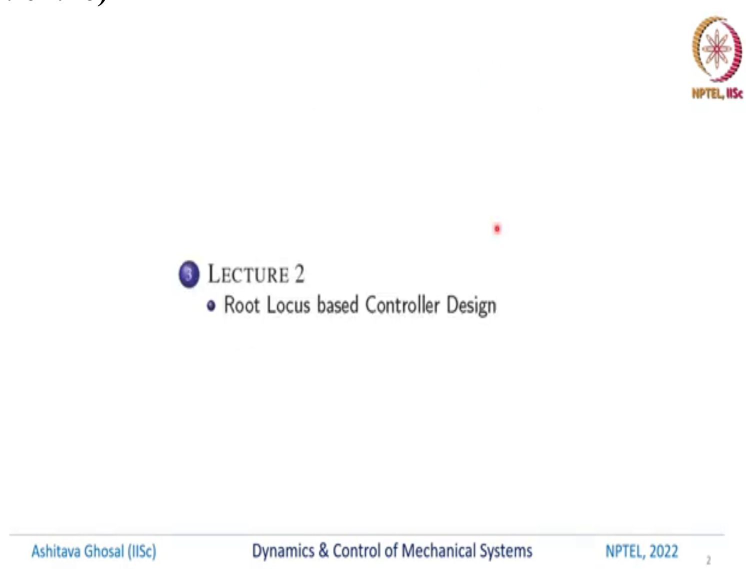
Dynamics and Control of Mechanical Systems
Prof. Ashitava Ghosal
Department of Mechanical Engineering
Indian Institute of Science – Bengaluru

Lecture –30
Root Locus based Controller Design1

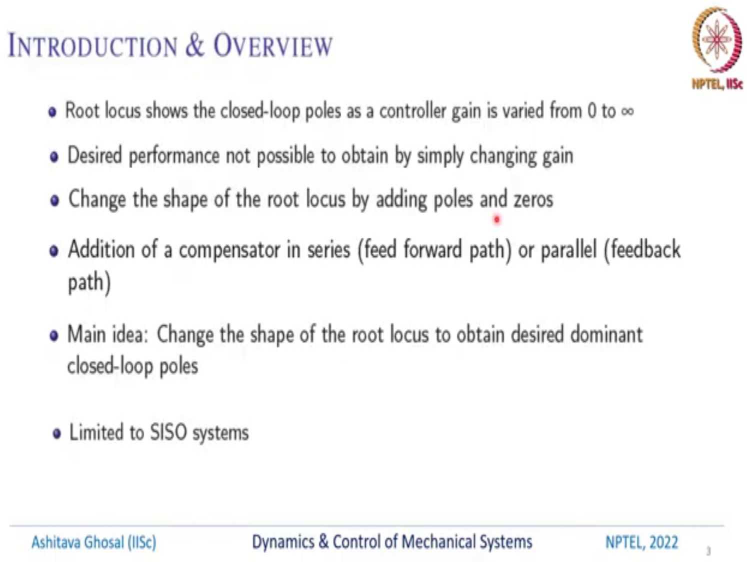
Welcome to this NPTEL lectures on Dynamics and Control of Mechanical Systems. This week we are looking at Design of Controllers. My name is Ashitava Ghosal. I am a Professor at the Department of Mechanical Engineering and in the Centre for Product Design and Manufacturing and also at the Robert Bosch Centre for Cyber Physical Systems, Indian Institute of Science, Bangalore.

In the last lecture we had looked at PID controllers and how we could go about choosing the proportional, derivative and integral gains to achieve a desired performance. In this week, we will look at root locus based controller design.

(Refer Slide Time: 01:16)



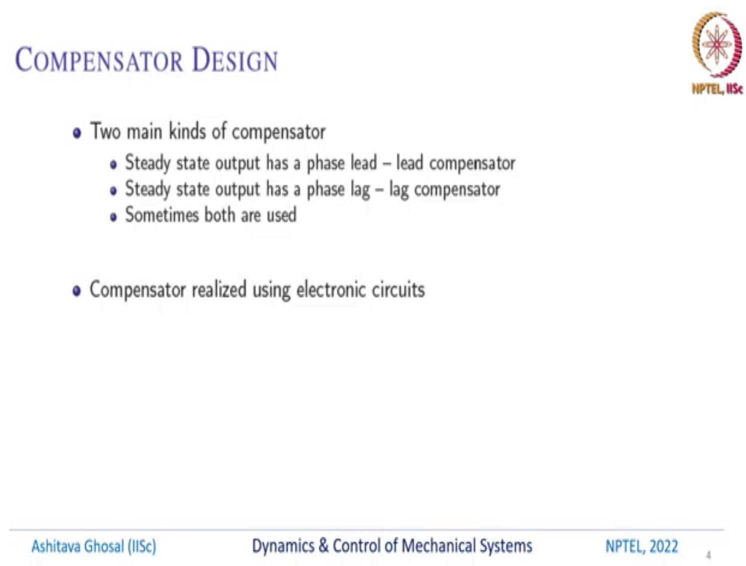
(Refer Slide Time: 01:19)



In the root locus based controller design, we will basically look at how the root locus can be changed or modified to achieve the desired performance. So, to recapitulate, a root locus shows the closed loop poles of a system as a controller gain is varied from 0 to ∞. The desired performance is often not possible by simply changing the gain and we need to change the shape of the root locus by adding poles and zeros.

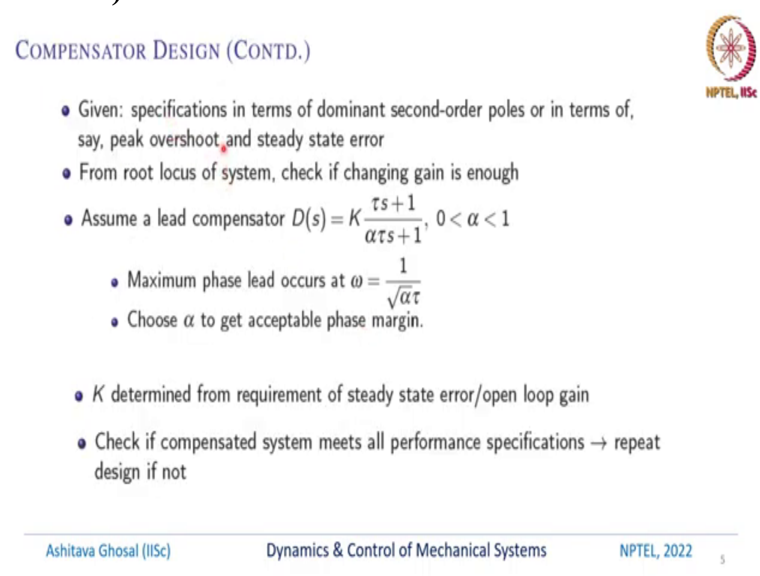
These are called compensators. These compensators can be in series or in parallel in the feedback path. The main idea is to change the shape of the root locus to obtain the desired dominant, closed loop pole and as usual or as a disadvantage, these are limited to single input, single output systems. So, most classical approaches to controller design can be used only for single input single output system easily.

(Refer Slide Time: 02:29)



There are two main kinds of compensators. If the steady state output has a phase lead, then we need a lead compensator. If the steady state output has a phase lag, then we will use a lag compensator Sometimes both are used. The compensators are normally realized using electronic circuit -- these need not be physical devices.

(Refer Slide Time: 02:56)

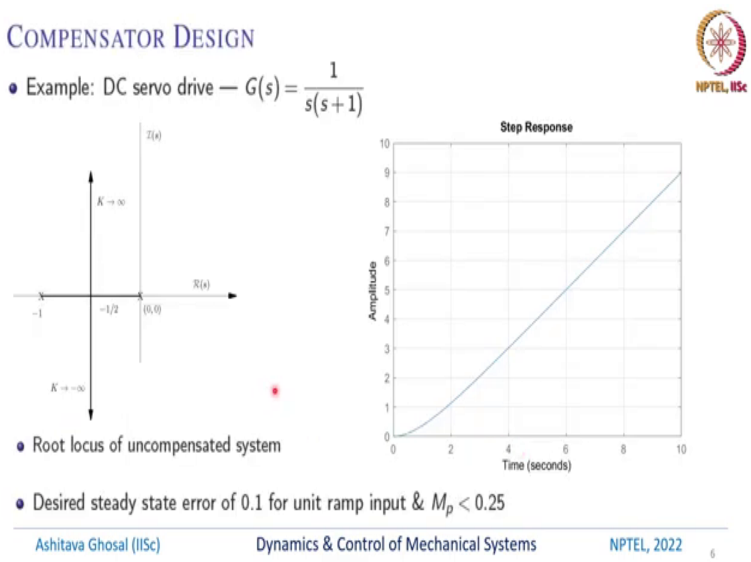


So, what is given? The specifications normally are in terms of dominant second order poles or in terms of, let us say, peak overshoot and steady state error. So, we would like the control system to achieve certain transient response and certain steady state error response. From root locus first, we can check if changing the gain is enough to ensure that the root locus passes through the desired dominant second order poles.

We are given some two second order poles which are the dominant second order poles. We know as we change the gains, the root locus will pass through certain points in the *s* plane, and we can just simply check whether it passes through the desired dominant second order poles. This is most of the time not enough. In that case we can assume a lead compensator. The lead compensator is given in this form.

The transfer function is $K(\tau s + 1) / (\alpha \tau s + 1)$ where α is between 0 and 1. So, as you can see from this transfer function, the maximum phase lead occurs when omega is 1/Ö α τ. So, we need to choose α to get the acceptable phase margin. *K* is determined from the requirement of a steady state error, or the open loop gain and it needs to be checked if the compensated system meets all the performance specification. If it does not meet, we need to go back and repeat the design.

(Refer Slide Time: 04:51)

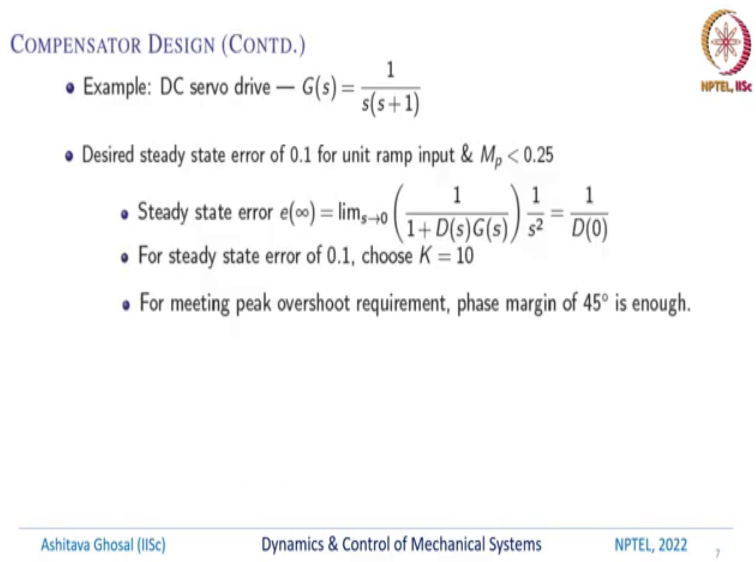


Let us take a simple example again we go back to our usual DC servo drive where the transfer function is given by $1/s(s + 1)$ again remember, *K* and then inertia, *J* and the friction all of them are chosen to be 1. In the case of this DC servo drive transfer function, there are two poles -- they are at 0 and 1 -- $s = 0$ and $s = -1$ and as the gain is increased from 0 to infinity, the branches of the root locus will come towards each other and then go off to infinity.

This is the root locus of the uncompensated system. If I want some other dominant poles, let us say not on this root locus. Then we need to do something else that is called as the compensated system. So, in this example, let us assume that we want a desired steady state error of 0.1 for a unit ramp input and we want a peak overshoot of less than 0.25.

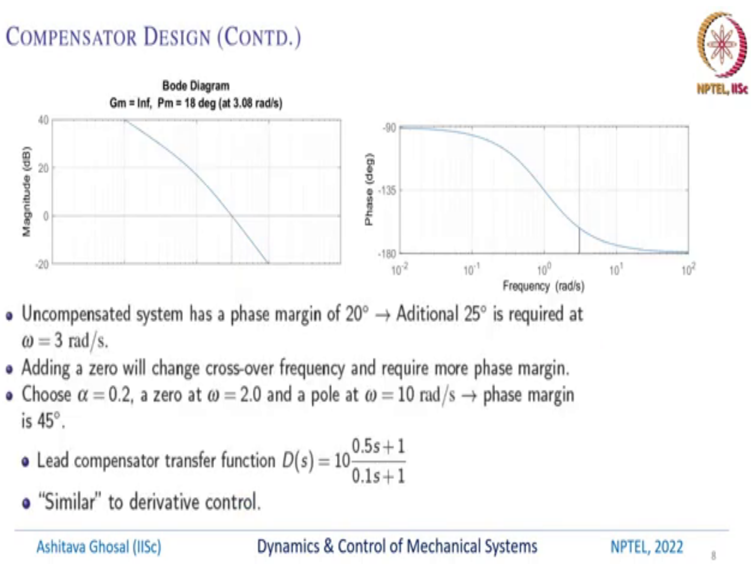
So, what is the response of this system to a step response? We have seen this earlier there will be a small exponential portion and then there is a linear, a straight-line behaviour. If you were to plot a line which is at 45 degrees, there will be a gap here which is determining the steady state error for the unit ramp input.

(Refer Slide Time: 06:29)



So, let us continue with this example, so, we have this transfer function which is 1/ s (s + 1). We want a steady state error of 0.1 for a unit ramp input and the peak overshoot should be less than 0.25. So, the steady state error is e with error which is $r - y$ and if you find the limit, as s tends to 0 because we want t tends to infinity. So, we will get 1/ 1 + $D(s)G(s)$ into (1 / s^2). We are giving a ramp input, so, this will give me 1/ D (0). So, D is the controller transfer function and whatever is the value of the controller transfer function for s = 0. That is what the steady state error is. For steady state error of 0.1. We can choose K as 10, so, if you choose K as 10, so, 1 / 10 will give me a steady state error of 0.1. Now, in order to meet the peak overshoot requirement, we need to have a phase margin and phase margin of 45 degrees is enough.

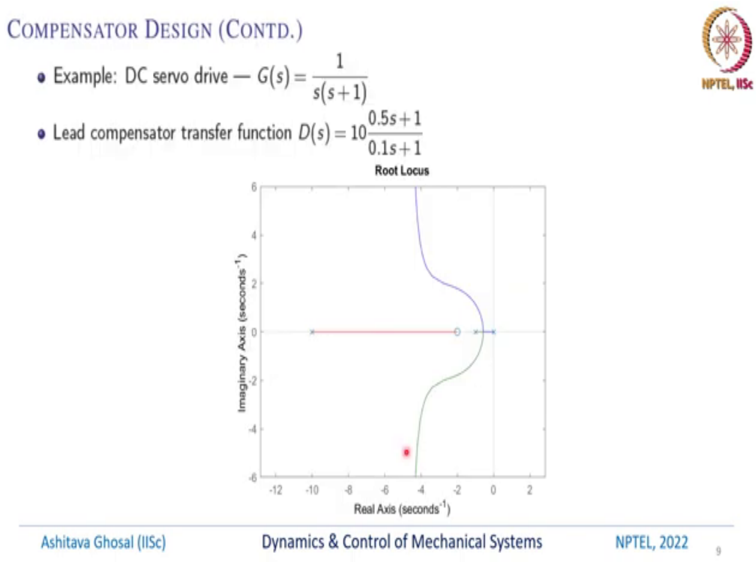
(Refer Slide Time: 07:47)



If you plot the Bode diagram, we can see that the gain margin is infinity, the phase margin is 18 degrees at 3.08 radians per second at this place. So, the uncompensated system has a phase margin of about 20 degrees. We need an additional 25 degrees at $\omega = 3$ radians per second and this can be done by adding a zero. But, however, if you add a zero, it will change the crossover frequency and just to be on the safe side, we require a little bit more phase margin. So, we choose α as 0.2, a zero at $\omega = 2$ and a pole at $\omega = 10$. The phase margin for such a system is 45 degrees. So, the lead compensator transfers function *D* (*s*) is 10 (which was *K)* and this is
0.5 (*s* + 1). So, there is a zero at 0.5 and a pole at 10 radians per second (0.1 *s* + 1).

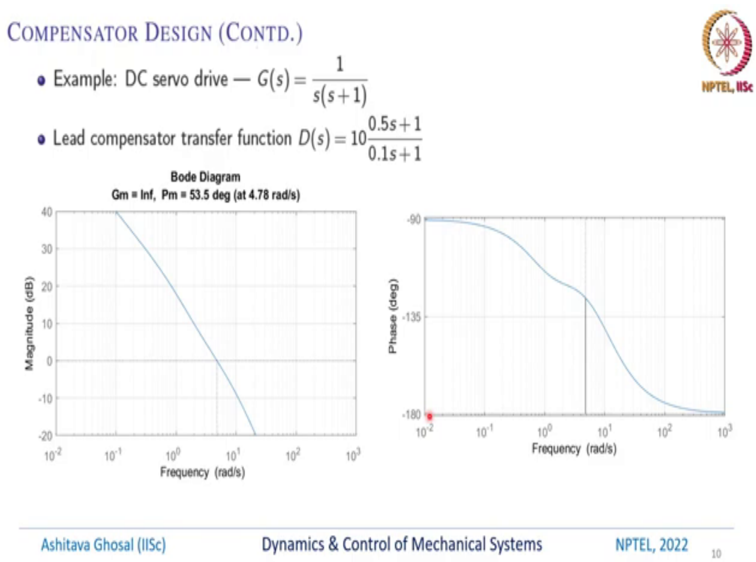
In a sense, this is sort of similar to a derivative controller. Why? Because you can see the numerator is like 0.5 *s* and the denominator is (0.1 *s* + 1). There is, of course, some + 1 effect is there. So, it is sort of like $K\, t_d\, s\, /\, (1 + t_v\, s)$. So, remember for the derivative part in a PID controller, we had some constant into something into $s / (1 + t_d\, s)$. So, it in some sense similar to a derivative controller but not exactly same because of this 1.

(Refer Slide Time: 09:40)



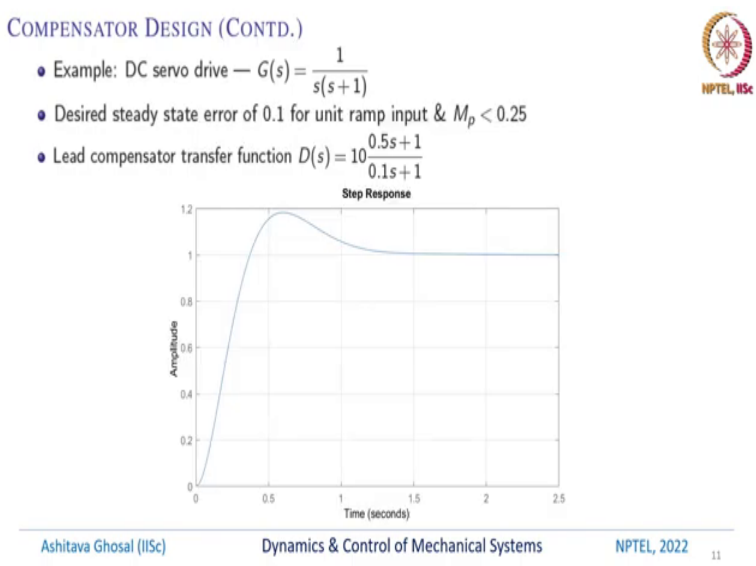
We can now, plot the root locus with this example, with the lead compensator given by 10 (0.5s + 1) / (0.1 s + 1) and you can see that the root locus looks like this. So, there is one pole and s = 0 and s = 1 and we have introduced one pole and one zero and the shape of the root locus now looks like this.

(Refer Slide Time: 10:13)



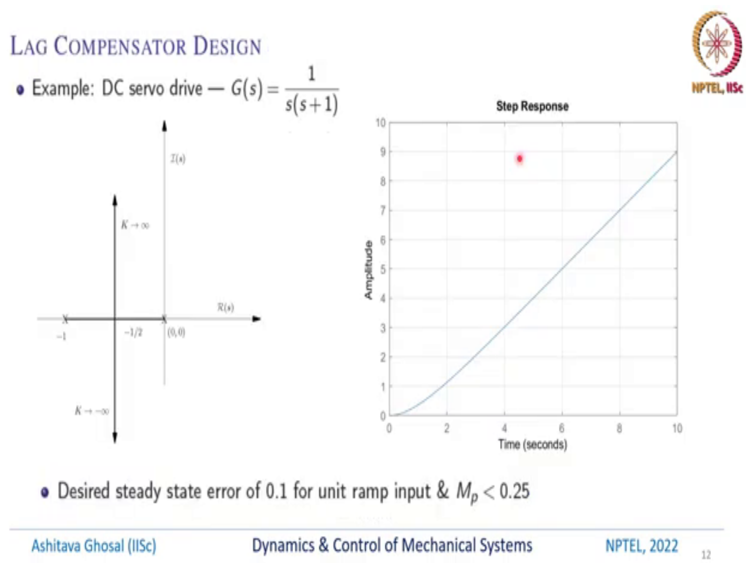
So, does it meet our requirements? We can go back and look at the Bode diagram and we can see that the gain margin is infinity, the phase margin is 53.5 degrees at 4.78 radians. So, it is more than 45 degrees which we were looking for. It looks like it will work.

(Refer Slide Time: 10:37)



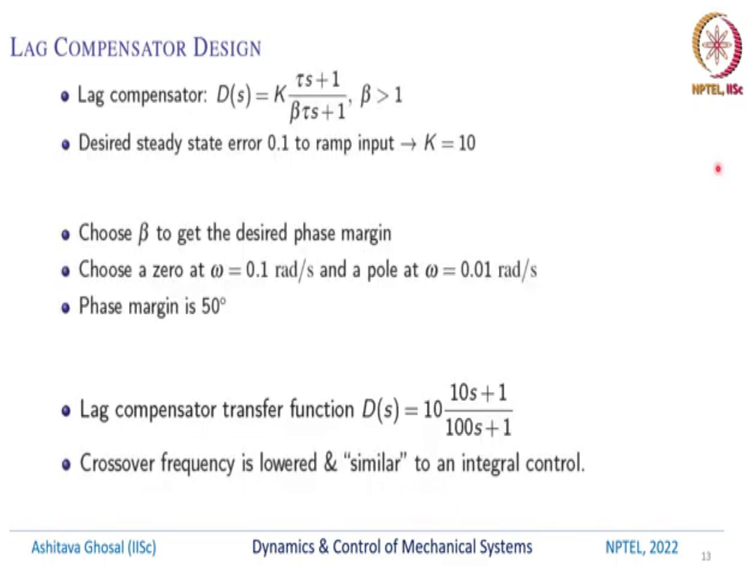
So, in order to finally, test we can do a step input to this system. So, G (s) is 1/ s (s + 1) and then we have a controller or a compensator in series which is 10 (0.5 s +1)/ (0.1 s + 1) and we can give a step input. The step input looks like this, so, as you can see that it settles down very close to 1, it does not overshoot - less than 0.2. So, we wanted overshoot of less than 0.25 which is fine, and the steady state error is also less than 0.1. It is very close to. It is much less than 0.1. It is some value which is little bit more than 1. So, our design which is this lead compensator which is D(s) is 10 (0.5 s + 1)/ (0.1 s + 1) satisfies these requirements of steady state error less than 0.1 and peak overshoot less than 0.25.

(Refer Slide Time: 11:47)



We can also see if a lag compensator works for this example. So, let us go back and take this example of the DC servo drive G(s) is again 1/ s (s + 1) and we will see whether the lag compensator can be used instead of the lead compensator. So, again we have this root locus which is 0 and –1 and as gain changes, it goes up to infinity. But clearly it does not meet the steady state error of 0.1 for a unit ramp input and M_p less than 0.5. The output for the step response for this system is given in this form.

(Refer Slide Time: 12:32)

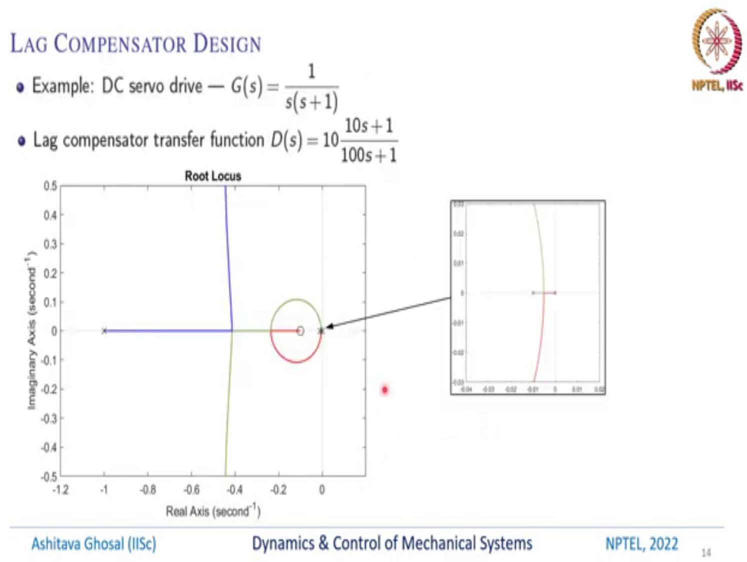


So, to obtain the lag compensator, we will choose the controller transfer function or the compensator transfer function as $K(\tau s + 1)/ (b \tau s + 1)$. So, apparently, it looks very similar to the lead compensator. Except in the case of lead compensator, this α here instead of b was between 0 and 1. However, for a lag compensator b is always greater than 1.

So, in order to obtain the desired steady state error of 0.1 to the ramp input again, we can see that K can be 10 and then we need to choose b to get the desired phase margin. So, we choose a zero at $\omega = 0.1$ and a pole at $\omega = 0.01$ and this gives a phase margin of 50 degrees - we can calculate that. So, the lag compensator for this system to meet the requirements is $D(s)$ is $10 (10s + 1)/ (100s + 1)$.

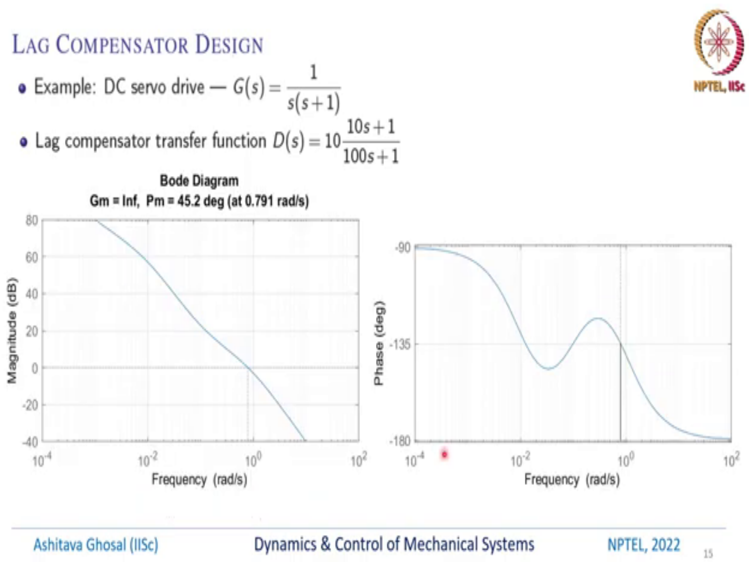
As you can see, this is very different from the lead compensator. So, what happens when you use such a lag compensator? The crossover frequency is lowered, and, in a sense, this is very similar to an integral control. So, basically, what you can see is this is $(10s + 1)$, here it is $(100s + 1)$. This is like much - the denominator is much larger. So, it is sort of like a integral control. If actually, the numerator was not there and then it would be $10/ (100s)$ sort of approximately. Then it would be a pure integral control, but this is similar to an integral control.

(Refer Slide Time: 14:31)



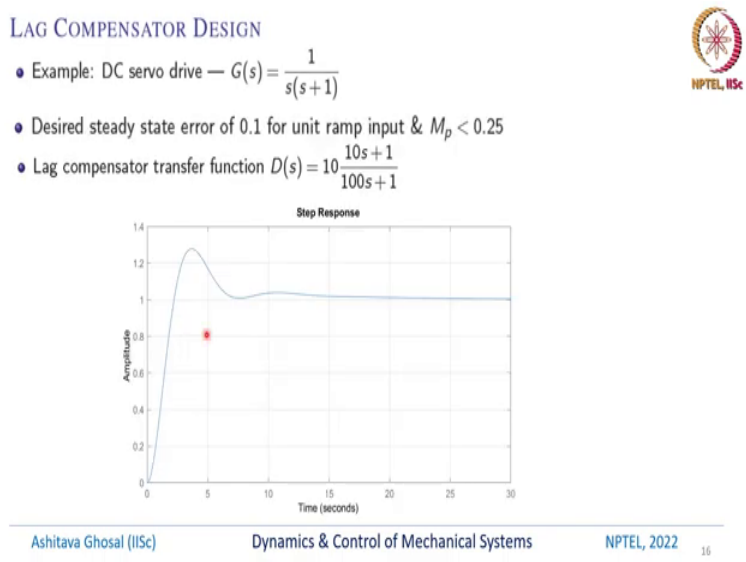
We can now plot the root locus with this lag compensator which is D (s) is 10 (10 s + 1)/ (100 s + 1) for this $G(s)$ and then you can see that the root locus looks like this. So, it is clearly a different root locus. So, we have originally s = 0 and s = –1 and then we have introduced one zero and one pole, and then the root locus now looks completely different. So, now we can check whether it again meets our requirements of steady state error and peak overshoot.

(Refer Slide Time: 15:16)



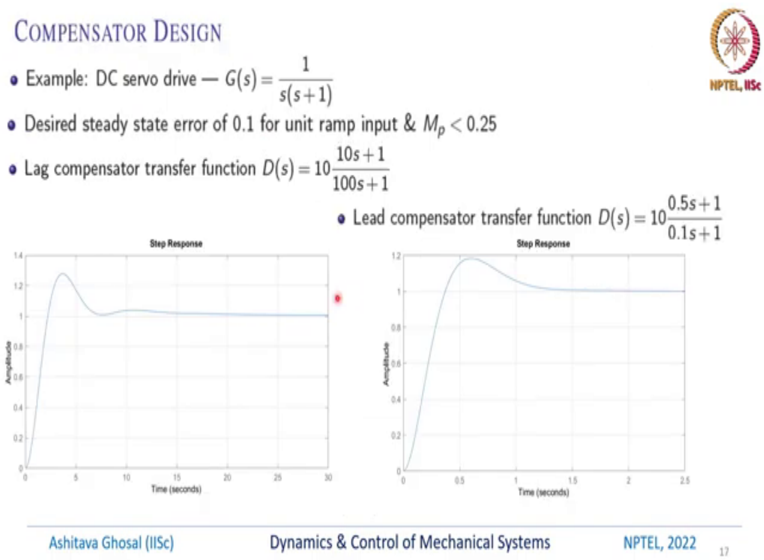
First let us look at the Bode diagram. So, you can see that the gain margin is infinity and Pm = 45.2 degrees at 0.71 radians per second for this. This is looks like it is acceptable. So, we want the phase margin more than 45 degrees.

(Refer Slide Time: 15:40)



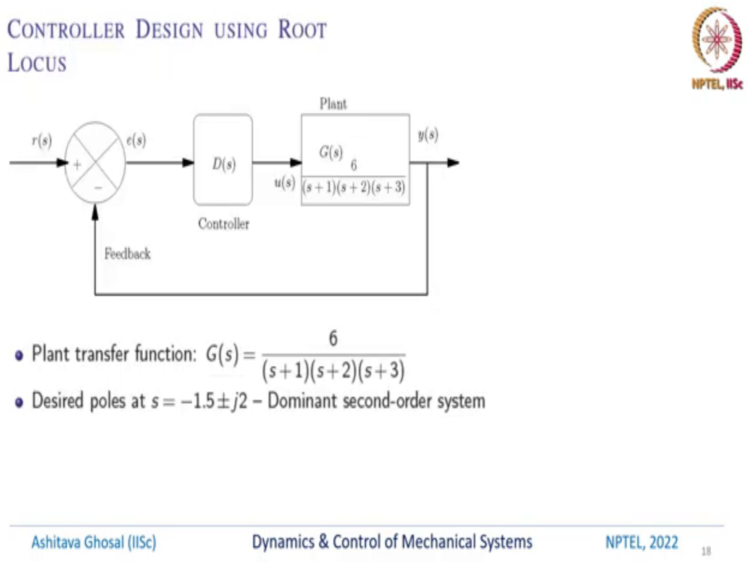
To eventually and finally check whether it meets our requirement, we give this system with this controller with this lag compensator a step input. So, if you give a step input here, the output looks like this. This is that output curve and again you can see it is less than 1.25. You have to believe me - this is less than 1.25. The peak as well as the steady state error is less or less than 0.1. However, the plot looks slightly different than with the lead compensator.

(Refer Slide Time: 16:20)



And I am showing you both side by side. So, for the lead compensator. This is what the plot was the lead compensator was 10 (0.5 s + 1)/ (0.1 s + 1) whereas the lag compensator is 10 (10 s + 1)/ (100 s + 1). So, in this case there is a little bit of oscillation here. The plot looks different but both of them meet the requirement of a steady state error of 0.1 to unit ramp input and peak overshoot less than 0.25.

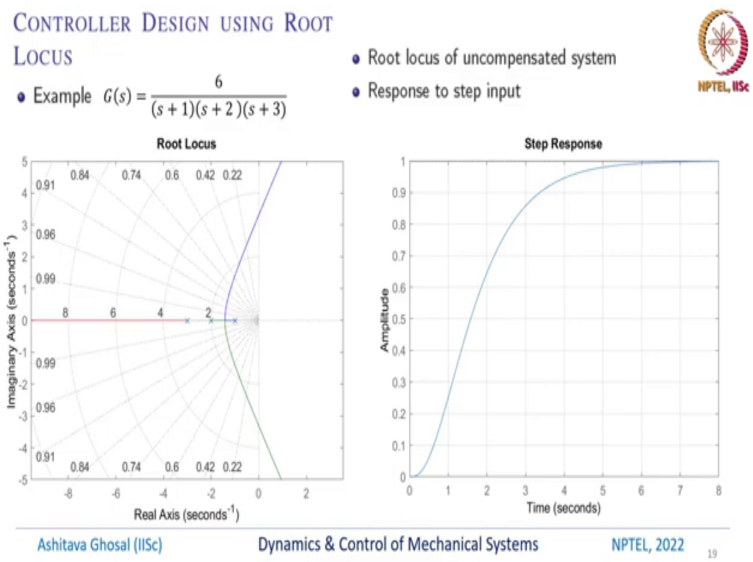
(Refer Slide Time: 17:02)



Let us now, look at a slightly more complicated system. So, what we have is a plant whose transfer function is given by 6 / (s + 1) (s + 2) (s + 3). If you go back and see the state space approach which we had studied earlier this as well this was s of the plans which we had discussed as an example. Now, we have this plant, the input is u (s), output is y(s) and we have a controller.

So, what the goal in this example is that we want the desired closed loop poles to be at s = –1.5 ± 2j. So, this is the dominant second order system which we want. So, we want to design a controller whose dominant second order system or dominant second order poles are at this place –1.5 ± 2j.

(Refer Slide Time: 18:08)

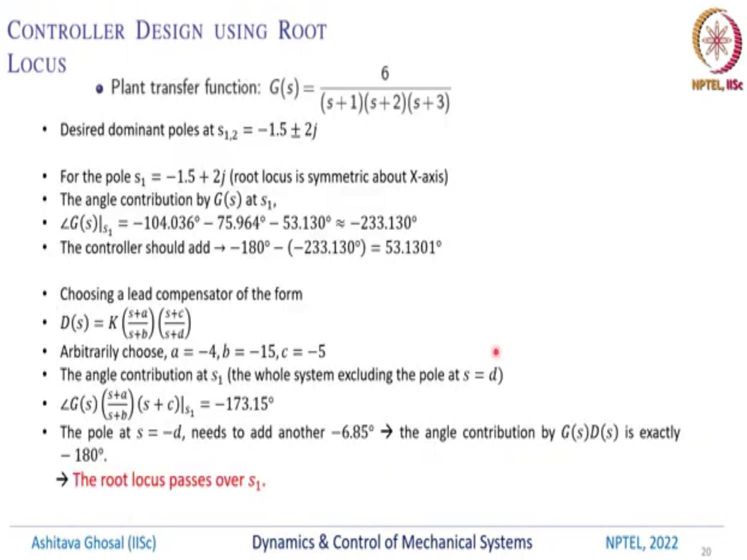


We can plot the root locus of this system, of the uncompensated system. So, we have just some gains which are increasing and then we have three poles at –1, –2, and –3. So, as the gain changes, the root locus goes off to like this. You can see one branch goes off to the right half plane and there is one branch which goes up to –∞. So, this system can be unstable for some values of gains.

So, if the gain is chosen such that it crosses this imaginary line here then it is unstable. In this is a plot which is derived from Matlab. It shows lots of things, it shows what is the root locus. It also shows you what are the damping values, and the circles are the natural frequencies. This point here has natural frequency of 6 and damping of 0.74. Not so, important but Matlab is a powerful tool which gives you lots of information.

We can also find, what is the response of this system to a step input. And the response look good that it goes off to settles at 1. It looks like with 0 error. It will be stable because it is like e^{-1t} and e^{-2t} and sort of like e^{-3t}. So, all the poles are in the left, half plane, hence it will be stable.

(Refer Slide Time: 18:08)



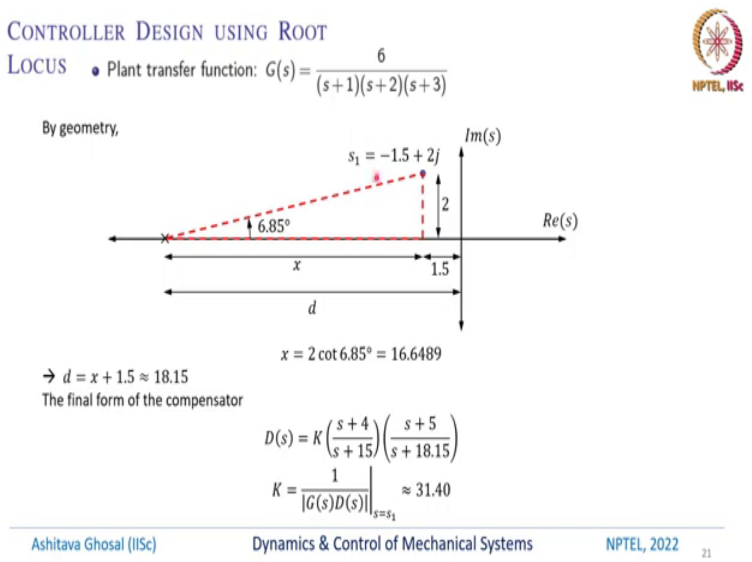
So, what we want is to design a controller whose desired dominant poles are at –1.5 ± 2. So, how do we do go about doing it? So, we pick one of the poles which is –1.5 + 2j. Remember the root locus is symmetric about the real axis or about the X axis. The angle contribution at this point, at s_1, we can find the angle from the different poles. These are –104.036, then this is –75.96 and –53.13. So, the total angle is –233.

So, we want at s_1 to be stable to meet the requirements we want. We should be adding a controller -- should be adding a phase of 53.13 because it should be –180 degrees. The angle at any point in the s plane for this transfer function, for the root locus to go through that point, the angle should satisfy that 180 degree criteria.

So, in order to achieve this additional phase angle, let us choose a lead compensator of the form $K (s + a)/ (s + b)\ (s + c)/ (s + d)$. So, note that this is slightly different from what we did earlier in the previous case, it was like $(s + a)/ (s + b)$ into K so that also works. However, it turns out that if you have a lead compensator, in which there are two of these, then the performance is better. You can try it out yourself with these two only or this four of them. Now we are adding two zeros and two poles. Then we arbitrarily choose a as –4, b as –15 and c as –5. You can again play around with these numbers, but it turns out that this is a good set of numbers to start with. Now the angle contribution at s_1 which is this point –1.5 + 2j and from everywhere except from $s = d$, except the pole, at $s = – d$. So, the angle contribution is the angle contribution of this term plus this term evaluated as s_1 and you will see this is –173.15.

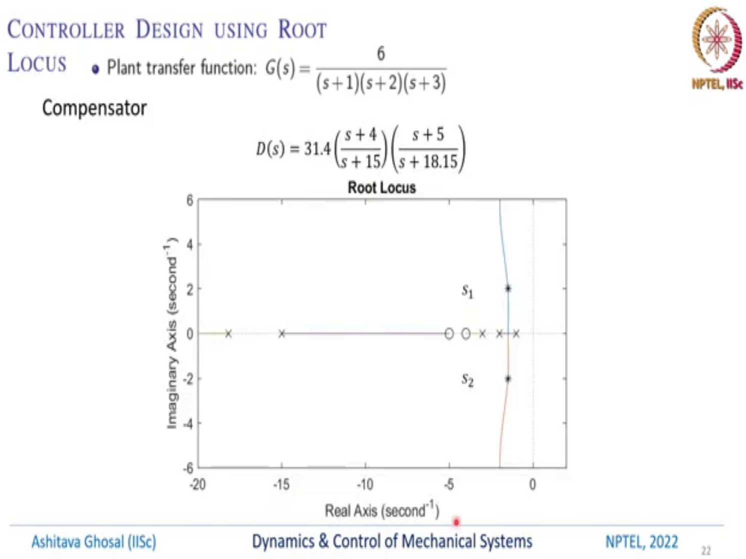
So, for the pole at $s = – d$, we need to add a smaller angle which is –6.85. Then it will make it 180 degrees. So then, the angle contribution is exactly –180 degrees and we then see that the root locus will pass through s_1. It will also pass through the conjugate because the root locus is always symmetric. How do I add this contribution of –6.85? By addition of a pole.

(Refer Slide Time: 23:22)



This can be done by geometry. So, we have s_1 is –1.5 + 2j. So, the imaginary distance is 2 and let us pick a pole at some point d which is here, and we want this angle to be 6.85. So, what does this get? That x = 2 cot 6.85 degrees, because this is 2 and this is 6.85 degrees, and we want to find out this distance. Where is the pole? So, initially it is at 1.5 and then we want to go some further to the left to get this 6.85 degrees. So, you can calculate x and we will see that x is 16.65 approximately. So, this distance d is x + 1.5 because originally itself it was at 1.5. So, this is like 18.5. So, the final form of the compensator is, remember we have chosen this as 4, 5, 15, so, this is s + 18.5. And the gain for this is 31.40. So, now we have a design in which the lead compensator has two zeros and two poles and gain value which is 31.4.

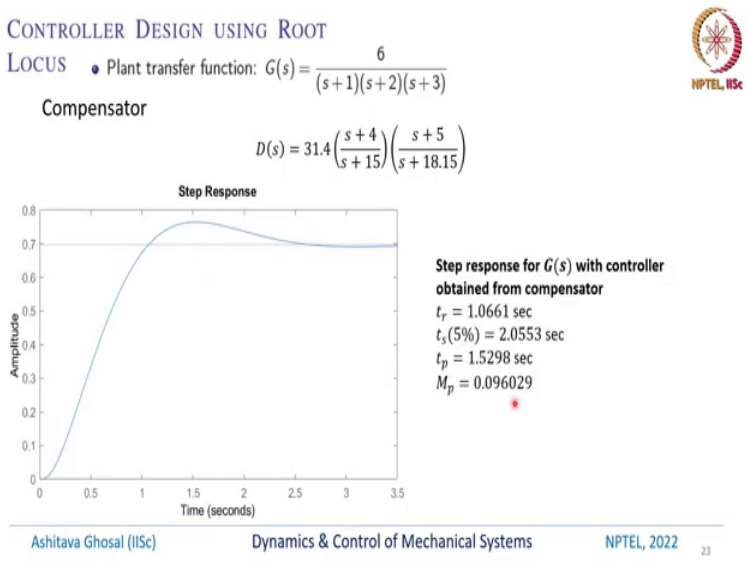
(Refer Slide Time: 25:46)



So, the controller transfer function or the compensator transfer function is 31.4(s + 4)/ (**s** + 15) into (s + 5)/ (s + 18.15). And we can again plot the root locus for this plant and then you can see that the root locus is now much more complicated. We have –1, –2, –3, these were the open loop poles and then we have added two zeros and two poles. And then, if you plot the root locus, you can see it is going through exactly the two points which we want - which is –1.5 ± 2j.

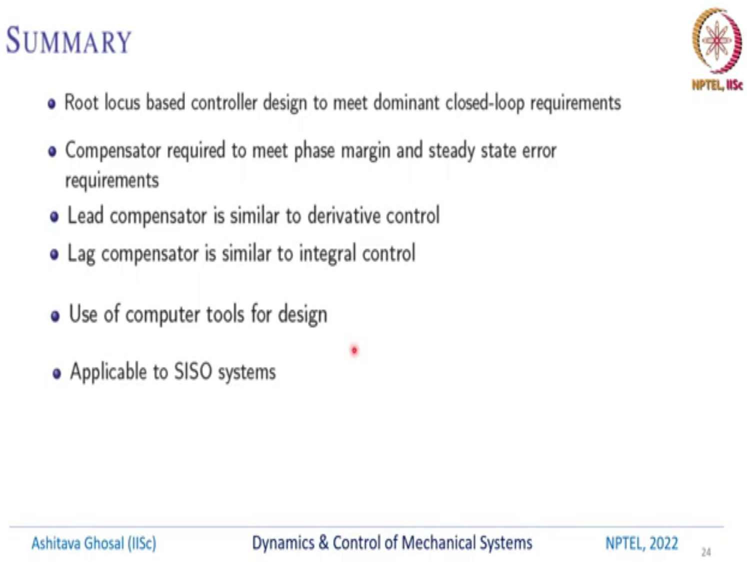
So, this compensator has now achieved these two dominant poles. Why is it dominant? Because all other poles are much to the left. So, one branch is going this way, one branch is going this way so and this other branch is going this way. So, the two dominant poles are these two and you can clearly see from this figure.

(Refer Slide Time: 26:00)



We can then look at what is the step response of this system? So, $G(s)$ is given by this. The controller or the compensator transfer function is given by whatever I showed you last time and if you give a step input, the output looks like this. So, here again there is some small overshoot but and there is some small steady state error, but this is what the dominant second order system is supposed to give. This sort of completes the design of a compensator for this third order system.

(Refer Slide Time: 26:40)



In summary, the root locus based controller designed to meet dominant closed loop requirements can be achieved. We need to change the shape of the root locus by adding a lead or a lag compensator, by basically adding zeros and poles, and this compensator is required to meet the phase margin and the steady state error requirements. The lead compensator is sort of similar to a derivative control. The lag compensator is sort of similar to an integral control.

They are not exactly derivative and integral control, but they sort of function like that and then we can play around with the location of the poles and zeros to meet the requirements. And I have showed you two kinds of requirements in one which is the steady state error, and the peak overshoot was given and then for another example, I said that we want to achieve these two dominant second order poles. And in both cases I showed you how to design a compensator which will achieve those things. Most of these things can be also done using computer tools. Nowadays, most design can be done using very sophisticated computer tools and again Matlab provides these Toolboxes for control system design. This root locus based controller design or using lead and lag compensator, is again applicable to SISO systems.