Dynamics and Control of Mechanical Systems
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Lecture – 24
Controllability and Observability of Linear Systems

In this lecture, we look at very important concepts in control -- these are called Controllability and Observability.

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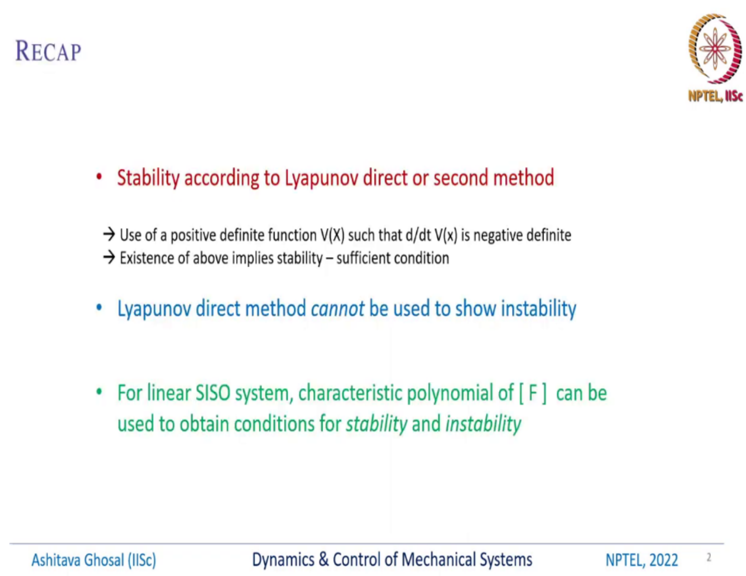

Before we start on these two important concepts of controllability and observability. Let us quickly recap and see what we have discussed till now. In the last lecture we looked at the stability according to Lyapunov and the direct or the second method. In that approach, we basically need to find the positive definite function V (**X**) where **X** are the state variables such that the derivative of V (**X**), time derivative of V (**X**), is negative definite. And according to Lyapunov if such a positive definite function and a negative definite *d*/*dt* (V (**X**)) exists then the system is stable. Then the state equations $\dot{X} = \boldsymbol{f}(\mathbf{X}, \boldsymbol{t})$. This existence of V (**X**) such that *d*/*dt* (V (**X**)) is negative definite is a sufficient condition. The Lyapunov direct method cannot be used to show instability and for a linear single input, single output system, the characteristic polynomial of [*F*] can be used to obtain the condition for stability and instability -- [*F*] here stands for the state equations $\dot{X} = [F]\mathbf{X} + [G]\mathbf{u}$.

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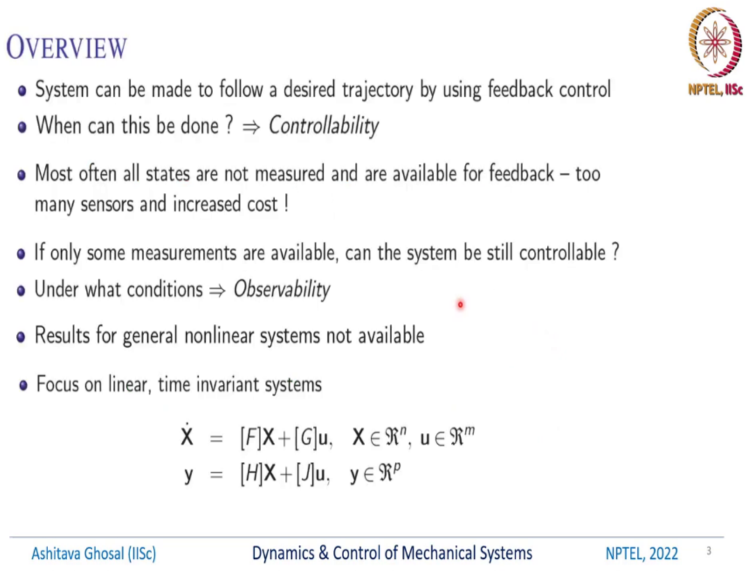

In the basic concept of control or feedback control, we have shown that a system can be made to follow a desired trajectory by using feedback control. A key question is when can this be done. This leads to the concept of controllability. And as we have seen in feedback control, we need to make measurements of the output using sensors. Most often, all states are not measured and are available for feedback. It requires too many sensors, and it also increases the cost of the system. Hence, if only some measurements are available, can the system be still controllable. And the conditions under which a smaller set of p measurements, where n is the dimension of the state space and p is the number of measurements that you are doing, is the topic of observability. So, under what conditions we can get away with making less measurements and still do feedback control and achieve the goals of control that is the topic of observability. There are very few results for general nonlinear systems that are available. Most of the time or most of the results are for linear time invariant systems, and just to recap, a linear time in variant system is one which is given by these two equations. We have the state equations which is $\dot{X} = [F]\mathbf{X} + [G]\mathbf{u}$, where $\mathbf{X}$ is an n dimensional state vector, and $\mathbf{u}$ is the input which could be m dimensional and then we have the output equation which is $\boldsymbol{y} = [H]\mathbf{X} + [J]\mathbf{u}$ and $\boldsymbol{y}$ could be p of them -- it is a p dimensional vector.

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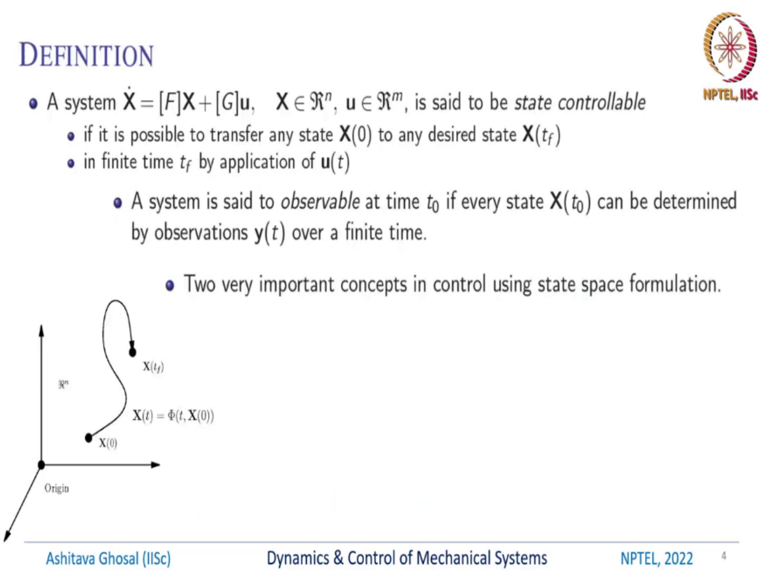

Let us look at a definition a system $\dot{X}$ = $[F]$**X** + $[G]$**u**, **X** Î R^n and **u** Î R^m is said to be state controllable, if it is possible to transfer any state **X**(0) to any desired state **X** (t_f) in finite time that is the important part. So, I want to take the system from **X** (0) to some **X** (t) but in finite time t_f. And how can we do that, we want to transfer from **X** (0) to **X**(t_f) by application of **u**(t). So, **u**(t) is the input, **X** (0) is some initial state, **X** (t) is some final state. I want to go from **X** (0) to **X** (t) in finite time t_f by applying **u**(t) likewise, a system is said to be observable at time t_0, if every state **X** (t_0) can be determined by observations **y**(t) over a finite time. Any state at some time t_0 can be determined by observations **y**(t) over a finite time. These are two very, very important concepts in control using state space formulation.

So, pictorially what is happening is -- we have a state space which is n dimensional. I want to go from **X** (0) to some **X**(t_f). So, this is the trajectory it could be some arbitrary trajectory and what is **X**(t)? This is that state transition matrix -- it is f(t, **X**(0)). This is the solution of the state equations in using the notion of a state transition matrix. And a system is said to be controllable if I can go from **X** (0) to this final **X** (t_f) and in finite time.

Likewise, a system is said to be observable if I can obtain **X** (t_0), at some time t_0, I can obtain **X** (t_0) by observing **y**(t) over a finite time.

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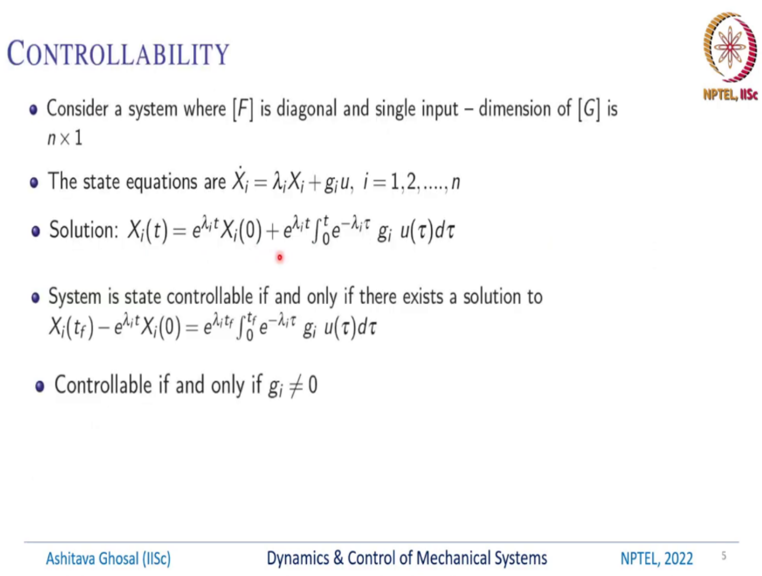

Let us consider a simple case where $[F]$ is diagonal and we have a single input. So, basically, the dimension of $[G]$ is n x 1. So then what do we have -- the state equations are $\dot{X}_i = \lambda_i X_i + g_i u$. So, remember, u is 1 -- it is a single input, X_i are the state vectors $i = 1$ through n and $[F]$ is diagonal. So, the diagonal elements are λ_1, λ_2 and so on.

So, what do we have here? We have a single equation, non-homogeneous equation, so, the solution of this equation is $X_i(t) = e^{\lambda_i t} X_i(0) + e^{\lambda_i t} \int_0^t e^{-\lambda_i \tau} g_i u(\tau)\, d\tau$ -- we have seen this. So, whenever we have a single non-homogeneous first order differential equation, the solution has one part which is from $\lambda_i X_i$ which is this homogeneous part, and then there is one part which is due to this u which is the non-homogeneous part. So, this is like a convolution and this is the solution to the homogeneous differential equation. System is said to be state controllable if and only if there exists a solution to this above equation. Or in particular, if you substitute $t = t_f$ and then you take this to the left-hand side, we have $X_i(t_f) - e^{\lambda_i t_f} X_i(0)$ is equal to this rest of it which is $e^{\lambda_i t_f} \int_0^{t_f} e^{-\lambda_i \tau} g_i u(\tau)\, d\tau$.

I want to go from $X_i(0)$ to $X_i(t_f)$ in finite time. So then if that is possible, then there must be a solution to this equation. And what you can clearly see is, if g_i were to be 0 then there is no solution. I can only go from $X_i(0)$ to $X_i(t_f)$ as long as this g_i (is not there) is non-zero because if g_i were to be 0, then whatever I do to u it is not going to change the states. The states are disconnected from the input u.

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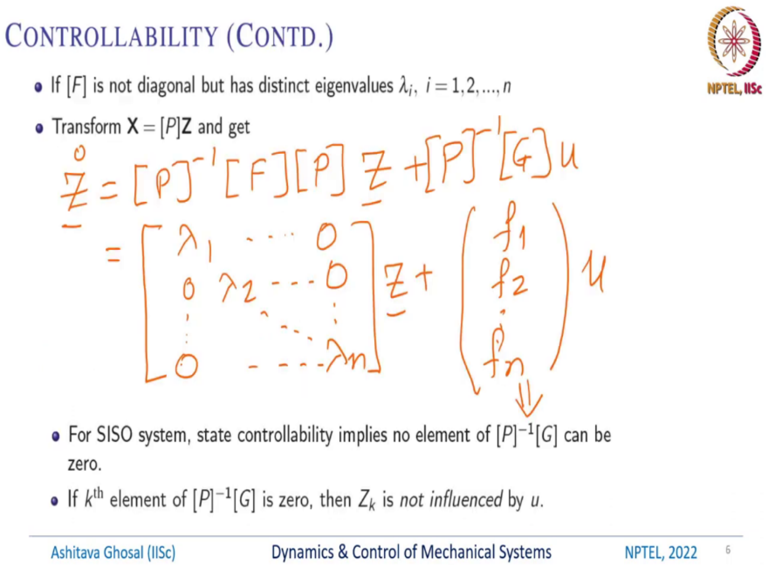

If $[F]$ is not diagonal but has distinct, eigenvalues λ_i, $i = 1,\ldots,n$ then we can transform the state equations using $\mathbf{X} = [P]\, \boldsymbol{Z}$, and we have done this before. I can write the state equations as $\dot{Z} = [P]^{-1} [F] [P] \boldsymbol{Z} + [P]^{-1} [G] u$, or we have a diagonal matrix, λ_1, λ_2 all the way till λ_n and all other elements are 0 into $\mathbf{Z}$ + $[P]^{-1}$ $[G]$ -- we can write it as some f_1, f_2 , .., f_n --the elements of $[P]^{-1}[G]$ are these numbers or scalars f_1 through f_n. As you can see for a single input single output system -- why? because u is only single input here -- the state controllability implies that no elements of $[P]^{-1}$ $[G]$ which is f_1 through f_n can be 0. If for example, if the kth element let us say f_k somewhere here is 0, then u will not influence Z_k because that equation will become $\dot{Z}_k = \lambda_k Z_k + 0\, u$. By application of u, I will not be able to change the states of the system. I will only get the natural dynamics. Whatever is λ, Z will be a function of e^{λ_k} $[P]$ Z (0). So, this is another way of intuitively looking at what is controllability -- no element of this $[P]^{-1}$ $[G]$ can be 0.

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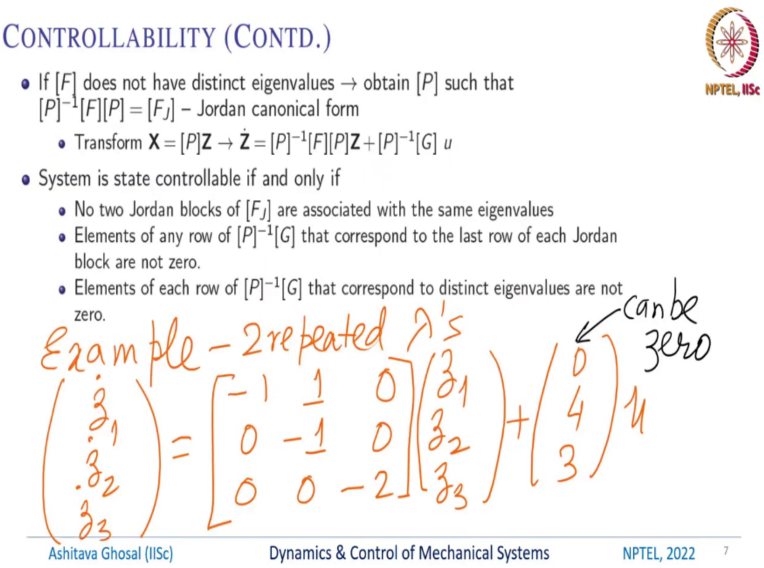

If $[F]$ does not have distinct eigenvalues, then it becomes a little bit more complicated but nevertheless the idea remains the same. We can again obtain a $[P]$ such that $[P]^{-1}$ $[F]$ $[P]$ is this Jordan canonical form of this matrix $[F]$. So, again, we transform $\mathbf{X} = [P]\,\boldsymbol{Z}$, again we get $\dot{Z}$ is $[P]^{-1}$ $[F]$ $[P]$ $\mathbf{Z} + [P]^{-1}$ $[G]$ u. Now, instead of this being a diagonal matrix, we have a Jordan canonical form, and we can say that this system is state controllable, if and only if, no two Jordan blocks of $[F_J]$ are associated with the same eigenvalues. The elements of any row of $[P]^{-1}$ $[G]$ that corresponds to the last row of each Jordan block are not 0 and elements of each row of $[P]^{-1}$ $[G]$ that corresponds to distinct eigenvalues are not 0. This one which corresponds to distinct eigenvalues non-zero is same as what we discussed earlier. So, if any of the elements of $[P]^{-1}$ $[G]$ corresponding to distinct eigenvalues is 0 then that state variable is not affected by u. However, if you have a Jordan block then it is a little bit more complicated. If you think about it, the second condition that elements of any row of $[P]^{-1}$ $[G]$ that corresponds to the last row of each Jordan block are not 0 -- that is sort of obvious.

As an example, let us consider two repeated λ's -- two repeated eigenvalues. So, this is a 3 x 3 system, the matrix $[F]$ is 3 x 3. We have $\dot{Z}_1$, $\dot{Z}_2$ and $\dot{Z}_3$. So, $\dot{Z}_1$ is given by $-1\,Z_1 + 1\,Z_2 + 0 \times Z_3$. So, $\dot{Z}_2$ is $0\,Z_1 - 1\,Z_2 + 0\,Z_3$ and $\dot{Z}_3$ is $-2\,Z_3$. And what you can see, here in the u, $[G]$ u that is (0 4 3). So, this 0 is still okay because this is the Jordan block, and the last row of the Jordan block -- this cannot be 0 -- this one can be 0 and the fact that this last equation which corresponds to an eigenvalue of –2 -- this 3 cannot be 0. This is an example of two repeated λ's -- the two repeated λ's gives this Jordan block and the third λ is –2 which is distinct. So, hence in this Jordan block I can have one 0 here but the second one cannot be 0.

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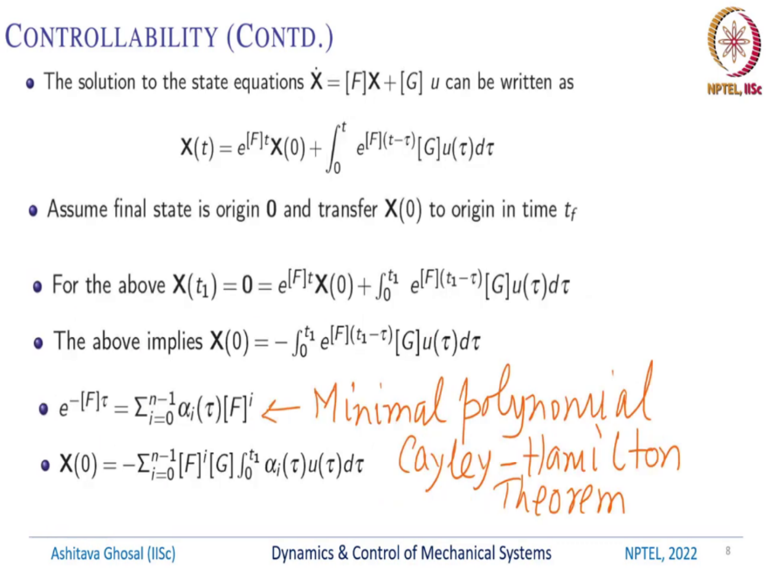

Now, let us get back to the original state equations which is $\dot{X} = [F]\mathbf{X} + [G]u$. The solution to the state equation can be written as $\mathbf{X}\ (t) = e^{[F]\ t}\ \mathbf{X}\ (0) \int_0^t e^{[F](t1-\tau)}\ [G]\ u\ (\tau)\ d\ \tau$, and let us now for the moment assume that the final state is the origin of the state space -- the final is **0** -- so, (0, 0, 0 and so on), and we want to transfer from some initial **X** (0) at $t = 0$ to the origin in time t_f.

For the above **X** (t_1) is **0** (and $e^{[F]\ t}$) this **0** is now equal to $e^{[F]\ t}$ **X** (0) + $\int_0^{t_1} e^{[F](t_1-\tau)}\ [G]\ u\ (\tau)\ d\ \tau$. Remember I want to go from **X** (0) which is my initial state to the origin. This is just a simplification and to bring out an important concept which we will see very soon. The above equation implies that if **X** (0) is $-\int_0^{t_1} e^{[F](t_1-\tau)}\ [G]\ u\ (\tau)\ d\ \tau$. You take to this side and then you do so then you will get a minus sign and then you can simplify and write **X** (0) is this.

$e^{-[F]\ \tau}$ can be written as a minimal polynomial. This is basically the Cayley-Hamilton theorem which says that a matrix satisfies it is characteristic polynomial -- $e^{[F]\ t}$ is some

[I] + $[F]t$ + $([F]^2\ t^2)/\ 2!$ and so on. That infinite series can be represented as a minimal polynomial $\sum_{i=0}^{n-1} \alpha_i(\tau)\,[F]^i$. This is the idea of a minimal polynomial which we have seen earlier when we wanted the solution of (or) how to find the state transition matrix, the concept of a minimal polynomial was introduced, and this is the same idea from there. Hence **X** (0) can be written as this polynomial with a - sign $\sum_{i=0}^{n-1} [F]^i.\,[G]\int_0^{t_1} \alpha_i(\tau)u(\tau)\,d\tau$.

All we are doing is we are writing $e^{-[F]\tau}$, which is here, in this polynomial form and then substituting that $e^{-[F]\tau}$. So, we are left with some $[F]^i$ $[G]$ which we can take it outside and then you have $\int_0^{t} \alpha_i(\tau)u(\tau)\,d\tau$. You can see there is all we are doing is we are simply substituting this in this equation.

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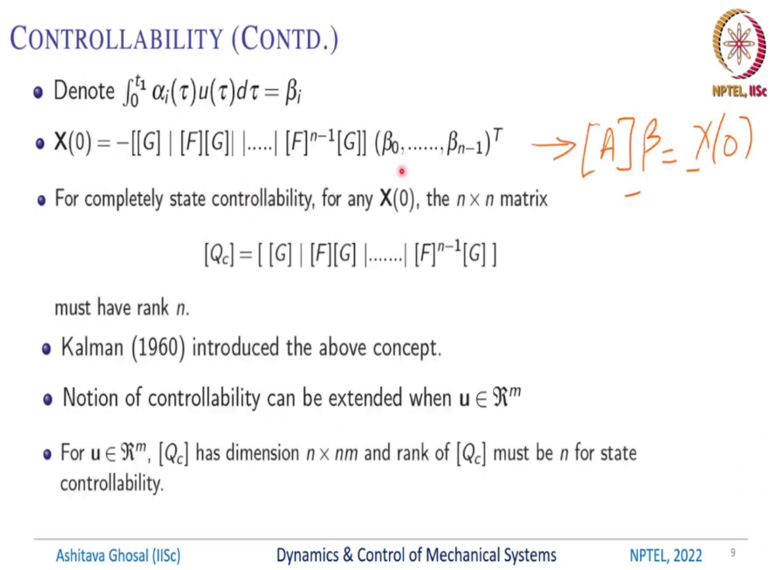

In the previous slide we had one integral $\int_0^{t} \alpha_i(\tau)u(\tau)\,d\tau$. So, let us denote that integral by b_i -- for each α_i, we have one b_i. So now, we can write this **X** (0) so, remember we start from some initial **X** (0) and we want to go to the origin, we can rewrite that previous expression for **X** (0) as minus and this is a matrix here, I will go over it slowly, this first column is [*G*], remember we are discussing [*G*] as *n* x 1 and [*F*] is *n* x *n*, so, we have *n* x 1, a column here then the next column is [*F*][*G*], the third column is $[F]^2[G]$ and the last column is $[F]^{n-1}[G]$. So, since this is the *n* x 1 vector, [*F*][*G*] is also an *n* x 1 vector and all of these columns are *n* x 1 column vectors. This matrix here fully is a n x n matrix. So, we can write **X** (0) as some n x n matrix into (b_0, …, all the way till b_{n-1}) another column vector. So, basically, what we have is, if I call this a matrix [A], we have [A] b = **X** (0). So, for complete state controllability, for any **X** (0) any initial state which we want to take to the origin, the n x n matrix [A] which is nothing but this quantity here, [G], [F][G], $[F]^2[G]$ and so on all the way to $[F]^{n-1}[G]$, which is denoted now by this [Q_c] which is sort of like –[A]. So then this [Q_c] matrix must have full rank. Why? Because we have some equation like [A]**X** = B, B is some arbitrary vector. For any solution of this to exist that means if I need to solve this linear equation [A]**X** = B, the rank of this matrix, [A] must be n because this is n x 1, b is also n x 1. So, this must be full rank, so that is what is mentioned here. The rank of this matrix [G], [F] [G] and so on, must have full rank. So, this derivation that system is state controllable if this matrix called [Q_c], which is [G], [F] [G] all the way till $[F]^{n-1}[G]$, has full rank was obtained by Kalman in 1960. He introduced this concept of this controllability matrix [Q_c] and he derived this result.

So, till now we have assumed u is single input. It can also be extended to multi-input case where, **u** is an m dimensional vector so, it is an element of R^m. So, for $\mathbf{u} \in R^m$, [Q_c] will have dimension n x nm and the rank of [Q_c] must be still n for state controllability. What is n? n is the number of state variables, n x n is the dimension of the matrix, [F] [G] as that if it is m dimensional then the rank, the dimension of [Q_c] must be n x nm and the rank of [Q_c] must still be n for state controllability.

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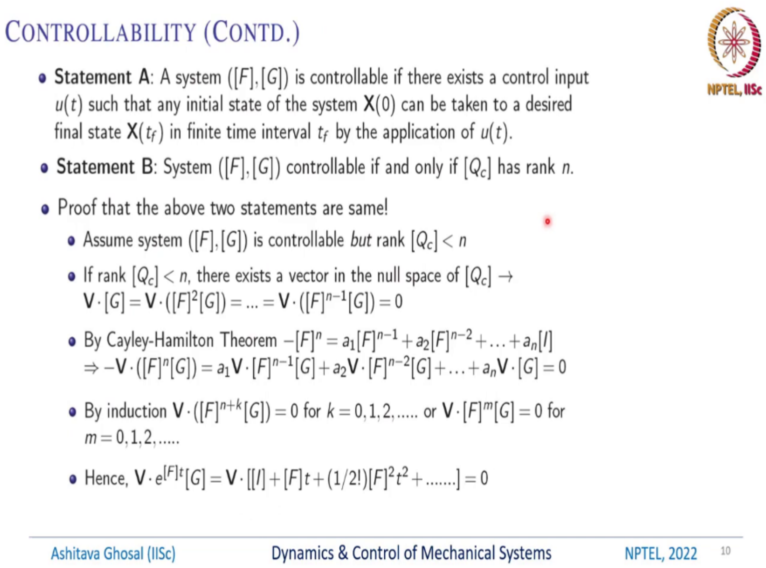

So now, we have two statements for controllability. Let us call the first one as statement A which is the following -- which is the very basic definition of when is a system state controllable. Statement A says the following -- a system $([F], [G])$ is controllable if there exists a control input $u(t)$ such that any initial state of the system $\mathbf{X}(0)$ can be taken to a desired final state $\mathbf{X}(t_f)$ in finite time interval t_f by the application of $u(t)$.

It is a very basic definition of system being controllable -- that I can go from any initial state to another desired final state in finite time by applying the control input $u(t)$.

In the previous slide, I also showed you another version of when a system is state controllable. So, let us call this statement B. The system $([F], [G])$ is controllable if and only if $[Q_c]$ has rank n and what was $[Q_c]$? $[Q_c]$ -- the first column was $[G]$, the second column was $[F][G]$, the third column was $[F]^2[G]$ and so on all the way to $[F]^{n-1}[G]$. So, $[Q_c]$ is of dimension n x n. So, the second statement was -- which I discussed in the last slide-- the system is state controllable if the rank of $[Q_c]$ is n. So now, let us prove that statement A and statement B are same.

This is one way of defining what is a controllable system. This is another way of defining a controllable system. This is the definition or the derivation according to Kalman and this is a very basic notion of what is something which is controllable. Let us start with the proof. So, first is we will prove that A implies B and then we will say B implies A so then A and B are identical if we can show both directions.

So, let us start so, we assume ($[F]$ and $[G]$) is controllable so, we will assume statement A is true but then we assume that statement B is not true. So, basically, we will show that if you assume ($[F]$, $[G]$) is controllable and then we say that rank of $[Q_c]$ is less than n, we will say that this is a not true -- this is false. Hence A will imply B. We first want to assume the opposite and we show that the opposite is not true. This is a very standard way of proving some theorems in maths. If you want to say A implies B, then we assume A to be true and then say that B is not true. Something in B is not correct or not valid and then we go through the steps and show that this assumption is not correct. Hence A implies B. That is the basic idea. So, we assume ($[F]$, $[G]$) is controllable but rank of $[Q_c]$ is less than n.

If rank of $[Q_c]$ is less than n, there exists a vector in the null space of $[Q_c]$. So, $[Q_c]$ if it is say rank is less than 3 -- I am assuming n is 3 -- then there will be a vector in the null space of $[Q_c]$. The null space of $[Q_c]$ will be one dimensional if the rank is 2. The basic idea is that if the rank of $[Q_c]$ is less than n there exists a vector in the null space of $[Q_c]$. This is a very basic result from linear algebra. If there is a vector in the null space of $[Q_c]$, then $V \times [G]$, remember $[G]$ is a column vector and V is that vector in the null space, $V \times [G]$, then $V \times [F][G]$ and all the way till this should be equal to 0.

Hence by Cayley-Hamilton theorem, we know - $[F]^n$ is some $a_1 [F]^{n-1} + a_2 [F]^{n-2}$ and so on + $a_n [I]$ -- Cayley-Hamilton theorem if you recollect is that the matrix satisfies it's characteristic polynomial. So, we have a_1 to the power some function of characteristic polynomial, λ^n plus something into λ^{n-1} plus all the way till something constant is equal to 0. Instead of λ, we substitute $[F]$ for λ and we get one polynomial in $[F]^n$ plus something into $[F]^{n-1}$ + something into $[F]^{n-2}$ and so on all the way till identity equal to 0. And then you take the $[F]^n$ on the other side, so, we will have - $[F]^n$ will be equal to all these other terms. So now, if you take a dot product - $V \times [F]^n[G]$, we will get $a_1 V \times [F]^{n-1}[G]$. So, what have we done? We are using this result from Cayley-Hamilton theorem. We are multiplying by $V \times [F]^n[G]$ and then we substitute all these things and you will get $a_1 V \times [F]^{n-1}[G] + a_2 V \times [F]^{n-2}[G]$ and so on all the way to $a_n V \times [G] = 0$. This equation right hand side you multiply by $[G]$ and left-hand side you do V dot that is all we are doing.

By induction now, $V \times [F]^{n+k}[G]$ is 0. So, for $k = 0, 1, 2$ or all the way till infinity or $V\times [F]^m[G]$ is 0 for $m = 0, 1, 2, 3$. Correct, because we have $V \times [F]^n[G]$ is 0. So then by induction we assume using induction we can show that $V \times [F]^{n+k}[G] = 0$ and then we rewrite this $n + k$ as m and we write $V \times [F]^m[G] = 0$. Hence $V \times e^{[F]\,t}\,[G]$ which is $V \times [I] + [F]\,t + [F]^2\, t^2\, (1/2!)$ all the way is equal to 0 because from this step. So, think about it, we have proved that $V \times [F]^m = 0$. So then we can write $V \times e^{[F]\,t}\,[G]$ and what is $e^{[F]t}$? This is all these quantities into $[G] = 0$.

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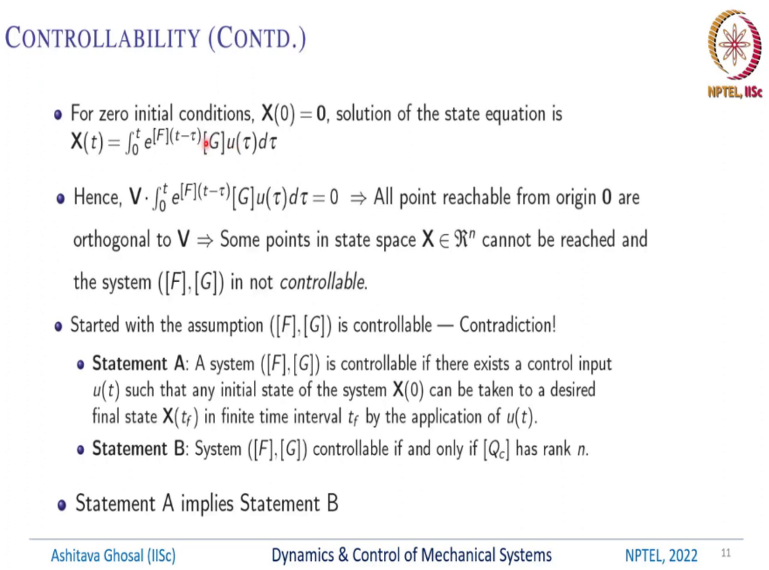

So, for 0 initial conditions **X** (0) = 0 the solution to the state equation is

$\mathbf{X}(t) = \int_0^{t_1} e^{[F](t_1 - \tau)}\,[G]\,u(\tau)\,d\tau$. So, hence V dot this right-hand side is now equal to 0. This implies all points reachable from origin **0** are orthogonal to V because these are the points which are reachable from origin. So, we start from **X** = **X** (0), as **0** then **X** (t) is the solution of the state equation which is the first term will go to **0** and we are left with only

$\int_0^t e^{[F]\,(t-\tau)}\,[G]\,u(\tau)\,d\tau$. You can go back and see the solution -- **X** (t) it is one part which is the initial, first or the initial condition which is $\mathbf{X}(0)\, e^{[F]t}$ plus the particular part. Now, **X**(0) is **0**, so, we are left with **X** (t) and then we can take a V dot of this and then it is clearly equal to **0** because that is what we have proved in the previous slide. So, hence all points reachable from the origin are orthogonal to V.

V is some vector and this is another vector and all these points are $\mathbf{X}(t)$ are orthogonal to V, which basically means some points in the state space $\mathbf{X}$ Î R^n cannot be reached. So because V dot of this equal to 0. So, think about it I have showed you that I start from origin, and this is the solution to the state equation and then there is a vector V which is in the null space of $[Q_c]$ and that V dot then this quantity is 0. Which basically means that there are some points in the state space which I cannot reach because that is orthogonal to the solution.

Hence $([F], [G])$ is not controllable but we started with the assumption that $([F], [G])$ is controllable. Hence this is a contradiction so, this is a proof by contradiction. So, hence statement A implies statement B. What did we start with - we assume statement A was true and we assume that statement B was false -- that the rank of $[Q_c]$ was less than n -- and then I have shown you that this cannot be true. If $([F], [G])$ is controllable, then the rank of $[Q_c]$ cannot be less than n. Hence A implies B. Now, we will continue and show you that the B also implies A and hence statement A and B are identical. It is a very useful thing to remember, I mean it is little bit complicated and little bit math and logic, but statement A is a very basic definition of system being controllable. Statement B is something which Kalman derived and at least till now I have shown you that A and B are similar in the sense that A implies B.

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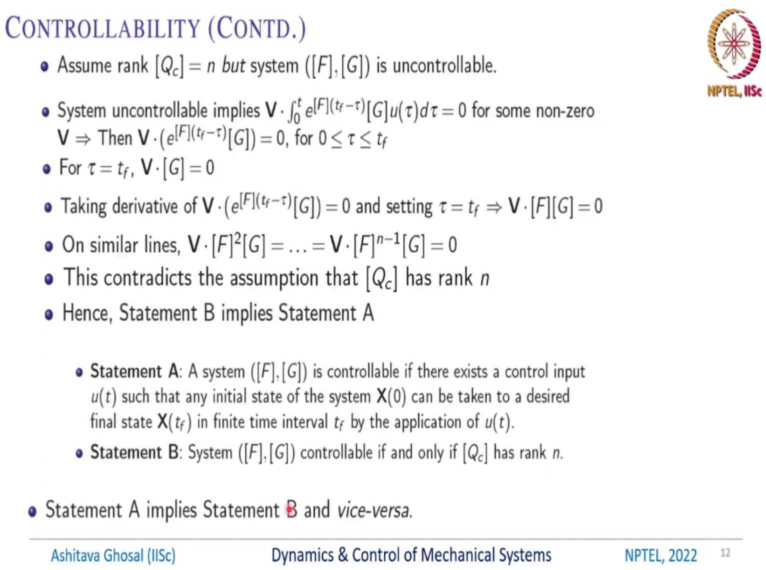

Let us do the backwards that B implies A. So, basically what do we have -- we assume that the rank of $[Q_c]$ is n but the system $([F],[G])$ is not controllable. We start with this assumption and we will show that this is false -- that there is a contradiction similar to last time. If system is uncontrollable, it implies that some *V* dot this solution of the state equation, will be equal to 0 for some non-zero *V*.

If the system is not controllable then there are some points which I cannot reach -- there is a vector *V* in the null space such that $V \times \mathbf{X}(t) = 0$. Then we can show this implies that $V \times \int_0^t e^{[F](t-\tau)} [G]\, u(\tau)\, d\tau$ is 0, for some 0 £ τ £ t_f. For $\tau = t_f$, $V \times [G]$ will be 0 then taking the derivative of *V* dot this equal to 0 and setting $\tau = t_f$ will have $V \times [F][G] = 0$, and on similar lines we have $V \times [F]^2[G]$ then all the way till $V \times [F]^{n-1}[G] = 0$. This contradicts the assumption that $[Q_c]$ has rank *n*. Remember what is $[Q_c]$ -- the first column was $[G]$, the second column was $[F][G]$, the third column was $[F]^2[G]$ and all these columns are linearly independent (but what so) or that $[Q_c]$ has rank *n*. But we are showing that there is a vector *V* such that $V \times [G] = 0$, $V \times [F][G] = 0$, $V \times [F]^2[G] = 0$ and so on. So, hence there is some non-zero *V* such that this system, I cannot reach that. The solution $\mathbf{X}(t) \times V = 0$ which basically contradicts the assumption that $[Q_c]$ has rank *n*. We started with assume $[Q_c]$ as rank *n* but the system is uncontrollable, and I showed you this contradicts that assumption. Hence statement B implies statement A.

Again what was statement A that was the very basic definition of controllability -- A system is system $([F], [G])$ is controllable if there exists a control input *u* such that any initial state of the system $\mathbf{X}(0)$ can be taken to a desired final $\mathbf{X}(t)$ in finite time interval t_f by the application of $u(t)$. And this is the definition statement B is what is derived by Kalman and he said the system $([F], [G])$ is controllable if and only if $[Q_c]$ has rank *n*. What we have shown you? That these two statements are exactly same. It is a long proof but it is important to realize that the very basic definition of controllability of a system $([F], [G])$ is the same as something which is very, very easy to figure out. We can always find out what is $[Q_c]$ -- first column is $[G]$, second column is $[F][G]$ and so on and to obtain the rank of that matrix is also fairly straightforward. Instead of looking at this sort of little bit abstract definition, we can easily check if a system $([F], [G])$ is controllable, just by checking the rank of $[Q_c]$ and both these definitions I have proved to you mean the same thing.

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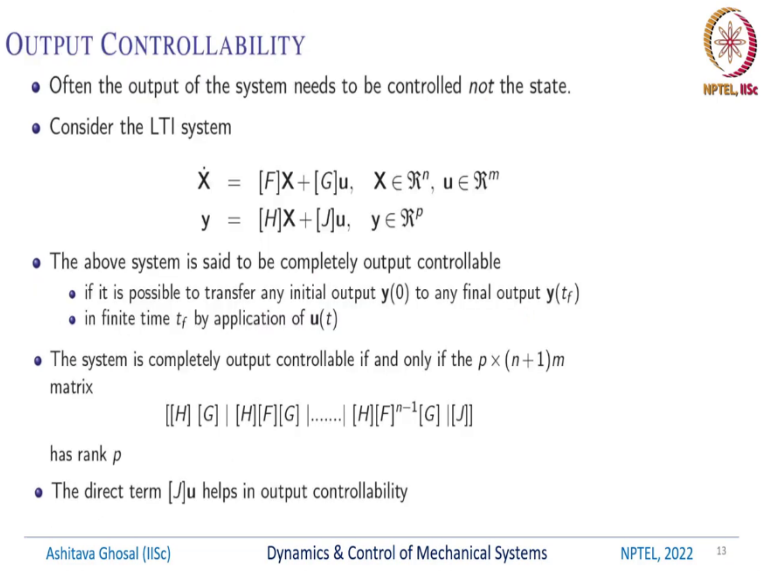

There is also something called output controllability. Often the output of the systems need to be controlled, not the state. If you consider this linear time, invariant system $\dot{X} = [F]\mathbf{X} + [G]\mathbf{u}$ and $\boldsymbol{y} = [H]\mathbf{X} + [J]\mathbf{u}$**,** the above system is said to be completely output controllable if it is possible to transfer any initial output $y\,(0)$ to final output $y\,(t_f)$ and in finite time by the application of $u\,(t)$.

We are not really interested in going from $\mathbf{X}\,(0)$ to $\mathbf{X}\,(t_f)$. We are more interested in going from $y\,(0)$ to $y(t_f)$ by applying u in finite time t_f. And it turns out that system is completely output, controllable if and only if the p x $(n + 1)\, m$ matrix, [$[H][G]$, $[H][F][G]$,…. $[H]\,[F]^{n-1}[G]$] and append it with the $[J]$ matrix has rank p. Remember there are $\boldsymbol{y}$ is $[H]\mathbf{X} + [J]\mathbf{u}$, the dimension of y is p x 1. So, we are making p measurements out of the available n states and then we can obtain a matrix which is now must include $[H]$ $[F]$ and $[G]$ and also $[J]$ and in this form. The first part is $[H][G]$ then $[H][F][G]$, then you concatenate these two and then all the way you concatenate $[H][F]^{n-1}[G]$ and finally $[J]$. So, this complicated looking matrix must have rank p.

I will not go into the proof of this, but this is what output controllability means. In output controllability you can see that there is also a term which is $[J]$. What was $[J]$? $[J]$ was the direct term there is a connection directly from **u** to the output. So, if you have a direct term $[J]$ into **u,** this helps in output controllability.

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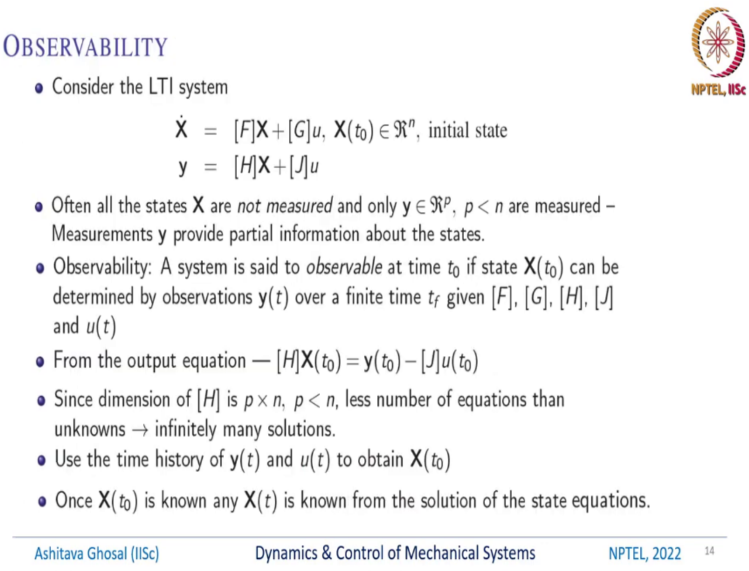

Let us continue, we also mentioned that there is a very important concept in control which is called observability. Let us go into a little bit more detail so, consider an LTI system which is again $\dot{X}$ = [F]**X** + [G]u and ***y*** = [H]**X** + [J]u. So, as mentioned in the past, often all the states **X** are not measured and only y Î R^p, where $p < n$ are measured. So, the measurements y provides partial information about the states. Remember p is less than n. So, the basic notion of observability is the following -- a system is said to be observable at time t_0 if state $\mathbf{X}(t_0)$ can be determined by observations y (t) over a finite time t_f given [F], [G], [H] and [J] and u(t). So, I want to know what is the state $\mathbf{X}(t_0)$ by measuring $y(t)$, but remember the number of measurements $y(t)$ which is $p < n$. From the output equation [H] $\mathbf{X}(t_0) = y(t_0) - [J]\ u\ (t_0)$. This is n x 1, [H] is p x n so, the dimension of [H] is p x n with $p < n$. Hence, there are less number of equations than unknowns. What are the unknowns? $\mathbf{X}(t_0)$. So, there are n unknowns, but the number of equations is only p. How do I find out these n unknowns when we have only p equation? There are infinitely many possible solutions from linear algebra.

It is like [A]**X** = B, the dimension of **X** is n but [A] is not n x n, so, we have infinitely many solutions. So, what can we do? We can use the time history of $y(t)$ and u (t) to obtain the $\mathbf{X}(t_0)$. So once $\mathbf{X}(t_0)$ is known any $\mathbf{X}(t)$ is also known. Why? Because the solution to the state equations are known. What is the solution? $\mathbf{X}(t)$ is some $e^{[F]t}$ $\mathbf{X}(t_0)$ plus some integral -- that convolution term. If I know what is the initial conditions, $\mathbf{X}(t_0)$ then I can use the state equations to find any other $\mathbf{X}(t)$.

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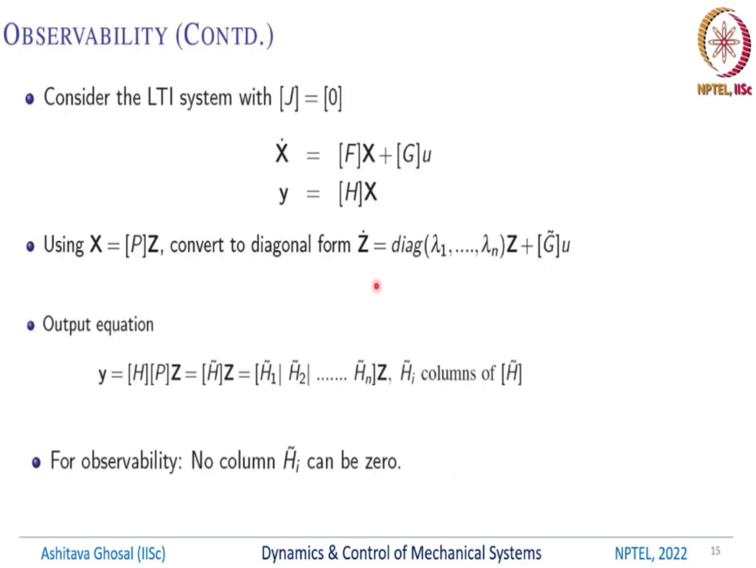

Before we go into the formal way of approaching observability, let us quickly see under what conditions a system is not observable. So, we will take some simple cases. First is we look at an LTI system, linear time invariant system with $[J] = 0$. So, then we have, the state equation, is $\dot{X}$ is $[F]\mathbf{X} + [G]u$ and we have $\boldsymbol{y} = [H]\mathbf{X}$. So, let us now like in the past, do a transformation from **X** to **Z** which is **X** = $[P]$ **Z** and then convert this $[F]$ into a diagonal form. So, we are looking at a very simple subset. so, in the diagonal form we will have $\dot{Z}$ is some diagonal matrix with λ_1, λ_2 all the way till λ_n into **Z** and then we will have some $[P]^{-1}[G]$ u. The output equation can be written as $y = [H]$ $[P]$ **Z**, so, remember, **X** is [P] **Z** -- **X** = [P] **Z**. So, we will have $[H]$[P] **Z** and let us call this $[H][P]$ as $[\widetilde{H}]$ and what is $[\widetilde{H}]$ -- so, **Z** is n x 1, there are n columns and let us call them $\widetilde{H}_1$, $\widetilde{H}_2$, all the way through $\widetilde{H}_n$, so, where $\widetilde{H}_i$ are the columns of this matrix $[\widetilde{H}]$. What you can clearly see is that for observability no column of this can be 0. Why? Because the first column is into Z_1, second column into Z_2 -- y is $\widetilde{H}_1 Z_1 + \widetilde{H}_2 Z_2$ and so on. This is a very intuitive way of looking at when a system is not observable. If any of these columns are 0 then that Z, say let us say the kth column is 0 then Z_k x 0 will not show up in y. There will be no connection between y and Z_k.

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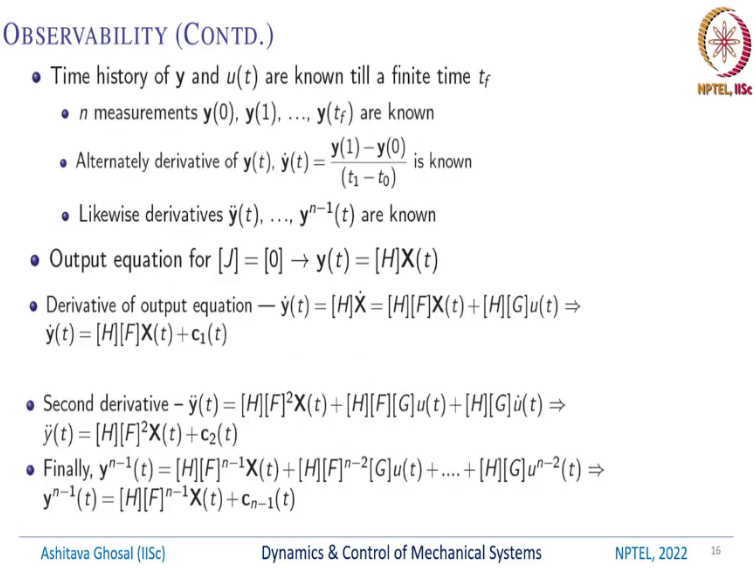

Now, let us go back to the basic idea that we want to use the time history of y and $u(t)$ and of course, we want to use it only for finite time t_f, and see whether we can obtain the initial condition $\mathbf{X}(t_0)$. So, we have n measurements $y(0)$, $y(1)$ all the way till $y(t_f)$. Alternatively, we can estimate the derivatives $y(t)$, $\dot{y}(t)$ and so on. So, what is $\dot{y}(t)$? Which is $[y(1) - y(0)]/t_1 - t_0$. So, I want to estimate the derivatives of $y(t)$ -- $\dot{y}$ is like this how about $\ddot{y}$ -- that also we can find out. So, basically, we can use some kind of a finite difference. So, $\ddot{y}$ will have $y(2)$ and then $y(1)$ and $y(0)$ and then it will have $t(2)$, $t(1)$ and $t(0)$. All the derivatives up to y^{n-1} (t) can be obtained from these n measurements.

Now, let us go back and see the output equation with $[J] = 0$. We have y (t) is $[H]$ $\mathbf{X}(t)$. So, the derivative of the output equation $\dot{y}$ will be $[H]$ $\dot{X}$ - remember [H] is a constant matrix. Now, $\dot{X}$ is same as $[F]\mathbf{X}$(t) + $[G]u(t)$. So, if you go back and substitute $\dot{X}$ from the state equations, we will get $[H][F]\mathbf{X}(t)$ + $[H][G]$ $u(t)$. So, we will have this. I want to write this as $\dot{y}$ (t) = $[H][F]\mathbf{X}(t)$ + $c_1(t)$. Why? Because this is known, if we know what is $[H]$, we know what is $[G]$, and we know what is u. So, when you are giving u so, the time history of y and $u(t)$ are known. So, for every given $u(t)$ I know what is the output y. I can rewrite this expression as $\dot{y} = [H][F]$ $\mathbf{X}(t) + c_1(t)$. From the second derivative also, I can write $\ddot{y} = [H][F]^2$ $\mathbf{X}(t)$ + $[H]$ $[F]$ $[G]$ u (t) + $[H]$ $[G]$ $\dot{u}$ (t). And again, we know what is $\dot{u}$ (t) -- $\dot{u}$ $(t) = u$ $(t_1) - u$ (t_0) divided by $(t_1 - t_0)$. Hence, I can write all these terms as some c_2 (t). So, I have $\ddot{y}$ as $[H][F]^2$, which is this term, into $\mathbf{X}(t)$ plus everything else under c_2 (t). And finally, I can write

y^{n-1} (t), $(n$-1)th derivative of y(t), as $[H]$ $[F]^{n-1}$ $\mathbf{X}(t)$ plus all these terms which can be again clubbed together as a c_{n-1} (t).

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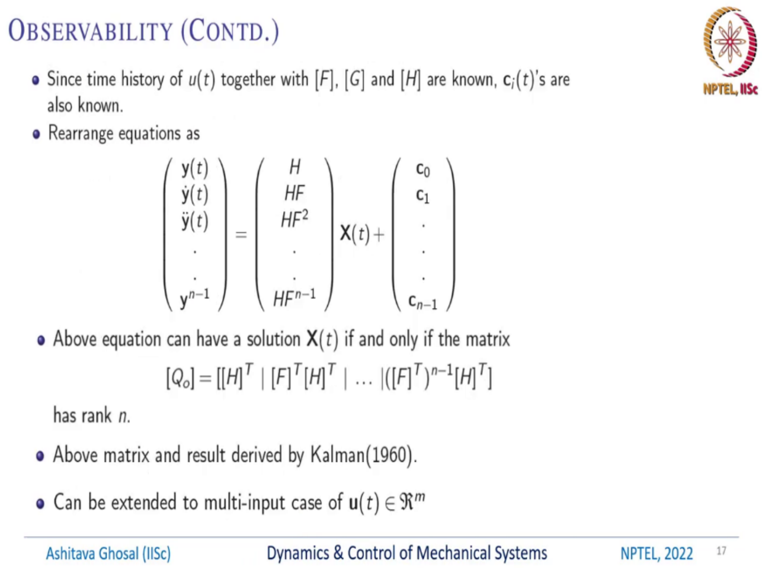

Since the time history of $u(t)$ together with $[F]$, $[G]$, $[H]$ are known, all the $c_i(t)$'s are known. So, we can rearrange all those equations as $y(t)$, $\dot{y}(t)$, $\ddot{y}(t)$, these are in one below the other is equal to $[H]$ $\mathbf{X}(t) + c_0$, $[H][F]$ $\mathbf{X}(t) + c_1$ and all the way to $[H][F]^{n-1}$ $\mathbf{X}(t)+ c_{n-1}$. This is like an expression which is y equals some matrix times $\mathbf{X}(t)$ plus some constant.

This above equation can only have a solution if and only if, this matrix here, $[H]$, $[H]$ $[F]$, $[H]$ $[F]^2$, $[H][F]^{n-1}$ is a full rank. Now, instead of writing it in this form, we are going to write it in this form which is $[H]^T$ then appended with $[F]^T[H^T$ then appended with $([F]^T)^2[H]^T$ and all the way till $([F]^T)^{n-1}$ $[H]^T$. This is a matrix which is exactly the same as this. It is written in a form such that it can fit into a line.

This matrix is called $[Q_0]$ or sorry $[Q_o]$ which is the observability matrix. The rank of this matrix here or $[Q_o]$ must be n only then this equation can be solved. This equation which is something like $y = [A]\mathbf{X} + B$ will have a solution only if the rank of this matrix $[A]\mathbf{X}$, the rank of $[A]$ is n - is full rank.

This is another of Kalman's derivation and he obtained this expression for this observability matrix, again in 1960, and he showed that a system is observable if the rank of this $[Q_o]$ is n. And remember what is observable? I can find the states by measuring output y. However, the number of outputs that I am measuring which is p of them is less than the number of states which is n of them.

We can also extend the same results if you have multi-input case. If you have m inputs instead of single input - which is what is shown here - and that has been done.

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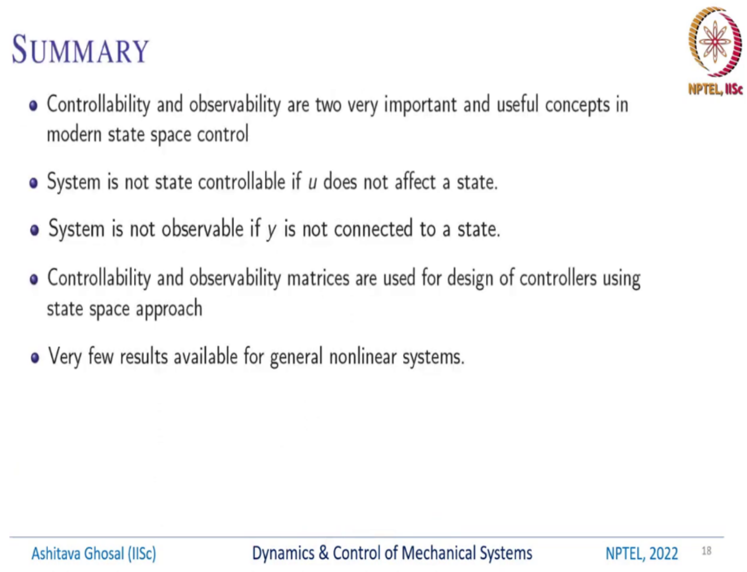

In summary, the controllability and observability are two very important and useful concepts in modern state space control. A system is not state controllable if u does not affect a state. A system is not observable if y is not connected to a state that is very intuitive and very basic. Kalman found two matrices $[Q_c]$ and $[Q_o]$ and if the rank of those matrices are n where n is the dimension of the state space, then we can say that the system is controllable or observable. And this controllability and observability matrices are used extensively for design of controllers using state space approach. We will see that later on in this course. Unfortunately, there are very few results available for general nonlinear systems. This is well known and well extensively studied and extensively used for linear systems -- not very nice or general results are available for nonlinear systems.