25 / 97

So, the reachable workspace is defined as all  $(x, y)$  between the maximum reach which is  $(l_1 + l_2 + l_3)$  and minimum reach which is  $(l_1 - l_2 - l_3)$ . We can also define something called as a dexterous workspace and this is all  $(x, y)$  between the radius  $(l_1 + l_2 - l_3)$  and  $(l_1 - l_2 + l_3)$  ok.

So, all points inside the dexterous workspace can be reached with any  $\phi$ . So, as the size  $l_3$  increases what you can see that the dexterous workspace will become smaller ok. The reachable workspace will increase because it is  $(l_1 + l_2 + l_3)$  ok, whereas the dexterous workspace is  $(l_1 + l_2 - l_3)$  ok. So, this is the well-known result which was obtained long time back and it is very nice result which says that as the size of the end effector  $l_3$  increases reachable workspace increases and dexterous workspace decreases. And this is also sort of intuitively correct. So, if you are holding a long stick you can reach far away you can reach the roof of the room, but if you use a long stick at the end you have very little freedom in orienting the tip. So, that is the whole idea of a dexterous workspace I can reach any point with whatever orientation I feel I want.

(Refer Slide Time: 16:13)

### 3R MANIPULATOR: UNIQUENESS OF IK SOLUTIONS



- Equation (6) revisited:

$$\theta_2 = \pm \cos^{-1} \left( \frac{X^2 + Y^2 - l_1^2 - l_2^2}{2l_1 l_2} \right)$$

For any  $(X, Y)$  two values of  $\theta_2$ . The two solutions merge at the workspace boundary.

- A given  $(X, Y)$  can be achieved by two configurations as shown in Figure.
- For planar 3R manipulator  $(x, y, \phi)$  yields two sets of values of  $\theta_i$ ,  $i = 1, 2, 3$ .
- Inverse kinematics problem *does not* give unique solution – Compare with direct kinematics!
- Existence and uniqueness issues important and non-trivial in solutions of *almost all* non-linear equations.

Navigation icons: back, forward, search, etc.

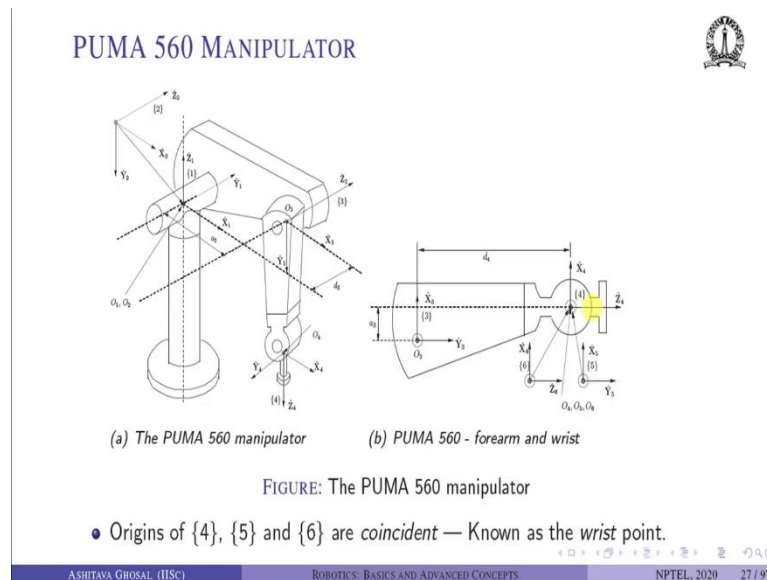
Let us go back to again that equation of  $\theta_2$  ok. So, I showed you  $\theta_2$  is  $\pm \cos$  inverse this. So, hence for any  $(X, Y)$ , there are two possible values of  $\theta_2$  and these 2 values merge at the workspace boundary. So, at a given  $(X, Y)$  can be achieved by 2 configurations as shown in the figure, I showed you this figure, that if I take any point  $(X, Y)$  here I can reach either like this or I can reach like this. So, one of them is  $+\theta_2$  and the other one is  $-\theta_2$ .

So, in general what we can say is for a planar 3R manipulator, given  $x, y$  and  $\phi$  we can obtain this  $x, y$  and  $\phi$  by 2 sets of values of  $\theta_1, 2$  and  $3$  ok. So, in conclusion the inverse kinematics problem does not give unique solutions ok. Remember in direct kinematics we always could get a unique solution ok. Multiplying matrices given  $\theta$ s I could find the position and orientation of the end effector uniquely. In the inverse kinematics problem given the position and orientation of the end effector, I get in this case of a planar 3R, 2 possible sets of values of  $\theta_1, \theta_2$  and  $\theta_3$ .

So, in general or more abstractly what we have is something called as a existence and uniqueness issues in solution of non-linear equation ok. So, it is quite hard and non-trivial to obtain the existing and uniqueness conditions for solution of non-linear equation. In this case uniqueness means what? We have 2 possible sets the solutions are not unique and existence means what we can obtain the values of  $x, y$  and  $\phi$  such that we can get a inverse kinematic solution ok. So, all  $x, y, \phi$  for which inverse kinematic solution exists, that gives us this existence criteria and uniqueness criteria is how many possible joint angles satisfies the given  $x, y$  and  $\phi$ .

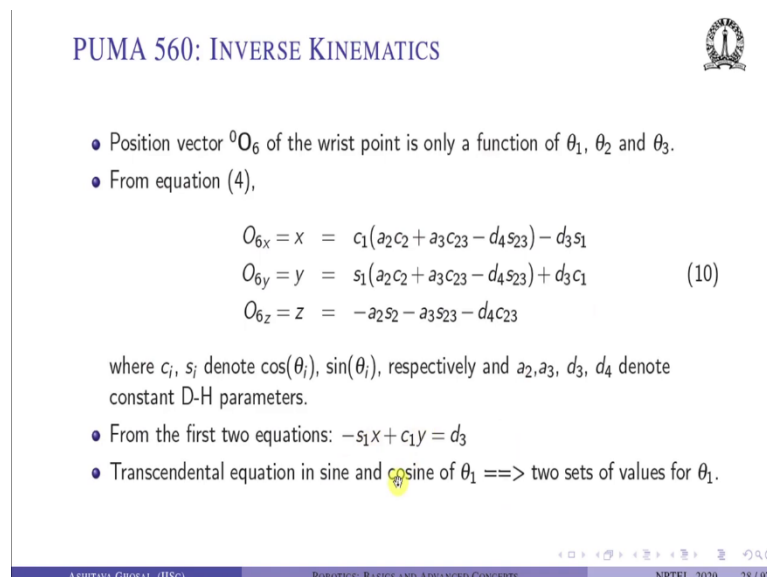


(Refer Slide Time: 18:55)



Now, let us continue we look at a 6 degree of freedom PUMA 560 robot. This has been shown earlier also there are 6 joints 1 is along Z<sub>1</sub>, one is along Z<sub>2</sub> then Z<sub>3</sub> and then there are 3 more which are intersecting at this point which is the point of intersection of axis 4, 5 and 6. So, this origins of {4}, {5} and {6} are coincident and this is called the wrist point.

(Refer Slide Time: 19:27)




So, it turns out that the position vector of this wrist point is only a function of  $\theta_1$ ,  $\theta_2$  and  $\theta_3$  ok. So, these are the 3 equations  $O_{6x}$  which is shortened as  $x$  is given by  $c_1(a_2c_2 + a_3c_{23} -$

$d_4 s_{23}$ )  $- d_3 s_1$  and so on ok. So, does this make sense? Yes if you think about it little bit that the links 4, 5 and 6 are after the joint axis 4, 5 and 6 ok. So, hence the position vector which is lying on the joint 4, axis 5, or 6 axis because all of them are at the same place, can only be a function of all the angles before this origin. And what are the angles? One is  $\theta_1$ , one is  $\theta_2$  and one is  $\theta_3$ , the  $\theta_4$ ,  $\theta_5$  and  $\theta_6$  affect the orientation of the end effector it does not affect the origin of the last link ok.

So, how do we solve the inverse kinematics of the PUMA? So, basically what are we given? We are given  $x$ ,  $y$  and  $z$  and fortunately we now have 3 equations in 3 unknowns. What are the unknown's -  $\theta_1$ ,  $\theta_2$  and  $\theta_3$ .

So, from this first 2 equations if you multiply the first equation by  $-s_1$  and the second equation by  $+c_1$  --  $s_1$  is  $\sin \theta_1$ ,  $c_1$  is  $\cos \theta_1$  -- and add them you can see everything drops out and we get a nice simple equation which is  $-s_1 x + c_1 y = d_3$ . So, you will get  $d_3 s_1^2$ ,  $d_3 c_1^2$ , when you add them, they will become 1 ok. So, we have a single transcendental equation in  $\theta_1$  ok. So, how do we solve this?

(Refer Slide Time: 21:43)

PUMA 560: IK (CONTD.)


- Substitute *tangent half-angle* formulas from trigonometry

$$x_1 = \tan \frac{\theta_1}{2}, \quad c_1 = \frac{1 - x_1^2}{1 + x_1^2}, \quad s_1 = \frac{2x_1}{1 + x_1^2} \quad (11)$$

in  $-s_1 x + c_1 y = d_3 \implies x_1^2(d_3 + y) + (2x)x_1 + (d_3 - y) = 0$

- Solve quadratic in  $x_1$  and take  $\tan^{-1}$  to get

$$\theta_1 = 2 \tan^{-1} \left( \frac{-x \pm \sqrt{x^2 + y^2 - d_3^2}}{y + d_3} \right) \quad (12)$$

- Note 1:  $\tan^{-1}$  gives an angle between 0 and  $\pi$  and hence  $0 \leq \theta_1 \leq 2\pi$ .
- Note 2: Two possible values of  $\theta_1$  due to the  $\pm$  before square root.

ASHITAVA GHOSAL (IISc)
ROBOTICS: BASICS AND ADVANCED CONCEPTS
NPTEL, 2020 29 / 97

So, the kinematics people have figured out this really nice way of trying to solve a single transcendental equation. So, the idea is that we convert a transcendental equation into a polynomial and how can we do that? Let us define a new variable  $x_1$  which is  $\tan \theta_1/2$  ok. So, if  $x_1$  is  $\tan \theta_1/2$ ,  $\cos \theta_1$  is  $(1 - x_1^2)/(1 + x_1^2)$  and  $\sin \theta_1$  is  $2x_1/(1 + x_1^2)$ .

So, hence  $-s_1 x + c_1 y = d_3$  can be written as a quadratic in  $x_1$  and remember  $x_1$  is  $\tan \theta_1/2$ . So, what have we done? We have taken a transcendental equation in  $\sin$  and  $\cos \theta_1$  and obtained a quadratic in  $\tan \theta_1/2$  ok. So, this is the well-known tangent half angle formulas from trigonometry.

So, once I have a quadratic of this form, I can easily solve for  $x_1$ . Now quadratic equations, we know the roots of a quadratic equation in closed form and then we can find out  $\theta_1$  which is  $\tan^{-1}$  of that quantity and since we are finding out  $\theta_1/2$ . So, actual  $\theta_1$  is  $2 \tan^{-1}$  of the roots of the quadratic equation ok.

So, a few observations so  $\tan$  inverse gives an angle between  $0$  and  $\pi$  normally. So, hence  $2 \tan^{-1}$  gives the value between  $0$  and  $2\pi$ , so we are getting the angle in the right quadrant. So, we do not have to use  $\text{atan2}$  here -- this idea of a tangent half angle makes ensures or makes us get an angle which is in the right quadrant.

The second observation is we get two possible values of  $\theta_1$  due to this  $\pm$  sign in the square root ok. So, again uniqueness no ok. So given  $x$ ,  $y$  and  $z$  in this form of equation of the wrist point, I am getting a value of  $\theta_1$  which are 2 of them. So, this is not unique - very similar to the previous case of the planar 3R case, where we could get two possible values of an angle which satisfies a given position and orientation. In this case two possible values of  $\theta_1$  which for a given wrist point ok.

(Refer Slide Time: 24:29)

### PUMA 560: IK (CONTD.)

- Squaring and adding expressions for  $x$ ,  $y$  and  $z$   

$$x^2 + y^2 + z^2 = d_3^2 + a_2^2 + a_3^2 + d_4^2 + 2a_2a_3c_3 - 2a_2d_4s_3$$
- Using tangent half-angle formulas

$$\theta_3 = 2 \tan^{-1} \left( \frac{-d_4 \pm \sqrt{d_4^2 + a_3^2 - K^2}}{K + a_3} \right), \quad \text{Two sets of values.} \quad (13)$$

$K = (1/2a_2)(x^2 + y^2 + z^2 - d_3^2 - a_2^2 - a_3^2 - d_4^2)$ , constant.

- $z = f(\theta_2, \theta_3)$ :  $-s_2(a_2 + a_3c_3 - d_4s_3) + c_2(-a_3s_3 - d_4c_3) = z$
- Solve for  $\theta_2$  (for known  $\theta_3$ ) using tangent half-angle substitutions

$$\theta_2 = 2 \tan^{-1} \left( \frac{-a_2 - a_3c_3 + d_4s_3 \pm \sqrt{a_2^2 + a_3^2 + d_4^2 + 2a_2(a_3c_3 - d_4s_3) - z^2}}{z - (a_3s_3 + d_4c_3)} \right) \quad (14)$$

- Four possible values of  $\theta_2$  in the range  $[0, 2\pi]$ .

◀ ▶ ⏪ ⏩ 🔍 🔄


Let us continue. If you square and add those 3 equations, which is  $x^2 + y^2 + z^2$ , you can see that you will end up with a single equation in  $\theta_3$ . So, this is correct? Yes. So,  $x^2 + y^2 + z^2$ , so  $\theta_1$  will vanish,  $\theta_2$  will also vanish. It turns out, because you can see the pattern - it is  $-a_2 s_2 - a_3 s_{23} - d_4 c_{23}$ ; whereas, here it is, you know, plus and minus, sort of very similar but here it is  $c_2$ , here it is  $s_2$  ok.

So, if you expand, the square and adding that expression and do a little bit of simplification you can get a single equation in cosine  $\theta_3$  and sin  $\theta_3$  ok. Again, this is a transcendental equation in cosine and sin  $\theta_3$ , we can again use the tangent half angle formulas to obtain  $\theta_3$ .

So, here also you can see that you get 2 sets of values of  $\theta_3$ . Again square root solutions of the quadratic equation where  $K$  is a constant ok. Finally, we can see that the last equation  $z$  is function of only  $\theta_2$  and  $\theta_3$ . So now that we know  $\theta_3$ , we can collect terms with a  $\theta_3$  inside this bracket and again we have a simple transcendental equation in  $\theta_2$ .

So,  $-s_2$  into something which is now known,  $c_2$  into something which is now known equal to  $z$  and we can solve this again using tangent half angle substitution. And we will get  $\theta_2$  as  $2 \tan^{-1}$  of really complicated long expression ok. So, how many values of  $\theta_2$  we get? We get four possible values of  $\theta_2$ . Why? Because  $\theta_3$  already had 2 possible values and  $\theta_2$  depends on  $\theta_3$ , so you can see it is  $a_3 c_3$  is there  $a_3 c_3$ ,  $d_4 s_3$  all these terms are here. So, we get four possible values of  $\theta_2$  in the range 0 to  $2\pi$ .

(Refer Slide Time: 26:55)

PUMA 560: IK (CONTD.)


- To obtain  $\theta_4$ ,  $\theta_5$  and  $\theta_6$ , form
 
$${}^3_6[R] = \begin{pmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & -c_4 s_5 \\ s_5 c_6 & -s_5 s_6 & c_5 \\ -s_4 c_5 c_6 - c_4 s_6 & s_4 c_5 s_6 - c_4 c_6 & s_4 s_5 \end{pmatrix} \quad (15)$$
- From matrix multiplication
 
$${}^3_6[R] = {}^0_3[R] {}^0_6[R] \quad (16)$$

\* 0-6-R = 0-3-R times 3-6-R

and since  $\theta_1$ ,  $\theta_2$  and  $\theta_3$  are known, right-hand side is known!

- Compare known right-hand side with elements of  ${}^3_6[R]$  and obtain  $\theta_4$ ,  $\theta_5$  and  $\theta_6$
- Similar to Z - Y - Z Euler angles with Y rotation of  $(-\theta_5)$

ASHITAVA GHOSAL (IISc)
ROBOTICS: BASICS AND ADVANCED CONCEPTS
NPTEL, 2020 31 / 97

To obtain  $\theta_4$ ,  $\theta_5$  and  $\theta_6$  which is basically the last 3 angles after the wrist, you can see that the rotation matrix  ${}^3_6[R]$  is of this form -- it is  $(c_4 c_5 c_6 - s_4 s_6)$   $r_{11}$  is this.  $r_{23}$  is  $c_5$ ,  $r_{21}$  is  $s_5 c_6$  and so on ok. This matrix  ${}^3_6[R]$  can be written as  ${}^0_3[R]^T {}^0_6[R]$  right, because  ${}^0_6[R] = {}^0_3[R] {}^3_6[R]$ . We multiply by the inverse which is the same as the transpose and we get this matrix equation.

The right-hand side is known because  ${}^0_6[R]$  is given to you for the inverse kinematics problem and  ${}^0_3[R]$  contains only  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$ . So, basically right-hand side is known and left-hand side has 3 variables  $\theta_4$ ,  $\theta_5$  and  $\theta_6$  ok.

(Refer Slide Time: 28:11)

PUMA 560: IK (CONTD.)

- To obtain  $\theta_4$ ,  $\theta_5$  and  $\theta_6$ , form

$${}^3_6[R] = \begin{pmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & -c_4 s_5 \\ s_5 c_6 & -s_5 s_6 & c_5 \\ -s_4 c_5 c_6 - c_4 s_6 & s_4 c_5 s_6 - c_4 c_6 & s_4 s_5 \end{pmatrix} \quad (15)$$

- From matrix multiplication

$${}^3_6[R] = {}^0_3[R]^T {}^0_6[R] \quad (16)$$

and since  $\theta_1$ ,  $\theta_2$  and  $\theta_3$  are known, right-hand side is known!

- Compare known right-hand side with elements of  ${}^3_6[R]$  and obtain  $\theta_4$ ,  $\theta_5$  and  $\theta_6$
- Similar to Z-Y-Z Euler angles with Y rotation of  $(-\theta_5)$

ASHITAVA GHOSAL (IISc)      ROBOTICS: BASICS AND ADVANCED CONCEPTS      NPTEL, 2020    31 / 97

So, we can just compare term by term and find out what is  $\theta_4$ ,  $\theta_5$  and  $\theta_6$ . So, for example,  $c_5$  will be equal to some known number. So,  $\theta_5$  will be  $\cos^{-1}$  of that known number. It turns out that in this case it is even simpler. Why? Because this matrix is very similar to what is called as the Z-Y-Z Euler angles with the Y rotation of  $-\theta_5$  ok.

So, last week we had looked at Euler angles representation of orientation of a rigid body using simple rotations. So, we have, in this example, a simple rotation about Z a simple rotation followed by a simple rotation about Y and another simple rotation about Z. However, unlike what we had done earlier the second rotation is by a minus angle  $\theta_5$ .

(Refer Slide Time: 29:11)

## PUMA 560: IK (CONTD.)



Algorithm  $r_{ij} \Rightarrow \theta_4, \theta_5$  and  $\theta_6$

If  $r_{23} \neq \pm 1$ , then

$$\theta_5 = \text{atan2}(\pm \sqrt{r_{21}^2 + r_{22}^2}, r_{23})$$

$$\theta_4 = \text{atan2}(r_{33}/s_5, -r_{13}/s_5),$$

$$\theta_6 = \text{atan2}(-r_{22}/s_5, r_{21}/s_5)$$

Else

If  $r_{23} = 1$ , then

$$\theta_4 = 0$$

$$\theta_5 = 0,$$

$$\theta_6 = \text{atan2}(-r_{12}, r_{11}),$$

If  $r_{23} = -1$ , then

$$\theta_4 = 0$$

$$\theta_5 = \pi,$$

$$\theta_6 = -\text{atan2}(r_{12}, -r_{11}).$$

Navigation icons

So, we can write the inverse Euler angle transformation just like we had done last week. So, given  $r_{ij}$  how do I find out  $\theta_4$ ,  $\theta_5$  and  $\theta_6$ ? Again, there is a singular configuration when  $r_{23}$  is  $\pm 1$ . So, if  $r_{23}$  is not equal to  $\pm 1$ ,  $\theta_5$  can be first found out by  $\text{atan2}(\pm \sqrt{r_{21}^2 + r_{22}^2}, r_{23})$ . Is it true? Yes,  $(r_{21}^2 + r_{22}^2)$  and then this will be a function of only  $\sin \theta_5$  and  $\cos \theta_5$  and we can use  $\text{atan2}$  to find  $\theta_5$ .

Then we can divide by  $\sin \theta_5$  and find  $\theta_4$  again using an  $\text{atan2}$ . And  $\theta_6$  - divided by  $\sin \theta_5$  these  $2 - r_{22}$  and  $r_{21}$  and find  $\theta_4, \theta_5$  and  $\theta_6$ . If  $r_{23}$  is  $+1$ , this is a special or a singular configuration, we set  $\theta_4$  as 0,  $\theta_5$  as 0 and  $\theta_6$  this. If  $r_{23}$  is  $-1$ , then we set  $\theta_4$  as 0  $\theta_5$  as  $\pi$  and  $\theta_6$  this ok.

(Refer Slide Time: 30:31)

## PUMA 560: UNIQUENESS OF IK SOLUTIONS




- From equation (12) two sets of  $\theta_1$ .
- From equation (13) two sets of  $\theta_3$ .
- Since  $\theta_3$  appears on the right-hand side of equation (14)  $\rightarrow$  Four possible values of  $\theta_2$ .
- Two possible sets of  $\theta_4, \theta_5$  and  $\theta_6$  from inverse Euler angle algorithm.
- Overall **eight** possible sets of joint angles  $\theta_i, i = 1, \dots, 6$  for a given  ${}^0_6[T]$ .

Navigation icons

So, in summary what have we done? We have obtained two sets of  $\theta_1$ , two sets of  $\theta_3$ , since  $\theta_3$  appears on the right-hand side of equation (14). So, four possible values of  $\theta_2$  and then we also have two possible sets of  $\theta_4$ ,  $\theta_5$  and  $\theta_6$  - because you can see  $\theta_5$  has  $\pm$  sign here. So, I can get two possible  $\theta_5$  and then when we divide it by  $\theta_5$  you will get two possible values of  $\theta_4$  and  $\theta_6$  ok. So, overall, what have we obtained? We have obtained eight possible sets of joint angles  $\theta_i$  for a given position and orientation of the sixth link with respect to the 0<sup>th</sup> link - given  ${}^0_6[T]$  ok.

(Refer Slide Time: 31:27)

### PUMA 560: WORKSPACE



- Usual definition: All  ${}^0_6[T]$  (position and orientation of {6}) such that inverse kinematics solution exists.
- Six dimensional entity — Difficult to imagine or describe!
- Possible to derive the 'position' workspace of 'wrist' point. Position vector of wrist point

$$\begin{aligned}
 x &= c_1(a_2c_2 + a_3c_{23} - d_4s_{23}) - d_3s_1 \\
 y &= s_1(a_2c_2 + a_3c_{23} - d_4s_{23}) + d_3c_1 \\
 z &= -a_2s_2 - a_3s_{23} - d_4c_{23}
 \end{aligned}
 \tag{17}$$


- $(x, y, z)$  are functions of three independent variables  $\theta_1, \theta_2$  and  $\theta_3 \Rightarrow$  Represents a **solid** in 3D space.
- Can obtain equations of the bounding surfaces.

ASHITAVA GHOSAL (IISc)
ROBOTICS: BASICS AND ADVANCED CONCEPTS
NPTEL, 2020 34 / 97

Now, let us see whether you can discuss a little bit about the workspace of this robot. So, basically under what condition the inverse kinematic solution exist. So, usual definition that all position and orientation of this 6<sup>th</sup> coordinate system such that the inverse kinematic solution exists. In this case it is very difficult to imagine or even visualize or describe, because it is a six-dimensional entity -- we have 3 positions  $x, y, z$  and 3 orientations.

It is possible to derive the position workspace of the wrist point. Why? Because we know that the position vector of the wrist point is  $x, y$  and  $z$  and they are functions of only  $\theta_1, \theta_2$  and  $\theta_3$  and the constant D-H parameters. So,  $x, y, z$  are functions of three independent variables  $\theta_1, \theta_2$  and  $\theta_3$ , so it represents a solid in 3D space. So, it is like a solid region in 3D space where the inverse kinematics exist solution exist. We can find the bounding surfaces of the solid region ok.

(Refer Slide Time: 32:49)

PUMA 560: WORKSPACE (CONTD.) 

- Squaring and adding the three equations in equation (17) gives
$$R^2 = x^2 + y^2 + z^2 = K_1 + K_2 c_3 - K_3 s_3, \quad K_1, K_2, \text{ and } K_3 \text{ constants.}$$
- The envelope of this family of surfaces must satisfy
$$\frac{\partial R^2}{\partial \theta_3} = 0, \quad \rightarrow \quad K_2 s_3 + K_3 c_3 = 0.$$
- Eliminating  $\theta_3$  and denoting  $a_3^2 + d_4^2$  by  $l^2$ , gives
$$[x^2 + y^2 + z^2 - ((a_2 + l)^2 + d_3^2)][x^2 + y^2 + z^2 - ((a_2 - l)^2 + d_3^2)] = 0 \quad (18)$$
which implies that the bounding surfaces are **spheres**.
- At all  $(x, y, z)$  all possible orientations, except at two special 'singular' configurations  $r_{23} = \pm 1$ .

ASHITAVA GHOSAL, (IISc)      ROBOTICS: BASICS AND ADVANCED CONCEPTS      NPTEL, 2020      35 / 97

And how do we find out that? So here are the steps, so if we square and add these three equations, let us call that as  $R^2$  which is  $x^2 + y^2 + z^2$  we can see it is a function of only  $\theta_3$ ,  $K_1$ ,  $K_2$ ,  $K_3$  are constant. So, this is a family of surfaces. So, if  $(x, y, z)$   $x^2 + y^2 + z^2$  was equal to constant; that is a sphere ok. But then as  $\theta_3$  changes we have a family of surfaces.

The envelope of this family of surfaces can be obtained by taking the partial derivative of this equation with respect to this variable  $\theta_3$  which is on the right-hand side. If you take this we will get one single equation  $K_2 s_3 + K_3 c_3$  equal to 0 and then we eliminate  $\theta_3$  from these 2 equations ok.

So, we can eliminate in formally using Sylvester's method, but we can just simply see that we can eliminate  $\theta_3$  and if you denote  $a_3^2 + d_4^2$  by  $l^2$  we will get this expression, which is  $x^2 + y^2 + z^2$ . So, basically this is the radius vector, the distance from the origin is minus some number and this is also minus, so bounding surfaces at 2 spheres. So, there is a sphere which is at a distance of surface is  $(a_2 + l)^2 + d_3^2$  and the other is  $(a_2 - l)^2 + d_3^2$  ok. So, we have 2 spheres which are the bounding surfaces of this solid region, where the inverse kinematic solution exists. And at all  $(x, y, z)$  where this inverse kinematic solution exist, we can find the orientation except two special singular configurations when  $r_{23}$  is  $\pm 1$  ok.



(Refer Slide Time: 35:05)

**PUMA 560: NUMERICAL EXAMPLE**

Chosen D-H parameters for the PUMA 560

$i$	$\alpha_{i-1}$	$a_{i-1}$	$d_i$	$\theta_i$
	degrees	m	m	degrees
1	0	0	0	45
2	-90	0	0	60
3	0	0.4318	0.125	135
4	-90	0.019	0.432	30
5	90	0	0	-45
6	-90	0	0	120


  
$${}^0_6T = \begin{bmatrix} 0.9749 & -0.2192 & -0.0388 & 0.1304 \\ 0.1643 & 0.8262 & -0.5388 & 0.3071 \\ 0.1502 & 0.5190 & 0.8415 & 0.0482 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ASHITAVA GHOSAL, IITSC      ROBOTICS: BASICS AND ADVANCED CONCEPTS      NPTEL, 2020      36 / 97

So, let us look at a numerical example ok, so this is taken from literature the DH parameter of a PUMA 560 the constant values are given in this table. We have chosen  $\theta_i$  arbitrarily as 45 degrees, 60 degrees, 135, 30, - 45, and 120 and these numbers 0.4318 for  $a_{i-1}$  for link 3 and  $d_i$  is 0.125 and so on this is from literature.

So, PUMA robot comes with these D-H tables. So, once we have this D-H table and once I give you this  $\theta$  then I can find out  ${}^0_6T$ . So, this is 4 by 4 homogeneous transformation matrix the last row is 0 0 0 , the rotation matrix is this top 3 by 3 and the position vector of the 6 th or the last coordinate system is given by 0.1304, 3071 and 0.0482. This is just multiplication of 6 matrices derived from taking each row of the D-H table.

(Refer Slide Time: 36:23)

PUMA 560: NUMERICAL EXAMPLE 

For the above  ${}^0_6[T]$ , the inverse kinematics solutions are

$i$	$\theta_1$	$\theta_2$	$\theta_3$	$\theta_4$	$\theta_5$	$\theta_6$
1	-91	120	50.04	177.51	-42.65	105.34
2	-91	120	50.04	-2.49	42.65	-74.66
3	45	-77.73	50.04	85.25	-159.22	-132.87
4	45	-77.73	50.04	-94.75	159.22	47.13
5	-91	-102.27	135	92.28	-178.31	15.79
6	-91	-102.27	135	-87.72	178.31	-164.21
7	45	60	135	30	-45	120
8	45	60	135	210	45	300

Note: Solutions (set 7) matches  $\theta_i$ ,  $i = 1, \dots, 6$ , chosen for direct kinematics.

ASHITAVA GHOSAL (IISc) ROBOTICS: BASICS AND ADVANCED CONCEPTS NPTEL, 2020 37 / 97

We can now take this  ${}^0_6[T]$  and run our inverse kinematic steps whatever we have discussed. What are the inverse kinematic steps? We first find out  $\theta_1$  then find out  $\theta_3$  then find out  $\theta_2$  and then find out  $\theta_4$ ,  $\theta_5$  and  $\theta_6$  using those steps which I have discussed few minutes back. So, it turns out that I will get 8 possible solution sets - we expect 8 possible solution sets so ok.

So,  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$ ,  $\theta_4$ ,  $\theta_5$ , and  $\theta_6$  and the solution sets - each row corresponding to a single solution set. So, as you can see the set 7, 45, 60, 135, 30, -45, 120 is what we started with here in the D-H table. So, that makes sense right because I took a set of constant D-H parameters and a set of values of  $\theta_s$ , I obtained the  ${}^0_6[T]$  transformation matrix and then I would run this took that same data set and run the inverse kinematics. So, one of the solution sets better be what we started with and that is indeed true.

(Refer Slide Time: 37:59)

## PUMA 560 MANIPULATOR



Workspace of **wrist point** of the PUMA shown in Figure.

Note: *Actual workspace is subset of shown workspace due to joint rotation limits.*

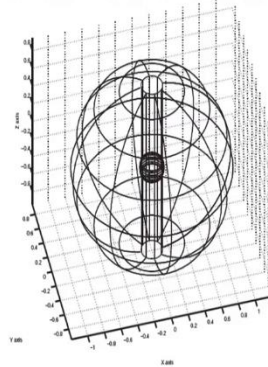


FIGURE: Workspace of the wrist point of the PUMA

What is the workspace looks like? We know it is bounded by 2 spheres. So, there is a outside sphere and there is an inside sphere, interestingly there is also some kind of a small hollow region where it is not reachable ok. So, it is like a cylinder with some spheres and so the workspace of a PUMA robot the wrist point looks like this 2 spheres bounded by 2 spheres.

(Refer Slide Time: 38:27)

## REVIEW OF IK



- Transcendental equations  $\rightarrow$  Polynomial equations using tangent half angle substitution.
- Polynomial equation of higher degree – Linear in  $\sin(\theta)$  or  $\cos(\theta)$   $\rightarrow$  Quadratic in  $x^2$  with  $x = \tan(\frac{\theta}{2})$ .
- For analytical solutions to IK  $\rightarrow$  *Eliminate* joint variable(s) from set of non-linear equations in several joint variables  $\rightarrow$  A single equation in *one* joint variable.
  - Planar 3R example – Three equations in three joint variables  $\rightarrow$  Two equations in  $\theta_1$  and  $\theta_2$   $\rightarrow$  One equation, equation (5), in  $\theta_2$  alone.
  - Single equation solved for  $\theta_2$  and then solve for  $\theta_1$  and  $\theta_3$ .
  - PUMA 560 – 3 equations in first 3 joint variables – *Decoupling* of position and orientation.
  - Solve for the first 3 joint joint variables and then for last 3 joint variables using orientation information.

So, let us just quickly review the inverse kinematics whatever we have done till now. So, first important observation is to solve the inverse kinematics problems we have to deal with transcendental equations. So, first step is we can obtain polynomial equations using

tangent half angle substitution. So, the polynomial equation is always of a higher degree. So, if you have  $\sin \theta$  and  $\cos \theta$  it will become quadratic in  $x^2$ , where  $x$  is  $\tan \theta/2$  ok.

For analytical solutions to the inverse kinematics problem, we have to eliminate joint variables from a set of non-linear equation in several joint variables ok. So, what do we want? We want a single equation in one joint variable that is very useful or important. So, in the case of the planar 3R example, we started with three equations in three joint variables, we obtained two equations in  $\theta_1$  and  $\theta_2$  - remember capital  $X$  and capital  $Y$  - and then we obtained one equation in  $\theta_2$  alone ok.

So, this single equation was solved for  $\theta_2$  and then we solve for  $\theta_1$  and  $\theta_3$ . For the PUMA 560 we have 3 equations in 3 joint variables - the wrist point. So, the position and orientation could be decoupled, we could just take the position equations and solve for 3 angles and then using those 3 angles we could solve for the last 3 joint angles, which is which uses the orientation information.

(Refer Slide Time: 40:19)

REVIEW OF IK (CONTD.)

- *Decoupling* of the position and orientation first noticed by Pieper (Pieper, 1968) for manipulators with intersecting wrist.
- Generalisation to *any* six DOF serial manipulators where three consecutive joint axes intersect.
  - At most a fourth-order polynomial in the tangent of a joint angle.
  - Wrist point can reach any position in the workspace in *at most 4* possible ways.
  - Fourth-degree polynomials can be solved in closed-form (Korn and Korn, 1968)  
→ IK of all six- degree-of-freedom serial manipulators with three consecutive intersecting axes can be solved in closed-form.
- PUMA 560 – Workspace of the wrist point is bounded by two spheres and require solution of *only* a quadratic due to special geometry.
- General geometry robot with intersecting wrist, boundaries traced by wrist point form a torus, a fourth-degree surface (Tsai and Soni, 1984).

ASHITAVA GHOSAL (IISc) ROBOTICS: BASICS AND ADVANCED CONCEPTS NPTEL, 2020 40 / 97

So, this decoupling of position and orientation was first noticed by Pieper in 1968 for manipulators with intersecting wrist - intersecting wrist means the last 3 joint axes joint axis intersect at a point. This was eventually generalized to any six degree of freedom serial manipulator, where three consecutive joint axes intersect. It was shown that at most a fourth order polynomial in the tangent of a joint angle is what we will get.


So, the wrist point can reach any position in the workspace in at most 4 possible ways, because we have 4 possible solutions of the tangent of the joint angle and this fourth degree polynomial which we get can be solved in closed form - this is very important. You know we can solve a quadratic equation in closed form, we can solve a cubic equation in closed form and we can also solve a quadratic equation in closed form. Any polynomial higher than 4 we cannot solve in closed form.

So, it turns out that the inverse kinematics of all six degree of freedom serial manipulators with three consecutive intersecting axis can be solved in closed form. It is a very useful and neat result. So, for the PUMA 560 the workspace of the wrist point is bounded by two spheres and requires the solution of only a quadratic. We do not have to solve a quartic equation for PUMA because of the special geometry. So, you know some axis are intersecting some axis are parallel and so on.

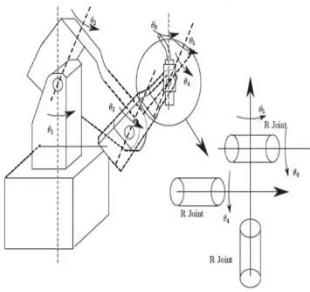
It was shown eventually that for general geometry robot with intersecting wrist the boundary is traced by the wrist point form a torus. So, they are not spheres, they are this solids or surfaces called torus, which is the fourth degree surface ok. A sphere is second order second degree torus is fourth degree.

(Refer Slide Time: 42:39)

### NON-INTERSECTING WRISTS



- Difficult to design/manufacture three intersecting axis wrist.
- Much easier if wrist has two intersecting axis.



$i$	$\alpha_{i-1}$	$a_{i-1}$	$d_i$	$\theta_i$
1	0	0	0	$\theta_1$
2	$-\pi/2$	0	0	$\theta_2$
3	0	$a_2$	$d_3$	$\theta_3$
4	$-\pi/2$	$a_3$	$d_4$	$\theta_4$
5	$\pi/2$	0	$d_5$	$\theta_5$
6	$-\pi/2$	0	0	$\theta_6$

FIGURE: A robot with non-intersecting wrist

ASHITAVA GHOSAL (IISc)
ROBOTICS: BASICS AND ADVANCED CONCEPTS
NPTEL, 2020 41 / 97

Let us continue. What happens if you have non intersecting wrist? So, intersecting wrist problem is more or less solved - you know we have 4 solutions quartic and then we can do

this inverse Euler angle transformation and find the last 3 angles. However, it is very very difficult to manufacture three intersecting axis wrist.


Why? Because it is sort of - you can - imagine that you have 3 lines which are meeting at a point and then we are going to manufacture this or locate these 3 motors such that their axis intersects at a point. It will never happen manufacturing wise ok. It is much easier if the wrist has two intersecting axis ok.

So, here is an example of a robot with the last two axis intersecting and then again previous two axis intersecting; but not all three intersecting at the same place. So, schematically it is shown here  $\theta_4$  and  $\theta_5$  intersects at the same place and  $\theta_5$  and  $\theta_6$  intersect at the same place, but all of them do not intersect at the same place.

So, the D-H table for this robot, this is well known welding robot, which was built long time back, it is very similar to the PUMA except there is a  $d_5$  here and this  $d_5$  is non zero. So, the  $d_5$  is the last this joint link offset for the 5 th link ok.

(Refer Slide Time: 44:23)

### NON-INTERSECTING WRISTS



- A six- degree-of-freedom robot – First 3 joint axis are similar to PUMA 560.
- Last three axes do not intersect and there is an offset  $d_5$ .
- From D-H table compute  ${}^0_1[T], \dots, {}^5_6[T]$  and then  ${}^0_6[T]$ .
- Last column of  ${}^0_6[T]$  is

$$\begin{aligned}
 x &= c_1(a_2c_2 + a_3c_{23} - d_4s_{23}) - d_3s_1 + d_5(s_1c_4 - c_1s_4c_{23}) \\
 y &= s_1(a_2c_2 + a_3c_{23} - d_4s_{23}) + d_3c_1 - d_5(c_1c_4 + s_1s_4c_{23}) \\
 z &= -a_2s_2 - a_3s_{23} - d_4c_{23} - d_5s_4s_{23}
 \end{aligned} \tag{19}$$

Note:  $(x, y, z)$  is a function of  $\theta_1, \theta_2, \theta_3$  and  $\theta_4$ .

- Need one more equation in the four joint variables!


ASHITAVA GHOSAL (IISc)
ROBOTICS: BASICS AND ADVANCED CONCEPTS
NPTEL, 2020 42/97

So, it is a six degree of freedom robot - first 3 joints are very similar to the PUMA 560, the last three axes do not intersect, there is an offset  $d_5$ . So, from the D-H table we can compute  ${}^0_1[T], {}^1_2[T]$  and all the way till  ${}^0_6[T]$ . So, if you compute  ${}^0_6[T]$  then last column of  ${}^0_6[T]$ , which is the position of the last link with respect to the fixed link can be shown to be a function of now 4 joint angles. It is  $\theta_1, \theta_2, \theta_3$  and also  $\theta_4$ .

So, if  $d_5$  were to be 0 we will get back the equations of the PUMA, but however there are these additional terms  $d_5 s_4 s_{23}$   $d_5 (s_1 c_4 - c_1 s_4 c_{23})$  and so on ok. So, what can we notice? We can see that the  $x$ ,  $y$  and  $z$  the  $n^{\text{th}}$  origin of the 6<sup>th</sup> coordinate system or the last link is now a function of  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$  and  $\theta_4$ . So, we have 3 equations in 4 unknowns. So we need one more equation in the fourth joint variables and how can we obtain this as follows.

(Refer Slide Time: 45:53)

### NON-INTERSECTING WRISTS



- From  ${}^3_6[R] = {}^0_3[R]^T {}^0_6[R]$ ,

$$\begin{pmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & -c_4 s_5 \\ s_5 c_6 & -s_5 s_6 & c_5 \\ -s_4 c_5 c_6 - c_4 s_6 & s_4 c_5 s_6 - c_4 c_6 & s_4 s_5 \end{pmatrix} = \begin{pmatrix} c_1 c_{23} & s_1 c_{23} & s_{23} \\ -c_1 s_{23} & -s_1 s_{23} & -c_{23} \\ -s_1 & c_1 & 0 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \quad (20)$$

- Divide the (1,3) and the (3,3) terms, to get (for  $\theta_5 \neq 0, \pi$ ),

$$s_4 (r_{13} c_1 c_{23} + r_{23} s_1 c_{23} + r_{33} s_{23}) = c_4 (r_{13} s_1 - r_{23} c_1) \quad (21)$$

- Equation (21) is the fourth equation!
- Solve numerically equations (19) and (21) to obtain  $\theta_i$ ,  $i = 1, 2, 3, 4$ .
- Solve for  $\theta_4$ ,  $\theta_5$  and  $\theta_6$  using Z - (-Y) - Z inverse Euler angle algorithm (similar to PUMA 560 example).

ASHITAVA GHOSAL (IISc)
ROBOTICS: BASICS AND ADVANCED CONCEPTS
NPTEL, 2020 43 / 97

We can rewrite the  ${}^3_6[R]$  which is the rotation matrix of the 6<sup>th</sup> link with respect to the third link as product of 2 rotation matrices  ${}^0_3[R]^T {}^0_6[R]$ . Symbolically it is shown here. So, this  ${}^3_6[R]$  is very similar to what we had for the PUMA,  ${}^0_3[R]^T$  will be a function of  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$  and  ${}^0_6[R]$  is given to us - we are trying to solve the inverse kinematics problem.


So, basically we have one side  $\theta_4$ ,  $\theta_5$ , and  $\theta_6$  and we have another side  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$ . So, if you divide the (1, 3) term which is  $-c_4 s_5$  and (3, 3) term which is  $c_4 s_5$ . So,  $\theta_5$  not equal to 0 - because we cannot divide 0 by 0- we can get one equation which is  $s_4$  into whatever is there on the right side which is  $r_{13} c_1 c_{23}$  and so on; will be equal to  $c_4$  into something else where the  $r_{ij}$  in this equation are known. But  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$  are unknowns, so this is an equation which involves  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$  and the given  $r_{ij}$ 's and hence this is the fourth equation ok.

We had 3 equations in  $x$ ,  $y$  and  $z$  and somehow we have managed to derive a fourth equation again in terms of  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$  and  $\theta_4$ . So, we have 4 equations and 4 unknowns, and we can at least solve numerically these equations to obtain  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$  and  $\theta_4$ . And once  $\theta_1$ ,

$\theta_2$ ,  $\theta_3$  and  $\theta_4$  is obtained we can find theta  $\theta_4$ ,  $\theta_5$ ,  $\theta_6$  by again this inverse Euler angle algorithm similar to the PUMA Z – (-Y) - Z.

(Refer Slide Time: 48:13)

### NON-INTERSECTING WRISTS



- Numerical values of D-H parameters same as PUMA 560 – Offset  $d_5 = 20$  mm.
- ${}^0_6[T]$  same as used for the PUMA 560 example

$${}^0_6[T] = \begin{bmatrix} 0.9749 & -0.2192 & -0.0388 & 0.1304 \\ 0.1643 & 0.8262 & -0.5388 & 0.3071 \\ 0.1502 & 0.5190 & 0.8415 & 0.0482 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Solve 4 non-linear equations numerically – *fsolve* in Matlab<sup>®</sup> used

$\theta_1 = 41.82, \theta_2 = 60.43, \theta_3 = 135.33, \theta_4 = 31.96$

- Using inverse Euler angle algorithm – 2 sets of values  
 $\theta_4 = 31.96, -148.04, \theta_5 = -45.22, +45.22$ , and  $\theta_6 = 121.57, -58.43$  — One  $\theta_4$  matches.
- $\theta_5 = 0, \pi$  — Singular configuration and only  $\theta_4 \pm \theta_6$  can be found.

ASHITAVA GHOSAL (IISc)
ROBOTICS: BASICS AND ADVANCED CONCEPTS
NPTEL, 2020 44 / 97

So, let us look at a numerical example the  ${}^0_6[T]$  is same as the PUMA example ok, we have taken the same set of angles and the same D-H parameters and we assume  $d_5$  is 20 mm. This is reasonable because the last link offset is small - it is not very large. So, if we solve these 4 equations in Matlab using this solve using the solution program called *fsolve*. How many if you have heard *fsolve*, I do not know but there is a way to solve non-linear equations in Matlab numerically using this routine called *fsolve* and we will get  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$  and  $\theta_4$  ok. So, it is a numerical procedure, so we will have to have a certain guess and then it will converge to the final solution and it turns out we will get  $\theta_1$  as 41.82,  $\theta_2$  as 60.43,  $\theta_3$  as 135.33,  $\theta_4$  as 31.96 ok.

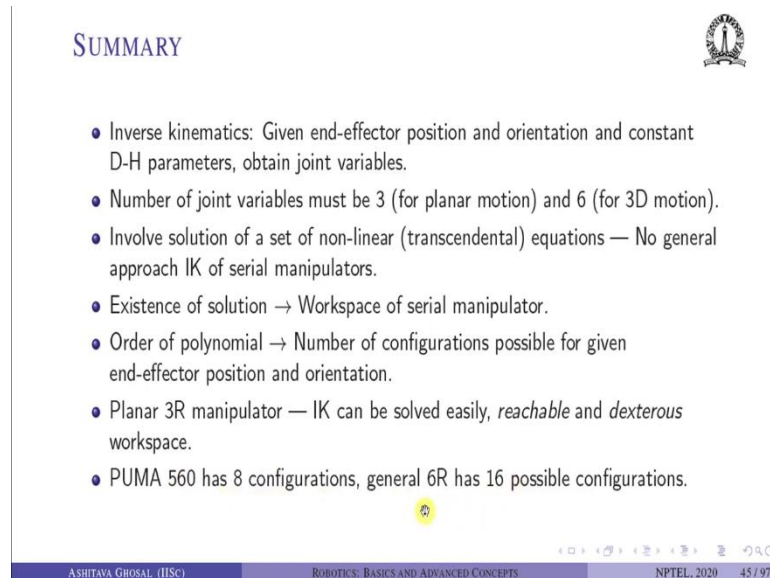
And using inverse Euler angle algorithm we get 2 sets of values of  $\theta_4$ ,  $\theta_5$ ,  $\theta_6$ , and we can see one  $\theta_4$  matches - just to give you some confidence that our numerical solution is ok. So,  $\theta_4$  is 31.96, here also one of the  $\theta_4$  is 31.96 ok. If  $\theta_5$  were 0 or  $\pi$ , this is the singular configuration, and we can only solve  $\theta_4 \pm \theta_6$  ok.

So, do we know these numbers are ok? Yes, because, what have we done? We have added at  $d_5$  which is the small number. What was  $\theta_1$  before for the PUMA? It was 45,  $\theta_2$  was 60,  $\theta_3$  was 135,  $\theta_4$  was 30. So, although we have added an offset, it is sort of close to what we



started with the PUMA example. So, it gives us more or less confidence that this numerical solution is correct.

(Refer Slide Time: 50:33)



**SUMMARY**

- Inverse kinematics: Given end-effector position and orientation and constant D-H parameters, obtain joint variables.
- Number of joint variables must be 3 (for planar motion) and 6 (for 3D motion).
- Involve solution of a set of non-linear (transcendental) equations — No general approach IK of serial manipulators.
- Existence of solution → Workspace of serial manipulator.
- Order of polynomial → Number of configurations possible for given end-effector position and orientation.
- Planar 3R manipulator — IK can be solved easily, *reachable* and *dexterous* workspace.
- PUMA 560 has 8 configurations, general 6R has 16 possible configurations.

ASHITAVA GHOSAL (IISC) ROBOTICS: BASICS AND ADVANCED CONCEPTS NPTEL, 2020 45 / 97

So, in summary the inverse kinematics problem is defined as given end effector position and orientation and all the constant D-H parameters, obtain the joint variables. So, the number of joint variables must be 3 for planar motion and 6 for 3D motion. Only then we have the equal number of equations and equal number of unknowns.

The inverse kinematics involve solutions of a set of non-linear transcendental equations. So, there are no general approach of inverse kinematics of serial robots. The existence of solution leads to the notion of workspace of a serial manipulator. So, we looked at the planar 3R example and stored that  $\cos^{-1}$  of something, that something should lie between  $\pm 1$  and then that gives this whole idea of a workspace of the planar robot.

The order of the polynomial, the single polynomial which you obtain to find one of the joint variables gives you the number of possible configurations for a given end effector position and orientation. So, in the case of planar 3R robot the IK inverse kinematics could be easily solved, we could also give this notion of a reachable workspace and dexterous workspace. So, we could find what is the furthest the robot can reach and what is the region in the workspace where you could achieve arbitrary orientation.

The PUMA 560 has 8 possible configurations and we will see later on in this week that the general 6 degree of freedom robot has 16 possible configurations. So, with this I will stop. In the next lecture we will look at 2 special kinds of serial robots, one in which the number of joint angles is less than 6 and the number of joint angle is greater than 6.