Sound and Structural Vibration

Prof. Venkata Sonti
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

Module No # 04
Lecture No # 20
A Schematic of Coupled Waves

Good morning and welcome to this next lecture on sound and structural vibration.

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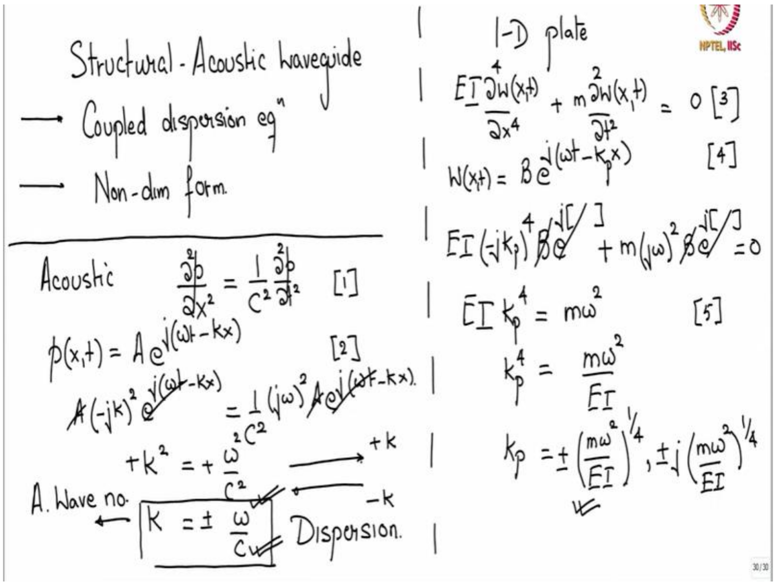

Last class we started looking at a structural acoustic wave guide. We derived the relevant equations and we found what we call the coupled dispersion equation and then I just presented the non-dimensional form. I said I will derive it in this class that is what we are going to see. But before that I would like to present something which I may have done it. But I will do it again because this happens to be the crux of all sound structural interactions.

So, if we look at the acoustic wave equation it looks like this

$$\frac{\partial^2 P}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2}. \qquad [1]$$

So, wave equation and we have seen with that crank and spring example that we can represent the pressure using this

$$P(x,t) = Ae^{j(\omega t - kx)}. \qquad [2]$$

If I substitute equation 2 into equation 1 on the left, there are 2 derivatives with space. So, I will get

$$A(-jk)^2 e^{j(\omega t - kx)} = \frac{1}{c^2}\,(j\omega)^2 A e^{j(\omega t - kx)},$$

$$k^2 = \frac{\omega^2}{c^2},$$

$$k = \pm\frac{\omega}{c}.$$

The k is the acoustic wave number ω is the frequency in the radians c is the speed of sound in the medium.

So $\pm$ because there can be a right going wave with the wave number $+k$ there can be a left going wave with a wave number $-k$. So, this is with respect to one dimensional sound wave and this equation is called the dispersion equation wave number given as a function of frequency is called the dispersion equation anywhere 1D sound is very simple. If you look at other systems, it can get very complicated.

Next if we have a 1 D plate equation is

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + m\frac{\partial^2 w(x,t)}{\partial t^2} = 0. \qquad [3]$$

So let us say the

$$w(x,t) = Be^{j(\omega t - k_p x)}. \qquad [4]$$

If we substitute equation 4 into equation 3, I have fourth derivative with space.

So, I will get

$$EI\left(-jk_p\right)^4 Be^{j(\omega t - k_p x)} + m(j\omega)^2 Be^{j(\omega t - k_p x)} = 0.$$

So B can cancel and this propagator can cancel I have

$$EI{k_p}^4 = m\omega^2. \qquad [5]$$

So, I have ${k_p}^4$ the wave number in the 1D plate given by $\frac{m\omega^2}{EI}$ or k_p given by so they 4 roots so let us

$$k_p = \pm\left(\frac{m\omega^2}{EI}\right)^{1/4}, \pm j\left(\frac{m\omega^2}{EI}\right)^{1/4}.$$

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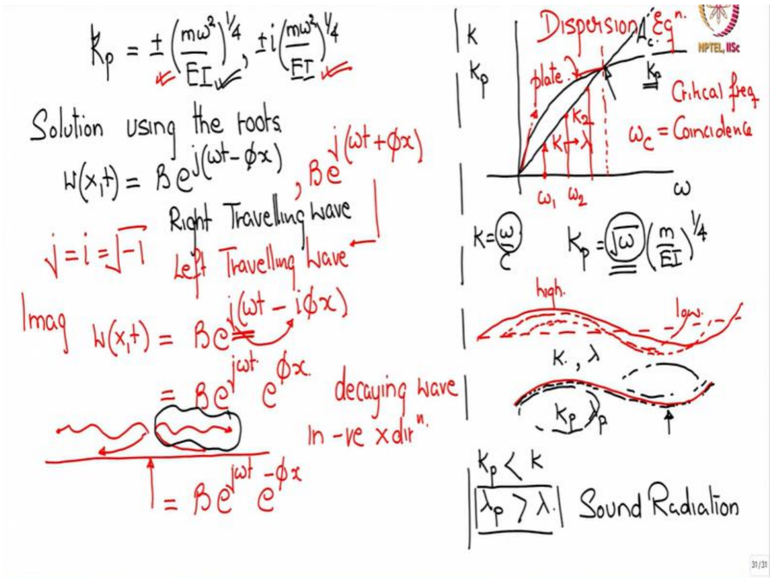

So let us look at this solution using each of these roots so I had B or rather w given by $w(x,t)$ given by $Be^{j(\omega t-\phi x)}$ let us say I take the plus value and I will ϕ. So ϕx this represents a rightward travelling wave or propagating wave similarly if I take the negative root over here what I will get is $Be^{j(\omega t+\phi x)}$.

And this represents a left travelling wave, so these are travelling waves if we choose the imaginary roots then I have $w(x,t) = Be^{j(\omega t-j\phi x)}$ let us say I take the positive imaginary. So $Be^{j(\omega t)}$ does not bother us. But this j over here I mean use j and i interchangeably so please do not mind that I am use it. Both are square root of -1 so now this j square or i square here is -1 so this becomes a plus.

So, what happens is this becomes $w(x,t) = Be^{j\omega t}e^{\phi x}$. So, what this means is this is a decaying wave in the negative x direction. So, I have this 1D plate over here suppose I excite it over here and I will get a positive propagating wave I will get a negative propagating wave that we have seen above. In addition, I will get a decaying wave on the left-hand side.

Similarly, if I choose negative imaginary root, I will get $w(x,t) = Be^{j\omega t}e^{-\phi x}$ and this represents a decaying wave in positive direction those are the 4 roots. Of these let us focus currently on this positive propagating wave which is the positive root here. Let us just focus on that now if we plot this if we plot both the wave numbers so I have this frequency axis this wave number axis for acoustic wave number and for the plate wave number.

The acoustic wave number was $\frac{\omega}{c}$ so it is a straight line with the slope related to c it is a straight line this is an acoustic wave number. Whereas k_p the plate wave number is given by separate the ω out it is $\sqrt{\omega}\left(\frac{m}{EI}\right)^{1/4}$. So, this is like a parabola, so it goes like this and somewhere those two meet these two cross each other somewhere that frequency is called coincidence I have mentioned this before or critical frequency.

Now what this means if I pick a frequency over here this is the acoustics wave number so let us say acoustic wave number is k_1 whatever at ω_1 it is k_1. k_1 meaning it is related to a certain λ_1 so there is let us say this is the acoustic wavelength this is the wavelength of the acoustic wavelength in the 1D. That means pressure starts with zero, pressure is high over here the acoustic wave. And again, pressure goes to zero and pressure is low over here.

Now if I take another ω I take ω_2 suppose I am just describing the plot of the dispersion equation how to read it. Because this is a physical quantity so at ω_2 my k_2 is higher and λ is, inverse of k. So λ is smaller so let us say it becomes small I am not bothering about scales or anything. So just get the feel for the size then if I take ω_3 it is further smaller, so the wave number goes up wavelength goes down.

Similarly, with k_p also the plate wave number but what happens is whereas the slope is high for the plate wavenumber with respect to ω initially it is you know slows down. So that at some point your acoustic wavelength and this structural plate wavelength are equal. And this is very important because this is when the best coupling can occur when the acoustic wavelength and the structural wavelength or acoustic wavenumber and structural wavenumber are equal.

That is when the best coupling can occur because here the structure is pulling, and the acoustic wave has a low-pressure region. Similarly, here the structure is pushing upward and by default the sound pressure is high over there. And they match perfectly so that is coincidence it happens because the dispersion equation for the acoustic wave is linear with frequency and because it is you know square root of ω dependence in flexural wave that is the reason this thing happens.

And as you go beyond the k_p is drops off quickly whereas the acoustic wave number continuous. So k_p here is lower beyond coincidence k_p is lower which means λ_p remains higher than $\lambda_{acoustic}$. And this condition is important for sound radiation.

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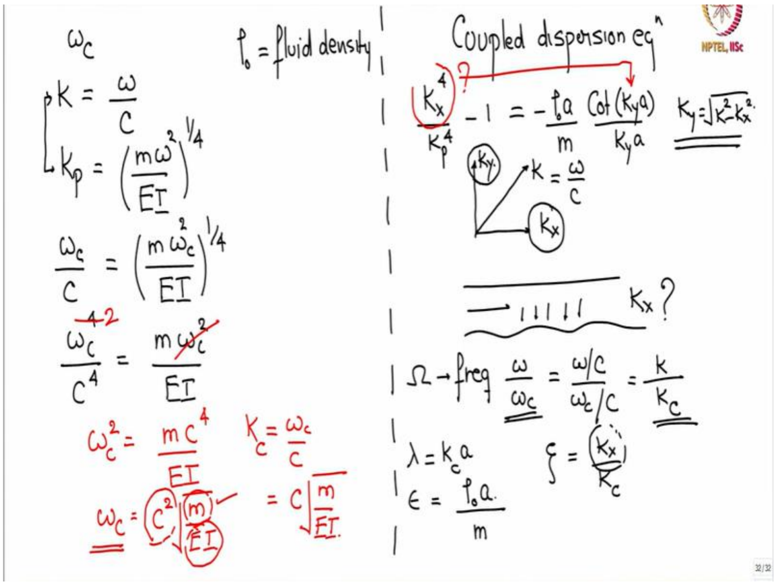

Now let us derive and expression for the coincidence frequency ω_c for the acoustic wave number

$$k = \frac{\omega}{c}.$$

And the k_p for the plate will take the positive propagating root

$$k_p = \left(\frac{m\omega^2}{EI}\right)^{1/4}.$$

So, at coincidence these two are equal that is what we say at coincidence this two are equal that means $\frac{\omega_c}{c} = \left(\frac{m{\omega_c}^2}{EI}\right)^{1/4}$.

So let us take it to match this let us take this 4 times so I have

$$\frac{{\omega_c}^4}{c^4} = \frac{m{\omega_c}^2}{EI}.$$

So ${\omega_c}^2 = \frac{mc^4}{EI}$ or ω_c is $c^2\sqrt{\frac{m}{EI}}$. So, it is a material and fluid property so if you take the speed of sound c of the fluid, you take the mass per unit area of your plate you take the stiffness EI bending stiffness for the plate they are related and they give ω_c depends on both.

Accordingly let me just write it here k_c wavenumber at coincidence is $\frac{\omega}{\omega_c}$ and therefore it is equal to $c\sqrt{\frac{m}{EI}}$. So, I thought should just repeat this part over here. So now in typically in structural acoustic non dimensionalizations are done with respect to the coincidence frequency it is a very common practice either coincidence frequency or coincidence wavenumber it is common practice.

And it gives physically meaningful results so if you have a non-dimensional frequency you can talk of beyond coincidence that is beyond one or below coincidence less than one ok. So, to remind you the coupled dispersion equation the coupled dispersion for the structural acoustic wave guide turned out to be

$$\frac{{k_x}^4}{{k_p}^4} - 1 = \frac{-\rho_0 a}{m}\,\frac{\cot(k_y a)}{k_y a}.$$

So ρ_0 is mean fluid density not the acoustic perturb density it is a mean fluid density. And how are k_y and k_x related k_y is equal to $\sqrt{k^2 - {k_x}^2}$. So k is the acoustic wave number in fluid, k_x is the axial direction wave number.

And then k_y is the vertical wave number at a given frequency k is known at a given frequency automatically k the total acoustic wave number is known. Whereas now because of the coupling the breakup into k_x and k_y is unknown. The two systems are interacting in a coupled manner top is rigid, bottom is flexing and there is a fluid. So fluid is interacting with the structure, structure is interacting with the fluid and therefore the k_x is now actually unknown we have to find that.

And k_x is related to k_y through this so we have to find that out anyway that is the equation and the non-dimensional parameters I said Ω the frequency a non-dimensional frequency I said is going to be $\frac{\omega}{\omega_c}$. So, this also means that it is $\frac{\omega/c}{\omega_c/c}$. ω/c is the acoustic wavenumber ω_c/c is the coincidence wavenumber.

$$\Omega = \frac{\omega}{\omega_c} = \frac{\omega/c}{\omega_c/c} = \frac{k}{k_c}.$$

Then λ we have the non-dimensional length which is again $k_c a$ and ϵ the fluid loading parameter I said is very important it is $\frac{\rho_0 a}{m}$. Then this ξ is another which is the one we want non-dimensionalized with k_c we want k_x now.

That is our plan in the whole of this equation the idea is to figure out k_x and k_x is related to k_y. So, k_y will be replaced in terms of k_x so that the whole equation has k_x and k_x is the only thing unknown the rest are all known. So, if we use this non-dimensionalization what do we get?

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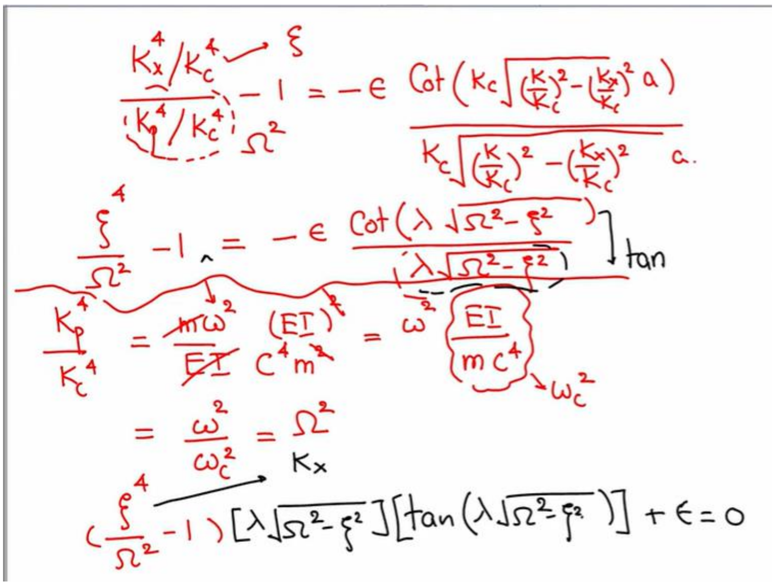

We get

$$\frac{{k_x}^4/{k_c}^4}{{k_p}^4/{k_c}^4} - 1 = -\epsilon \frac{\cot\left(k_c\sqrt{\left(\frac{k}{k_c}\right)^2 - \left(\frac{k_x}{k_c}\right)^2}\ a\right)}{k_c\sqrt{\left(\frac{k}{k_c}\right)^2 - \left(\frac{k_x}{k_c}\right)^2}\ a}.$$

So now k_x by k_c this is ξ we said so I have psi ξ^4 and now what is that thing ${k_p}^4/{k_c}^4$. So, ${k_p}^4$ is $\frac{m\omega^2}{EI}$ as we have seen before the plate wave number at ω. Free plate wave number at ω in vacuum and then what is ${k_c}^4$ in the denominator?

It is $c^4 \frac{m^2}{(EI)^2}$ you can check the previous page.

$$\frac{{k_p}^4}{{k_c}^4} = \frac{m\omega^2}{EI}\frac{(EI)^2}{c^4m^2}.$$

So now if I cancel of things whatever I get EI cancelled and I get an m cancelled. So, this is $\omega^2 \frac{EI}{mc^4}$. So, this quantity $(\frac{EI}{mc^4})$ if you again see is my ${\omega_C}^2$ so what I have here this entity is ω^2 by ${\omega_C}^2$ which is my non-dimensional frequency squared.

$$\frac{{k_p}^4}{{k_c}^4} = \Omega^2.$$

So, I get here

$$\frac{\xi^4}{\Omega^2} - 1 = -\epsilon\frac{\cot\left(\lambda\sqrt{\Omega^2 - \xi^2}\right)}{\lambda\sqrt{\Omega^2 - \xi^2}},$$

$$\left(\frac{\xi^4}{\Omega^2} - 1\right)\left[\lambda\sqrt{\Omega^2 - \xi^2}\right]\left[\tan\left(\lambda\sqrt{\Omega^2 - \xi^2}\right)\right] + \epsilon = 0.$$

Time has run out for this lecture I will close this here and we will start looking at how to solve for ξ? ξ was k_x in disguise non-dimensionalized k_x. So, we will solve for ξ in the next class thank you.