Dynamics and Control of Mechanical Systems
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Lecture – 31
Stability of Dynamical Systems

Welcome to these NPTEL lectures on Dynamics and Control of Mechanical Systems. In this week we have been looking at Design of Controllers my name is Ashitava Ghosal. I am a Professor at the Department of Mechanical Engineering, Centre for Product Design and Manufacturing and also in the Robert Bosch Centre for Cyber Physical Systems at the Indian Institute of Science Bangalore.

In the last two lectures we had looked at the design of a PID controller how to find the gains – P, I and D gains. In the second lecture this week we had looked at how to design a controller using root locus. In this lecture we will look at design of controllers using state space design.

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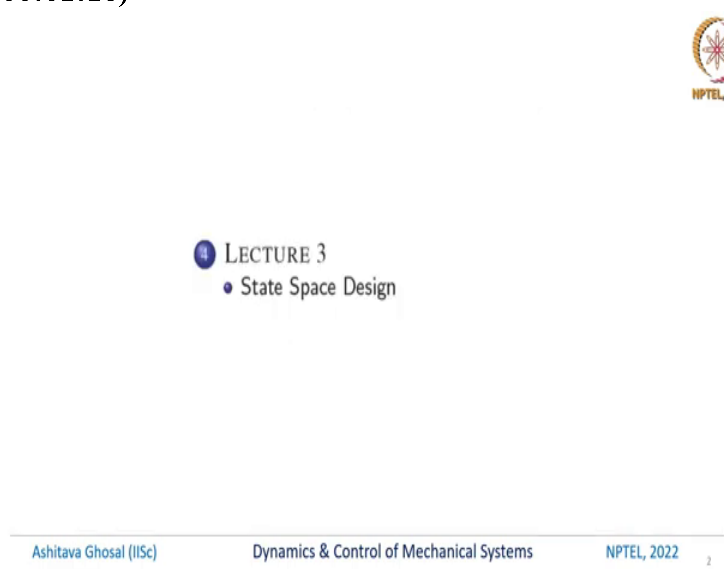

In this lecture we will look at the design of controllers for mechanical systems using the state space formulation.

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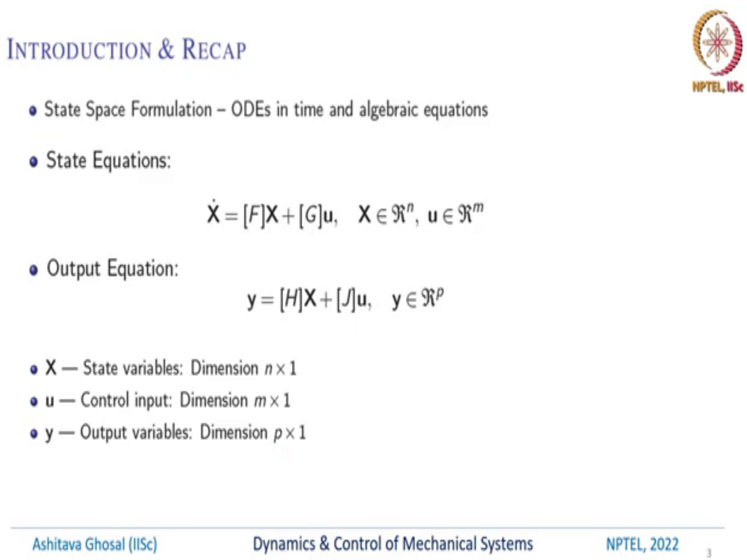

Before we start on the topic of design, let us quickly recap what we know about state space and what exactly is the state space formulation. The state space formulation is basically the equations of motion linearized and then we have ordinary differential equations in time domain. We also have algebraic equations which relate the output to the states. So, the state equations are given by $\dot{X} = [F]\mathbf{X} + [G]\mathbf{u}$. X are the state variables they could be n of them, **u** are the inputs to this plant and they could be m of them. And, we also have an output equation which is $\boldsymbol{y} = [H]\mathbf{X} + [J]\mathbf{u}$. $[H]$ is a constant matrix, $[J]$ is also a constant matrix, $[F]$ and $[G]$ are also constant matrices obtained from linearization of the equations of motion about an operating point. As I said the state variables **X** have dimension n x 1, the control inputs to the plant have dimension m x 1 and the output variables have dimension p x 1.

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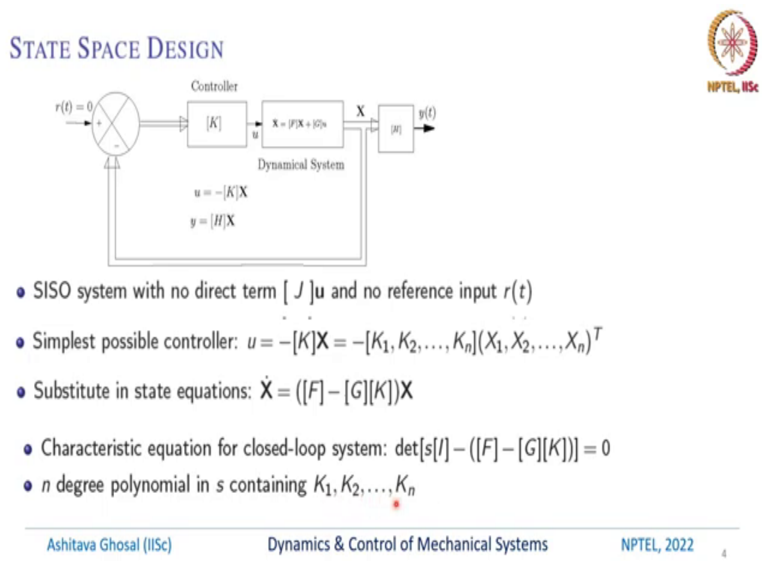

In a block diagram form, what we have is this dynamical system which is given by $\dot{X} = [F]\mathbf{X} + [G]\mathbf{u}$. The input is u and u could be in from a controller and one of the simplest controller is when u is given by $u = -[K]\mathbf{X}$, where, [K] is the constant matrix. The input to the controller is typically the error which is $r(t) - y(t)$, however, in the state space formulation for the moment, we will assume that this $r(t)$ is 0.

We also do not have the direct term $[J]\mathbf{u}$. So, you can see y is just $[H]\mathbf{X}$. There was a $y = [H]\mathbf{X} + [J]u$ so, we are not going to consider the direct term. And as I said, there is no reference input to start with -- later on we will introduce the reference input in the state space design. The simplest possible controller could be $u = -[K]\mathbf{X}$, X's are the state variables. So, $[K]$ is a matrix of K_1, K_2 all the way till K_n.

We also remember that these are linear systems. These are single input, single output systems. So, the reference trajectory when we introduce will be 1 dimension. The output variable y is also 1 dimension and u is also of 1 x 1. So, hence this $[K]$ is a 1 x n matrix into $\mathbf{X}$ which is n x 1. So, u will be a single input to the dynamical system. If you substitute $u = -[K]\mathbf{X}$ in the equation of the dynamical system which is $\dot{X} = [F]\mathbf{X} + [G]u$, we will get $\dot{X} = ([F] - [G][K])\mathbf{X}$.

The characteristic polynomial for the closed loop system is determinant of $[s[I] - ([F] - [G][K])]$. This is the n^{th} degree polynomial in s containing K_1, K_2, all the way through till K_n.

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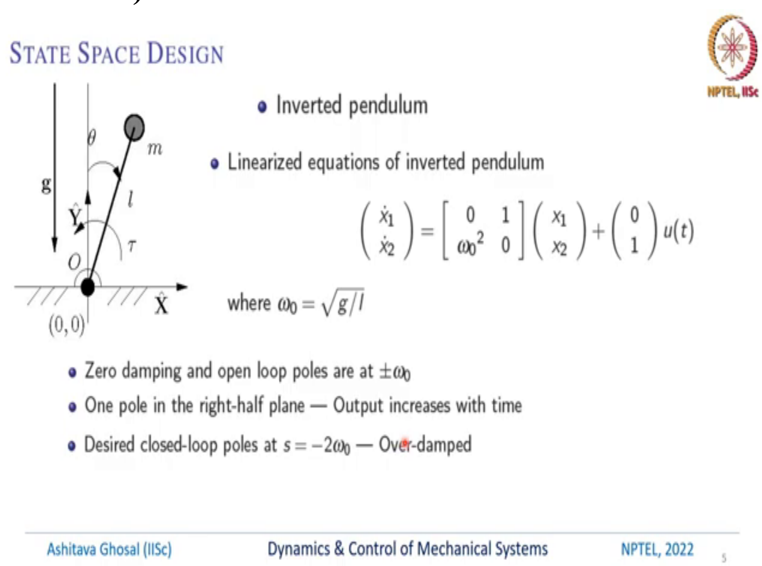

Let us take an example to see each of these terms more clearly. So, we have this inverted pendulum. This is a pendulum which is opposite to this gravity vector it is making an angle θ from the vertical we can derive the equations of motion. We have derived it earlier and then we can linearize about θ = 0. So then, what you will get is $\dot{X}_1 = \dot{X}_2$ and $\dot{X}_2 = \omega_0^2 X_1 + u(t)$.

We have assumed that there is an input which is like a torque and this ω_0 is $\sqrt{g/l}$. These equations were derived earlier using the Lagrangian formulation, where we found the kinetic energy, the potential energy and then we did some differentiation and then we could derive the nonlinear equations of this inverted pendulum. If you write it in this form, linearized about θ = 0, where θ is measured from the vertical we will get these state equations.

In these equations there is no damping and you can see that the characteristic polynomial of this $[F]$ term, so, this is $[F]\boldsymbol{X} + [G]u$. The eigen values of this $[F]$ matrix are $\pm\,\omega_0$ and recall, I have shown you, that the eigenvalues of $[F]$ are also the open loop poles. In this case the poles are at $\pm\,\omega_0$ -- one pole is in the right-half plane and hence it is like an unstable system. If you give some input, the output will continuously increase with time.

Let us say now that we want to design a controller such that the desired closed loop poles are at $s = -2\ \omega_0$. So, basically, in the s plane, we had $+\ \omega_0$, $-\ \omega_0$ and the controller poles should be in at $-2\ \omega_0$ -- there are two of them.

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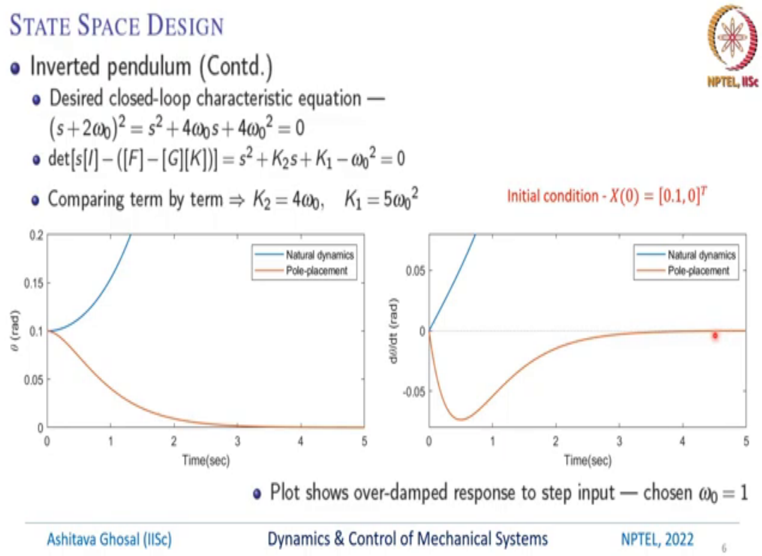

So, continuing so, the desired closed loop characteristic polynomial is $(s + 2\ \omega_0)^2$ and we can expand this as $s^2 + 4\ \omega_0\ s + 4\ \omega_0^{\ 2} = 0$. The determinant of the closed loop system or the characteristic polynomial of the closed loop system is det (s [I] – ([F] – [G][K])) and remember what is [F]? [F] is [0 1; ω_0^2 0] and [K] is [K_1, K_2] and [G] is [0 1].

So, when you substitute all that and we compute the determinant, you will get $s^2 + K_2\ s + K_1 - \omega_0^{\ 2} = 0$. If you compare these two equations, this is the desired closed loop characteristic equation and this is the characteristic equation with K_1 and K_2. What you can see is K_2 is 4 ω_0 – so, K_2 is 4 ω_0 and K_1 is 5 $\omega_0^{\ 2}$.

We have a system of equations now, where $\dot{X} = [F]\mathbf{X} + [G]u$ and $u = -[K]\mathbf{X}$. So, we can substitute all these things and solve the equations using Matlab. So, we have to do it numerically. We can also probably do it symbolically because it is a very simple equation but let us do it using Matlab. So, if you give some initial condition X (0) as 0.1 and 0 -- 0.1 means the pendulum is slightly perturbed from the vertical.

So, this remember X_1 is θ and X_2 is $\dot{\theta}$ -- they are the state variables. So, if you just give a small perturbation, the normal natural dynamics of the system is that θ will increase. That is obvious. So, this is a pendulum which is inverted and if you just perturb it little bit from the vertical the angle θ which is measured from the vertical, will keep on increasing and this blue curve shows that perturbed θ -- it increases with time.

However, when we use $u = -[K]X$, where K_1 is $5\ \omega_0{}^2$, K_2 is $4\ \omega_0$ and in these examples we have chosen $\omega_0 = 1$. So then you can see that the plot of θ goes back to 0. So, we have an unstable system, so, small perturbations will increase with time but when we use $u = -[K]\mathbf{X}$, we use a user controller which is designed such that the closed loop poles are at $-2\ \omega_0$ -- so on the left half plane -- then the output θ goes to 0. We can also see what is $(d\ \theta/\ d\ t)$ -- So, the $(d\ \theta/\ d\ t)$ will increase, whereas after you use the controller, it will decrease for a while and then it goes back to 0. So, $\dot{\theta}$ goes to 0 means the pendulum is no longer moving and θ goes to also 0.

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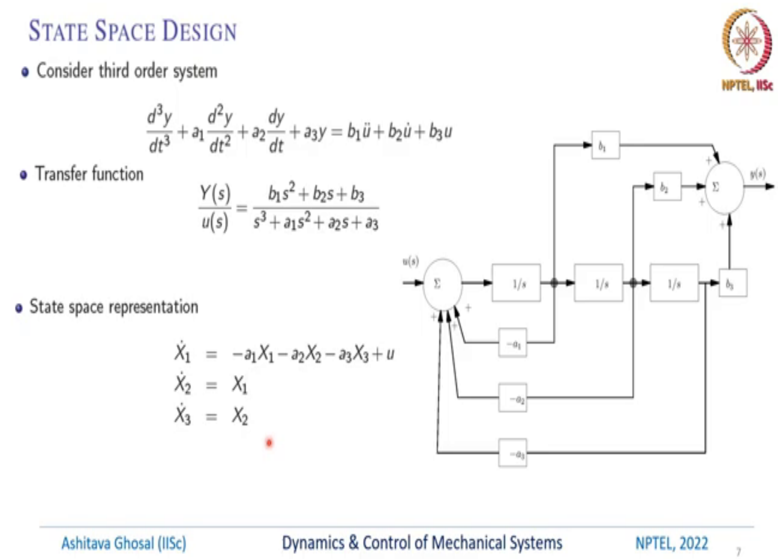

So, continuing with state space design let us consider a little bit more complex system, a third order system which is given by this $\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y$ and the right-hand side now has both u, $\dot{u}$ and $\ddot{u}$. So, it is $b_1\ddot{u} + b_2\dot{u} + b_3 u$. We can obtain the transfer function after taking the Laplace transform of this ordinary differential equation.

So, $Y(s)/u(s)$ will be given by $(b_1 s^2 + b_2 s + b_3)/(s^3 + a_1 s^2 + a_2 s + a_3)$. This can be represented in this block diagram. So, we have an input u, then, this is integrated, once more integrated, once more integrated and then we at this place, you have u. So, $b_3 u$ then $b_2 \dot{u}$ and $b_1 \ddot{u}$, whereas if you go back from this side, you will see that you will have $a_1 \ddot{y}$ then $a_2 \dot{y}$ and $a_3 y$. That will be given by. So, this block diagram represents basically the same transfer function. In the state space representation, we can also write $\dot{X}_1$ is $-a_1 X_1 - a_2 X_2 - a_3 X_3 +. u$ and $\dot{X}_2$ is X_1 and $\dot{X}_3$ is X_2. This is another way of writing this differential equation.

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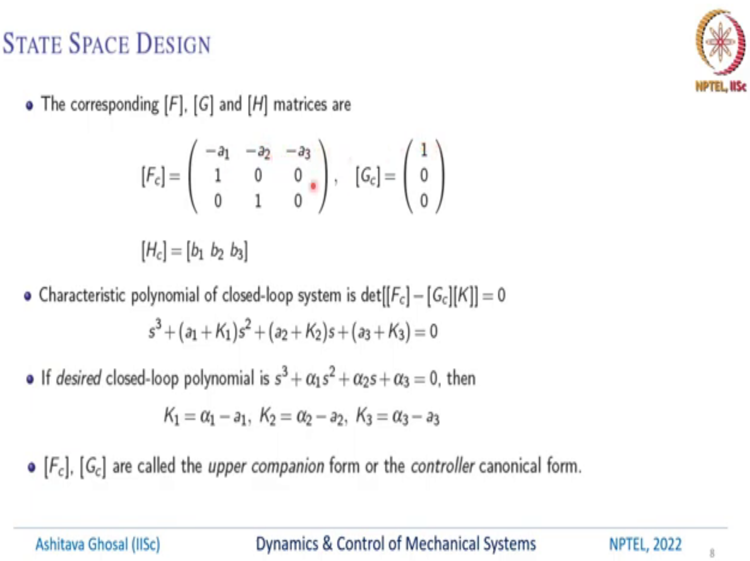

So, the corresponding $[F]$, $[G]$ and $[H]$ matrices can be written in this form. So, there is a subscript c and we will see why I am using the subscript c in a short while. This $[F_c]$ is $[-a_1 \; -a_2 \; -a_3]$, the first row. The second row is [1 0 0], third row is [0 1 0] and the $[Gc]$ matrix is (1 0 0) and the $[Hc]$ matrix is $[b_1 \; b_2 \; b_3]$. Now, if you want to find the characteristic polynomial of the closed loop system with u is $-[G_c][K_c]$. So that is determinant of $[[F_c] - [G_c][K]] = 0$. You will get this cubic polynomial which is s^3, the coefficient of s^2 is $(a_1 + K_1)$, s is $(a_2 + K_2)$ and this is $(a_3 + K_3)$. So, $[K]$ is $[K_1 \; K_2 \; K_3]$ a row vector. So, if the desired polynomial is, let us say another cubic which is $s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = 0$. Then we can just compare these two cubic equations, and we can clearly see very easily see that K_1 is $\alpha_1 - a_1$, $a_2 + K_2$ will be α_2 which is K_2 is $\alpha_2 - a_2$ and K_3 is $\alpha_3 - a_3$. So, hence, if the matrix $[F]$ and the matrix $[G]$ and the matrix $[H]$ can be written in this form, these are very easily used to obtained K_1, K_2, K_3. So, a matrix [F] written in this form, which is first row, [$- a_1\ -a_2\ -a_3$] and so on. These are called as the upper companion form or also the controller canonical form.

So, this c basically means that [F] is in the controller canonical form, [G_c] means it is in the controller canonical form. So, whenever you have a controller canonical form, this [G_c] is (1 0 0), and [F_c] is in this form [$- a_1\ -a_2\ -a_3$], second row is [1 0 0] and [0 1 0]. I have purposely taken this simple cubic example because very easily we can see what is the controller canonical form or the controller [G] form - the [G] in the controller canonical form. If you have n-dimensional, if you have a nth degree ordinary differential equation, you can see this will become $- a_1\ -a_2$ all the way till $-a_n$ and similarly this is [1 0 0] all the way, and then you can keep on shifting.

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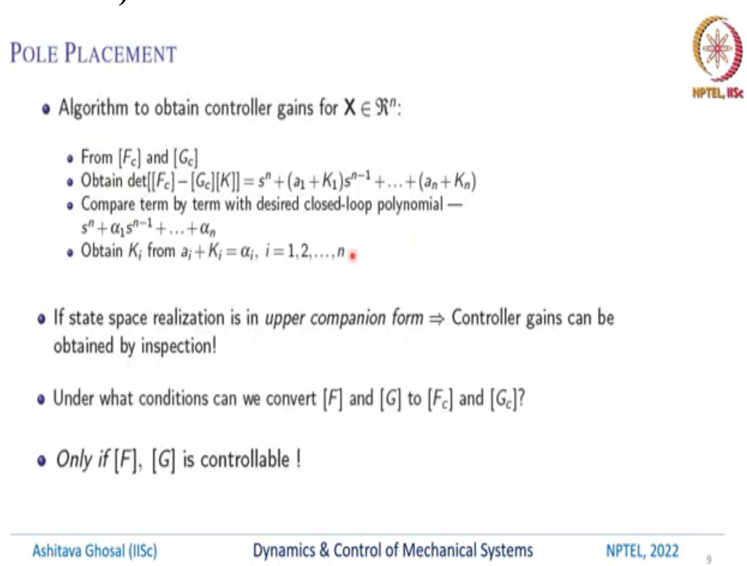

The algorithm to obtain controller gains when it is n-dimensional is first form the [F_c] and [G_c] matrix. So, find [F] and [G] in the controller canonical form then obtain determinant of [[F_c] – [G_c] [K]] which will be a nth degree polynomial of the form s^n and then $(a_1 + K_1)\ s^{n-1}$ and so on $(a_n + K_n)$. And then, if you are given a desired, closed loop polynomial of the form $s^n + \alpha_1\ s^{n-1} + \ldots + \alpha_n$, then we can just by inspection, obtain K_i as from this expression which is $a_i + K_i = \alpha_i$, for $i = 1$ through n. In summary, if the state space realization is in upper canonical form or in the controller canonical form, the controller gains can be obtained by inspection. So, the natural question is under what conditions we can convert a normal [F] and [G] to the controller, canonical form [F_c] and [G_c] and it turns out that this can be done only if the system given by [F], [G] is controllable.

Remember we had discussed something called controllability. We had discussed a matrix called Kalman matrix [Q_c], that the first column was [G], second column was [F] [G] and so on. So, if the rank of that matrix is n we said that the system given by $\dot{X} = [F]\mathbf{X} + [G]\mathbf{u}$ is controllable. And it turns out and this is very interesting and not very obvious why it should be, that I can convert and space state space realization which is $\dot{X} = [F]\mathbf{X} + [G]\mathbf{u}$ into $\dot{X} = [Fc]\mathbf{X} + [Gc]\mathbf{u}$, meaning in the controller canonical form, only if this [F] and [G] is controllable.

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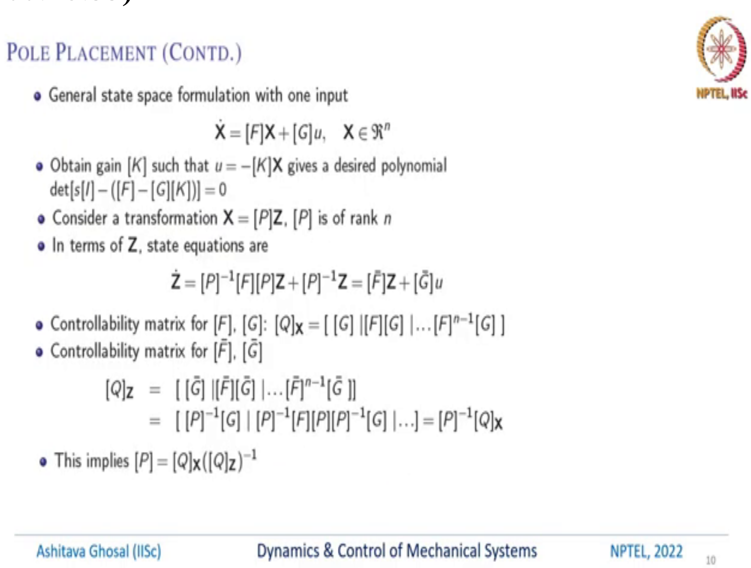

Let us continue little bit. The general state space formulation with one input can be written as $\dot{X}$ = [F]$\mathbf{X}$ + [G] u. Now, $\mathbf{X}$ is a n x 1 vector. We can obtain gain [K] such that $u = -[K]\,\mathbf{X}$ gives a desired polynomial which is det [$s[I]$ – ([F] – [G][K]] = 0. Now let us consider the transformation $\mathbf{X} = [P]\,\mathbf{Z}$. So, we are doing a linear transformation, from the state variables $\mathbf{X}$, we want to go to state variable $\mathbf{Z}$ by means of $\mathbf{X} = [P]\,\mathbf{Z}$ and this matrix [P] must be of rank n. There is no problem with of converting from $\mathbf{X}$ to $\mathbf{Z}$ or $\mathbf{Z}$ to $\mathbf{X}$, so $[P]^{-1}$ must exist. In terms of $\mathbf{Z}$

the state equations can be written, so, we can write $\dot{X}$ is $[P]\,\dot{Z}$. So, we can write $[P]\,\dot{Z} = [F]\,[P]Z$ and so on. And then we can pre-multiply by $[P]^{-1}$, we can write $\dot{Z} = [P]^{-1}[F][P]\,\mathbf{Z} + [P]^{-1}\mathbf{Z}$.

We have seen this earlier. So, when we first discussed state space, we discussed how we can use a matrix $[P]$ to diagonalize this $[F]$ matrix. Here we are not going to do diagonalization but basically, we are doing some transformation from **X** to **Z**. So, let us call this $[P]^{-1}[F][P]$ as some $[\overline{F}]\ \mathrm{Z} + [\overline{G}]\ u$.

The controllability matrix for $[F]$ and $[G]$ which is $[Q]_{\mathbf{X}}$. It is $[[G], [F][G], \ldots, [F]^{n-1}\,[G]]$, all these column vectors. The controllability matrix for $[\overline{F}]$, $[\overline{G}]$ can be similarly written as $[\ [\overline{G}]$, $[\overline{F}]\ [\overline{G}]$ and then $[\overline{F}]^{n-1}\ [\overline{G}]\ ]$ and we can rewrite $[\overline{G}]$. So, what is $[\overline{G}]$? It is $[P]^{-1}[G]$. So, $[\overline{G}]$ is $[P]^{-1}[G]$, $[\overline{F}]$ is $[P]^{-1}[F][P]$ and so on. So, hence we can rewrite all these terms $[\overline{F}]$ and $[\overline{G}]$ in terms of original $[G]$ and $[F]$ and it turns out that you will get terms like $[P]^{-1}[F][P][P]^{-1}[G]$. $[P]\,[P]^{-1}$ is identity, so, we will left with $[P]^{-1}[F][G]$. So, we can take out $[P]^{-1}$ outside because every term will have a $[P]^{-1}$, we will have $[P]^{-1}[G]$, then $[P]^{-1}[F][G]$ and so on. So, it will get $[P]^{-1}[Q]_{\mathbf{X}}$. This implies that $[P]$ is nothing but $[Q]_{\mathbf{X}}\,[Q]_{\mathbf{Z}}^{-1}$ -- you can see, if I write take it on this sides, $[P]\,[Q]_{\mathbf{Z}}$ will be equal to $[Q]_{\mathbf{X}}$ and then $[P]$ will be $[Q]_{\mathbf{X}}\,[Q]_{\mathbf{Z}}^{-1}$.

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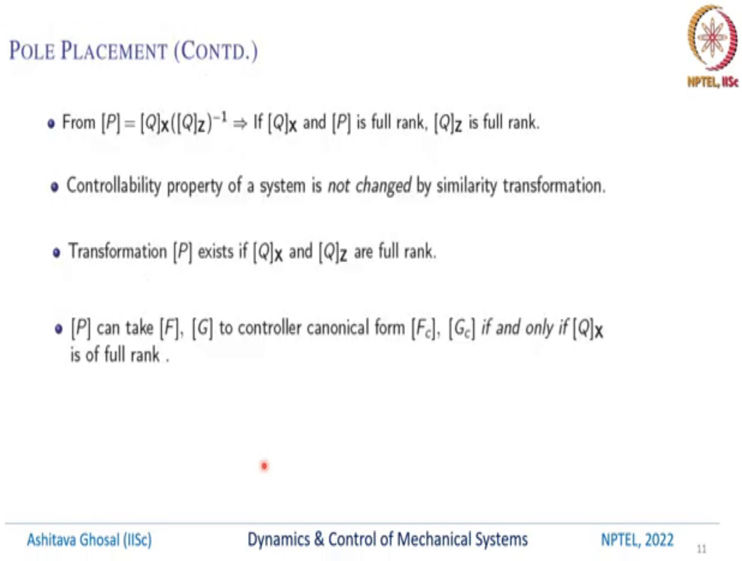

So, from $[P] = [Q]_X [Q]_Z^{-1}$, we can conclude that if $[Q]_X$ and $[P]$ is a full rank, which means rank *n,* then $[Q]_Z$ is also full rank. So, we remember, we started with $[P]$ of full rank, if $[Q]_X$ is also of rank n or then $[Q]_Z$ will also be a full rank n. So, the controllability property of a system is not changed by this transformation -- similarity transformation, which is $\mathbf{X} = [P]\ \mathbf{Z}$. So, the transformation $[P]$ exists only if $[Q]_X$ and $[Q]_Z$ are full rank. $[P] = [Q]_X [Q]_Z^{-1}$, if there is an inverse of $[Q]_Z$ or there is the inverse of $[Q]_X$ only then $[P]$ will exist. Hence, we can conclude that $[P]$ can take $[F]$ and $[G]$ to controller canonical form $[F_c]$ and $[G_c]$ if and only if $[Q]_X$ is a full rank. So, I hope this is clear that we are going to transform from **X** to **Z** using a matrix $[P]$ and then I showed you that the controllability property does not change. So, in terms of $[Q]_Z$ we can check that the transformation $[P]$ exists if $[Q]_X$ and $[Q]_Z$. and hence $[P]$ can take $[F]$ and $[G]$ to controller canonical form only if or if and only if $[Q]_X$ is a full rank. And what is $[Q]_X$ -- it is the controllability matrix with $[F]$ and [G].

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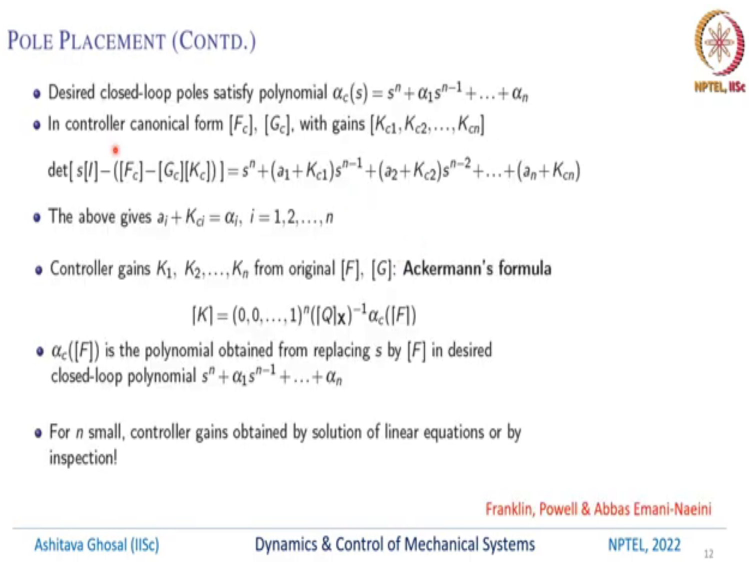

Let us continue with pole placement. Let us say that the desired closed loop polynomial satisfy a polynomial which is $s^n + \alpha_1 s^{n-1}$ all the way to α_n and we are going to denote this polynomial as $\alpha_c(s)$. So, this is the closed loop desired polynomial. In controller canonical form using $[F_c]$ and $[G_c]$ with gains Kc_1, Kc_2, all the way till Kc_n, the det $[s[I] – ([F_c] – [G_c][K_c]]$ can be written as $s^n + (a_1 + K_{c1}) s^{n-1}$ and so on. This we have seen earlier, at least for the cubic example I worked out all the steps. Here we are doing it -- we are extending it to n-dimensions or sorry n when **X** is n x 1. So, the above gives that $a_i + K_{ci} = \alpha_i$. So, you can easily find the gains if it is given in the controller canonical form. How do we find the controller gains when it is given in the $[F]$ and $[G]$ -- not in the canonical form?

Let us call this K_1, K_2 all the way till K_n. So, these are for the original, $[F]$ and $[G]$ matrix. This is a very well, known formula in linear controls. This is called Ackerman's formula. I am not going to go into the proof of this formula, and you can see in many textbooks. For example, in Franklin, Powell and Abbasi Emani-Naeni, Feedback Control of Dynamical Systems.

You can find $[K]$, the gains from this expression -- which is 0 0, all the way till n. This is like a column vector with the last element as 1 x $[Q]_X^{-1}$ $[Q]_X$ and then $\alpha_c([F])$. What is $\alpha_c([F])$?

$\alpha_c([F])$ is nothing but this closed loop polynomial α_c, where s is now replaced by $[F]$. There is this very famous theorem called Cayley-Hamilton theorem which says that the polynomial, this should also be satisfied, if you substitute [F] instead of s. So, all we need to do is s^n, we will write $[F]^n$, s^{n-1} is $[F]^{n-1}$. So, we can obtain this α_c with s substituted by [F], then we find the inverse of this controllability matrix and then we multiply or take the nth component of this and then we get [K]. This is a very well-known formula which is available in many textbooks, and I do not want to go into details.

So, for n small the controller gains can be obtained by solution of linear equations or by inspection. If you have a large n then we can go back and use this Ackermann's formula to obtain the controller gains.

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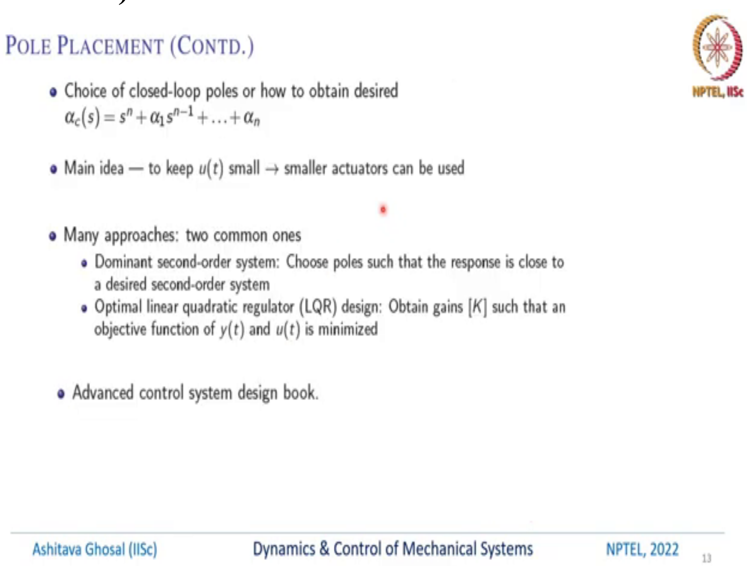

So, the natural question is how do we choose the closed loop poles? Or how do we obtain the desired polynomial α_c (**s**)? So, if you have a desired polynomial then I showed you how we can find α_c ([F]). Then we know $[Q]_{\mathbf{X}}^{-1}$ and then we take this component and then we can find [K] matrix. But what is this? The natural question is what is the desired closed loop polynomial and how do we obtain or how do we choose this desired closed loop polynomial?

The main idea is to keep u (t) small. What is u (t)? u is remember –[K] **X**. So, we want to keep the controller input or the input to the plant small. Why? Because then we can use smaller actuators? u has to be supplied by some actuator. In the case of the single link being rotated by a DC motor u was actually the voltage which we are applying and sending to the DC motor.

We would like to make sure or try to make sure that the voltage which you are applying is small. Hence, we can use the smaller motor.

There are many approaches to choosing this desired closed loop polynomial. The two most common ones are we can decide on what is called as a dominant second order system. So, we choose two poles near to the imaginary axis in the s plane such that the response is close to a desired second order system. So, if you have two poles close to the imaginary axis, we can easily see what is the settling time? What is the peak overshoot? What is the natural frequency? What is the damping? All those things are clearly defined for a second order system. But if you have a nth order system, you pick two poles or we say that my system should behave like these two dominant poles.

There is also another way which is called optimal linear quadratic regulator or LQR design. So, we obtain the gains $[K]$ such that an objective function of $y(t)$ and $u(t)$ is minimized. So, we can create one objective function then we can minimize that objective function of y and u and then we can obtain $[K]$. I am not going to go into the details of this optimal LQR design. If anybody is interested, they can look at some advanced control system design book.

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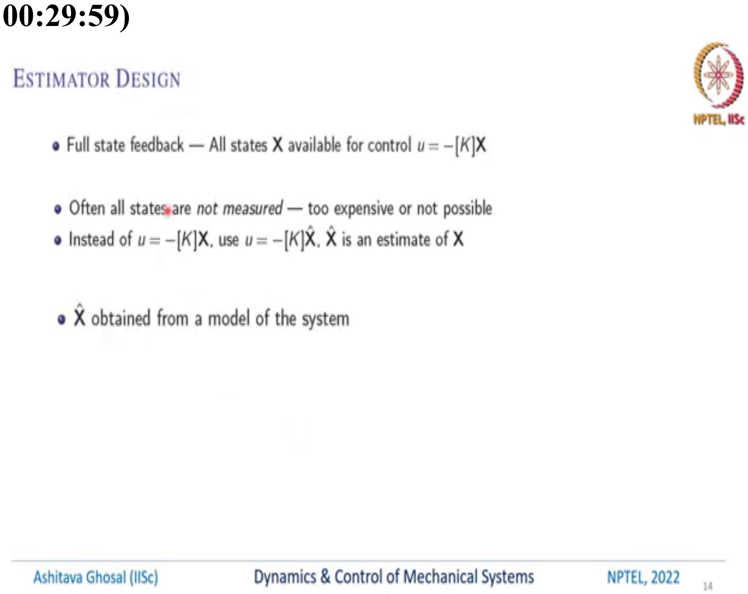

Let's now continue and we come to this very important topic called estimator design. So, till now we assume that all the state variables are available for control, this is also called full state feedback. So, basically, we have $u = -[K]\mathbf{X}$, so, we could measure all the state variables. There are n of them and we could find K_1 through K_n to do some desired closed loop polynomial. Often all states are not measured. Basically, it is too expensive or sometimes not even possible.

So, if you have a complicated plant, so, some of the states are not so easy to measure. So, instead of $u = -[K]\mathbf{X}$, we would like to use $u = -[K]\hat{X}$. So, $\hat{X}$ an estimate of X. We will measure only some of the X's and then we would like to obtain an estimate of all the **X**. So, this is called an estimator and how do we design a system which can estimate all the state variables from few measurements? So, one natural way is to obtain $\hat{X}$ from the model of a system.

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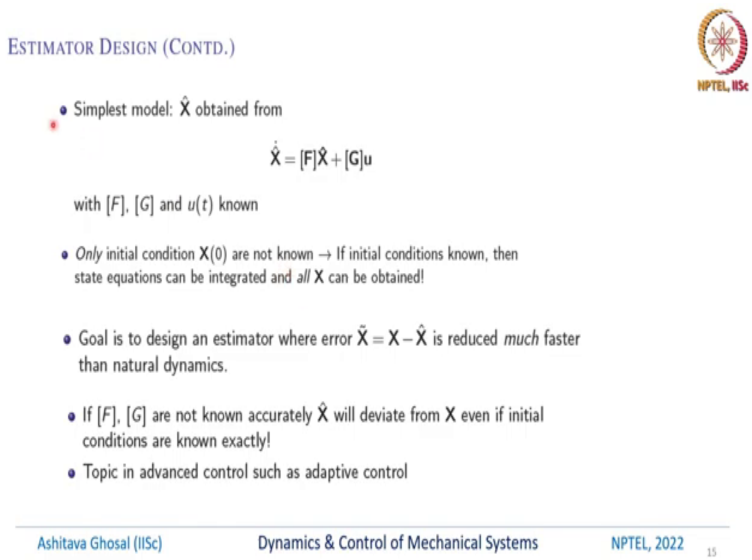

So, what do we mean by model of a system? The simplest possible model is X hat can be obtained by this differential equation which is $\dot{\hat{X}} = [F]\hat{X} + [G]u$. So, what is this assuming? That we know the plant $[F]$, $[G]$ and u exactly. So, basically then all what is unknown. The initial conditions of **X** (0) is not known. So, if the initial conditions were known exactly then the state equations $\dot{X} = [F]\mathbf{X} + [G]u$ can be in integrated and then all the **X**'s can be obtained. But in this estimator design in this part of the estimator design or the very simple estimator design which we are discussing, we do not know the initial conditions. We know everything else. We know what is exactly $[F]$, $[G]$ and $u(t)$. So, the goal is to design an estimator where the error which is $\mathbf{X} - \hat{X}$ is reduced much faster than the natural dynamics.

So, we will define a new variable $\tilde{X}$ which is $\mathbf{X} - \hat{X}$. We will design a system such that this $\tilde{X}$ goes to **0** very quickly which means basically **X** will become $\hat{X}$ or $\hat{X}$ will become **X**. So, we know **X**. If [*F*] and [*G*] are not known exactly or not known, accurately, $\hat{X}$ will deviate from **X** even if the initial conditions are known exactly because if this differential equation. If this [*F*] is different than the plant [*F*] or this [*G*] is different than the plant [*G*] then, of course $\hat{X}$ will deviate from **X**. This is a topic in advance control, such as adaptive control, and we will not discuss this in this lecture.

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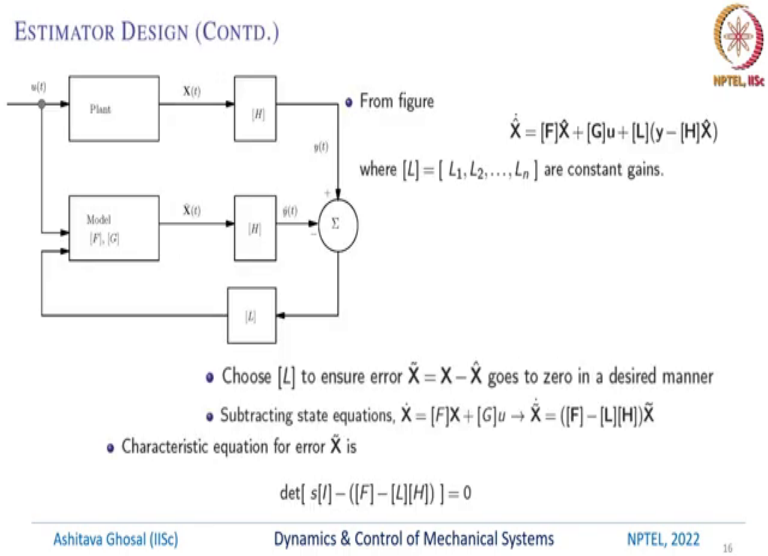

So, how does the estimator looks like? So, basically, what we have is *u*(*t*) coming into the plant the output of the plant is **X**(*t*). Then we have this transfer function or [*H*] matrix. So then [*H*]**X** will give me *y* and we are going to measure *y*. So, same *u* can be sent to a model which is [*F*] and [*G*] and we get an output which is $\hat{X}$ and then $\hat{X}$ into [*H*] is $\hat{y}$ and we can find the difference between *y* and $\hat{y}$. And then this error is sent back into the model so and through this block which is containing [*L*]. So, this [*L*] is called the estimator gains so, the equation from this figure is $\dot{\hat{X}}$ = [*F*]**X** + [*G*] *u* + [*L*] (y – [H] $\hat{X}$). So, y – [H] $\hat{X}$, [H] $\hat{X}$ is actually $\hat{y}$. So, $y - \hat{y}$ is this quantity here you multiply that by [*L*] and then you send it feed it back to this plant or to this model.

And what is the basic idea? We are going to choose $[L]$ to ensure that the difference between **X** and $\hat{X}$ goes to zero in a desired manner. So, this $\tilde{X}$ goes to zero in quickly, so that we have **X** if we know $\hat{X}$. So, if you look at these two expressions $\dot{X} = [F]\mathbf{X} + [G]\,u$ and $\dot{\hat{X}}$ is this, then if you subtract the state equations from here, what you will get $\dot{\tilde{X}} = ([\mathrm{F}] - [L][H])\,\tilde{X}$.

So, you can see that if I have subtracted $\dot{X} - \dot{\hat{X}}$ so, we will get $\tilde{X}$. So, the left-hand side is $\dot{\tilde{X}}$ and then we have $[F]\mathbf{X} - [F]\,\hat{X}$. So, this will be $[F]\,\tilde{X}$ and so on. This is the error equation or error in the estimate. So, what is the characteristic equation for the error $\tilde{X}$? That is standard by now, we are very familiar. It is det $[\, s[I] - ([F] - [L][H])] = 0$. The characteristic equation for $\tilde{X}$ is given by this determinant.

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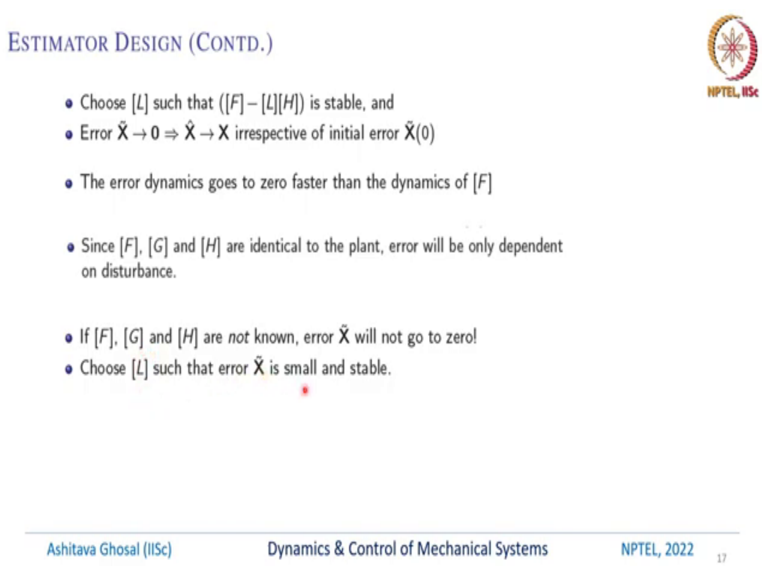

So, we can choose $[L]$ such that $([F] - [L][H])$ is stable so, basically $\tilde{X}$ does not go off to infinity. So, if $\tilde{X}$ goes to zero, $\tilde{X}$ will go to **X** and this is irrespective of the initial error $\tilde{X}$. So, the error dynamics needs to go to zero faster than the dynamics of $[F]$ and since $[F]$, $[G]$ and $[H]$ are identical to the actual plant, the error will be only dependent on the disturbance. If $[F]$, $[G]$ and

$[H]$ are not known, the error $\tilde{X}$ will not go to zero. But however, we can choose $[L]$ such that the error $\tilde{X}$ is small and stable. This is the best that we can do with this kind of estimator.

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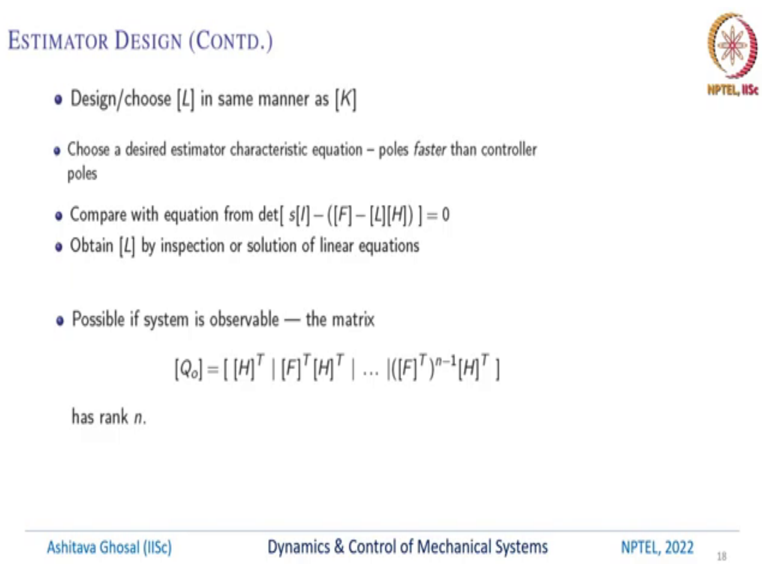

And how do we choose or design $[L]$? In exactly the same manner as we did for the controller gains $[K]$. So, we choose the desired estimator characteristic equation and the poles need to be faster than the controller poles because the controller requires **X**. So, $u = -[K]\mathbf{X}$ and we need to ensure that $\hat{X}$ becomes **X** before it can be used by the controller. And again just like the way we chose the controller gains.

We will compare the desired estimator characteristic equation with det $[s[I] - ([F] - [L][H]) = 0$. And then we can obtain $[L]$ by inspection or solution of linear equations. Exactly the same way as we did for controller gains, we can compare term by term and find out L_1, L_2, L_3 all the way to L_n and is this possible? Can we do this always? Again, the interesting answer is this is possible, only if the system is observable.

And again recall, what is the observability matrix? We have $[Q_O]$ which is $[H]^T$, the next term is $[F]^T\,[H]^T$ and all the way to $([F]^T)^{n-1}[H]^T$. So, the rank of this observability matrix must be n, and only then we can obtain this $[L]$ or we can easily see that we can obtain L_1, L_2 all the way till L_n.

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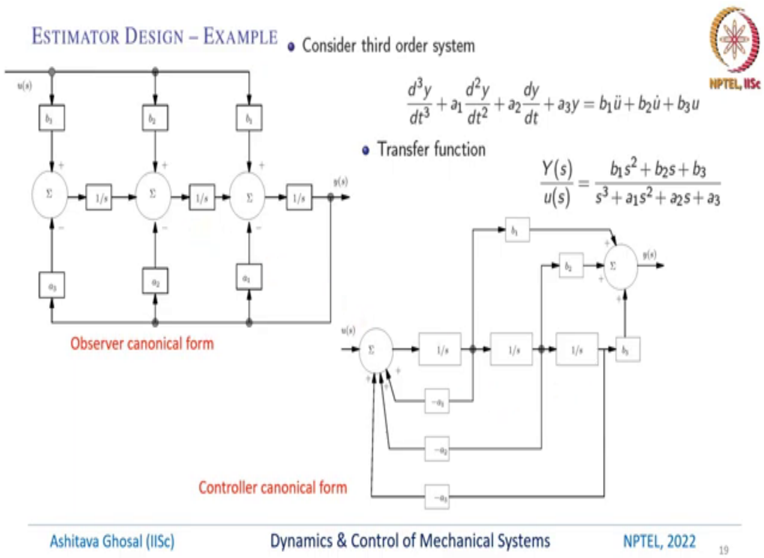

So, just let us look at the estimator design example. So, let us go back and consider the third order system that we had which is $d^3 y/ dt^3 + a_1 \ddot{y} + a_2 \dot{y} + a_3 y = b_1 \ddot{u} + b_2 \dot{u} + b_3 u$. And we had obtained this transfer function $Y(s)/ u(s)$ is $b_1 s^2 + b_2 s + b_3$ and denominator is $s^3 + a_1 s^2 + a_2 s + a_3$. So, in the controller canonical form, if you go back and see your notes or previous slides, the block diagram look like this.

So, we have an integrator, another one, another one and then we have b_1 then this is b_2 and then this is b_3 and then there are this feedback of $-a_1$, $-a_2$, $-a_3$. We can also derive another block diagram which looks like this. So, basically you have u and then $b_3 u$ so, $b_3 u$ then we have $u b_2$ and $u b_1$ but then you have to do all these integrations. And then you have to add a_3 coming from here, a_2 coming from here and then this is y.

So, this block diagram is what is called as observable canonical form. This block diagram is what is called as a controller canonical form. So, we can derive both these block diagrams again with suitable transformations -- in one case, we did that some $[Q]_X$ and so on. Here, we will do with the observability matrix, let us not go into that detail. But at least for this example, we can get one block diagram which looks like this and the previous controller canonical form block diagram look like this. So, there is some difference.

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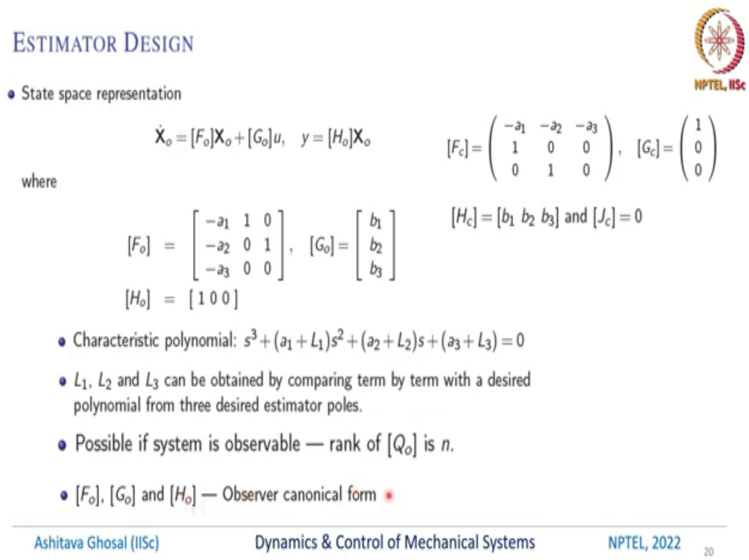

So, the state space representation using the observer, the new block diagram can be written in this form. This $\dot{X}$,and I am putting o for the observer canonical form, is $[F_o]$ $\mathbf{X}_o$ + $[G_o]$ u

and $y = [H_o]$ $[\mathbf{X}_o]$. **X** and $\mathbf{X}_o$ are different. In the controller canonical form, the $[F]$ matrix was $[-a_1\ -a_2\ -a_3,\ 1\ 0\ 0,\ 0\ 1\ 0]$ and $[G]$ was (1 0 0), $[H]$ was $[b_1\ b_2\ b_3]$ and $[J]$ was 0 -- $[J]$ is 0 in both these examples.

But in the observer canonical form, the $[F]$ matrix is $(-a_1\ -a_2\ -a_3)$. So, here it was the first row, here it is the first column. The second column is (1 0 0) this is (0 1 0), here the rows are like [1 0 0, 0 1 0]. The $[G]$ matrix in the observer canonical form is $[b_1\ b_2\ b_3]$ and the $[H]$ matrix is [1 0 0]. So, one interesting thing that you can observe is the $[H]$ matrix in the controller canonical form was $[b_1\ b_2\ b_3]$ -- row matrix. Here the $[G]$ matrix is a column matrix $[b_1, b_2, b_3]$ the $[G]$ matrix in the controller canonical form was [1 0 0], here the $[H]$ matrix in the observer canonical form is [1 0 0]. So, if you go back and see some books in control theory -- these observer and canonical form and the controller canonical form are in some sense related to each other. So, in some books you will see that they are called dual.

Again, I do not want to go into some more advanced control theory terms but let us continue. So, the characteristic polynomial from the observer canonical form can be written as $s^3 + (a_1+L_1)\, s^2 + (a_2+ L_2)\, s + (a_3 +L_3) = 0$. So, L_1, L_2, L_3 can be obtained by comparing term by term with the desired polynomial, because the desired polynomial will be $s^3 + b_1\, s^2 +b_2\, s + b_3 = 0$. And we can compare and see that L_1 will be some $b_1 - a_1$ and so on. So, from the desired estimator poles we can find out, what are these L_1, L_2, L_3. And again, I have mentioned this, I have not proved this, if you can see some advanced control theory book, this is always possible if the system is observable or the rank of this $[Q_o]$ observability matrix is n.

And because it is so easy to obtain the L_1, L_2, L_3 or the estimator gains, $[F_o]$, $[G_o]$, $[H_o]$ these are called the observer canonical form.

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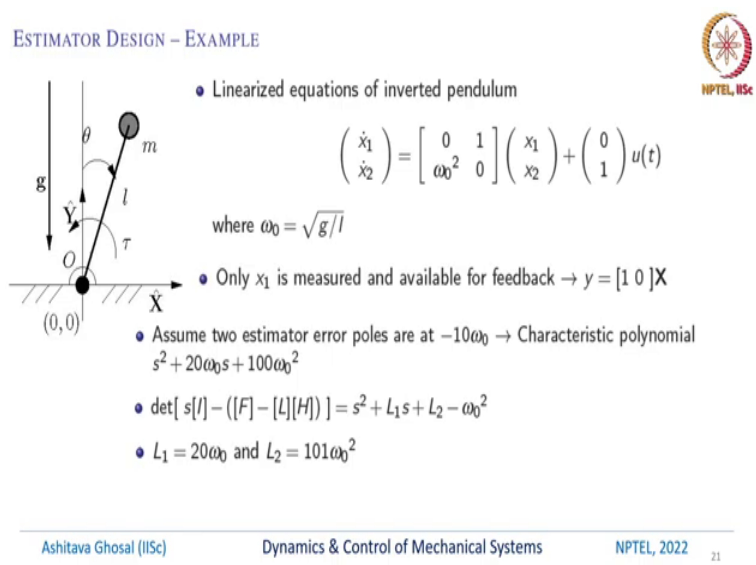

So, after all this theory and general expressions just let us look at a simple example. So, we have this inverted pendulum. The linearized equation of an inverted pendulum can be written in this form. So $\dot{X}_1$, $\dot{X}_2$ so, X_1 is θ, X_2 is $\dot{\theta}$. So, $\dot{X}_1$ is x_2 and then $\dot{X}$ is $\omega_0^2 X_1 + u(t)$, $u(t)$ is like this torque and this ω_0 is Ög / l.

So, what you can see is that this is an unstable system, why? Because the eigenvalues of this are $\pm\ \omega_0$. So, it is on the real axis, there is one which is on the left half plane and one which is on the right half plane. Let us continue, we will only measure x_1. So, remember for full state feedback, we have $u = -[K]\mathbf{X}$. But here we are only going to measure x_1, we are not going to measure X_2. And the available feedback is $y = [1\ 0]\ \mathbf{X}$ so, $\mathbf{X}$ is X_1, X_2 so, y is only X_1. Let us assume that the two estimator poles are at $-10\ \omega_0$, so, it is ten times to the left of the natural dynamics which is $\pm\ \omega_0$. The characteristic polynomial in that case is $s^2 + 20\ \omega_0\ s + 100\ \omega_0^2$. So, why did we choose $10\ \omega_0$? Remember, I had hinted or I had said that the estimator should work faster than the controller.

So, the $\hat{X}$ should approach X quickly so that it can be used in the controller which is $u = -[K]\ \hat{X}$. So, the characteristic polynomial is $s[I] - ([F] - [L][H])$, determinant of that. So that is given by $s^2 + L_1\ s + L_2 - \omega_0^2$. So, again $[F]$ is this $[0\ 1,\ \omega_0^2\ 0]$, $[L]$ is L_1, L_2, $[H]$ is $[1\ 0]$. So, if you just substitute and expand all this this is what you will get and then you can compare this polynomial with this polynomial and you can easily see L_1 is $20\ \omega_0$ and L_2 is $101\ \omega_0^2$. So, just by simple inspection this is a very simple system, second order system, we can obtain the gains estimator gains L_1, and L_2. Such that the two estimator poles are at $-10\ \omega_0$, quite a lot from the original poles which are $\pm\ \omega_0$.

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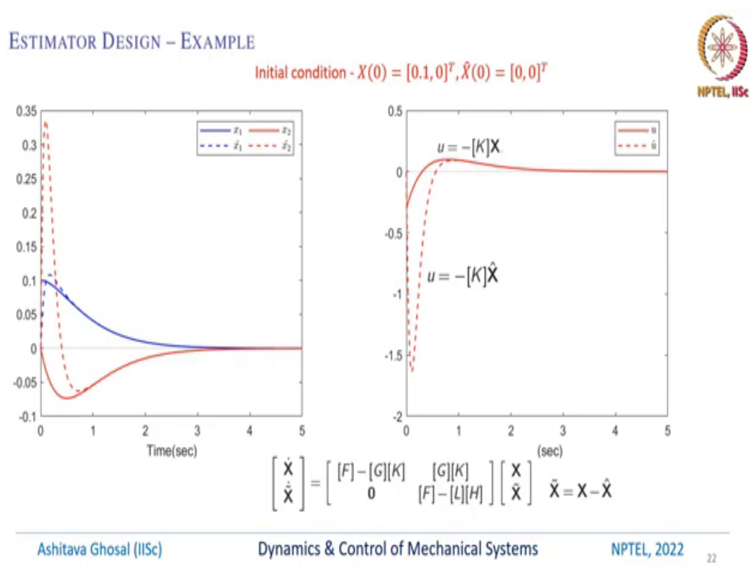

We can program this in Matlab and then we can give some initial conditions and initial condition for **X** is some let us say (0.1, 0). So, the pendulum is tilted little bit from the zero position - θ is 0.1 - and we do not know what is $\hat{X}$. So, let us assume that the initial conditions for $\hat{X}$ is (0, 0) - you know we can start from (0, 0) always. So, if you solve these equations, if you solve the state equations and the estimator equations then you can see that the plot of X_1 and $\hat{X}_1$, dark and this dotted line. So, initially they will be different but very soon X_1 and $\hat{X}_1$ will approach. In less than 1 second they are becoming same. Similarly, X_2 and $\hat{X}_2$, initially they are different, the initial conditions are different, but very soon they will approach each other and they will follow, they will be exactly each other. Some small difference might be there in many, many places of decimal but X_1 is very close to and $\hat{X}_1$, X_2 is very close to $\hat{X}_2$. So, we can now also find what is the u? So u is normally $-[K]\mathbf{X}$. The dark line shows u so, this is the input which is required to the plant. Now, if you were to use $u = -[K]\ \hat{X}$ then what you can see is the u required is more than if there is a full state feedback.

So, what have we done? We have an estimator which is using only some measurements. So that is clearly good - I do not need sensors for both X_1 and X_2 - I can only measure X_1. But at what cost? The cost is that the u which you require or the control input that you require is larger. In an estimator initially the control input will be more but again as you can see after some time because **X** and $\hat{X}$ are close to each other, this will also become same.

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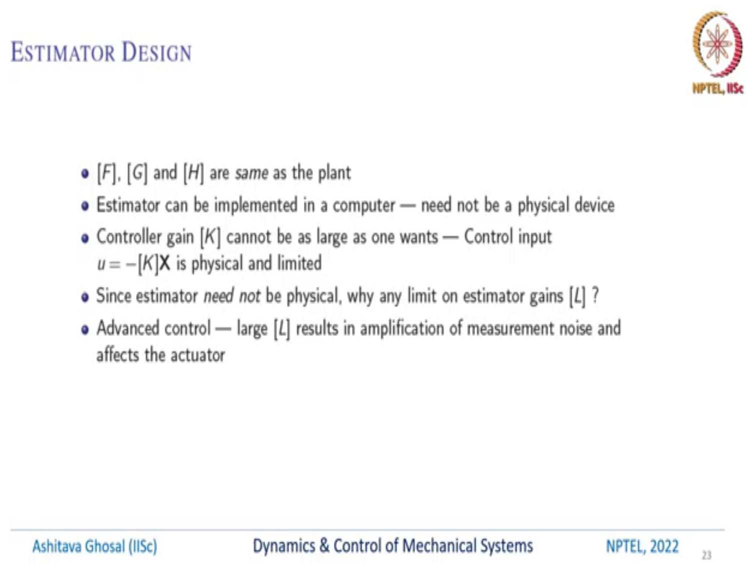

In estimator design, we assume that [*F*], [*G*] and [*H*] are exactly the same as in the plant. So, [*F*], [*G*] and [*H*] form the model and they are exactly the same as it is in the plant. The estimator can be implemented in a computer - it need not be a physical device. The controller gain [*K*] cannot be as large as one wants, why? Because the control input is $u = -[K]\mathbf{X}$ is physical. So, it is like basically a torque supplied by a motor or some voltage or some current. And we would like to keep *u* small and as small as possible. However, since the estimator need not be a physical device, why should there be any limit on the estimator gains [*L*]? So, we can choose the estimator gains very large. So then $\hat{X}$ will approach **X** much, much faster which is good. However, there is a drawback, and it turns out that if you have very large [*L*] it also results in the amplification of the measurement noise and it affects the actuator.

So, this is a topic in advanced control which I do not want to go into in this lecture. But if anyone is interested, they can look at what is the drawback in choosing a large [*L*]?

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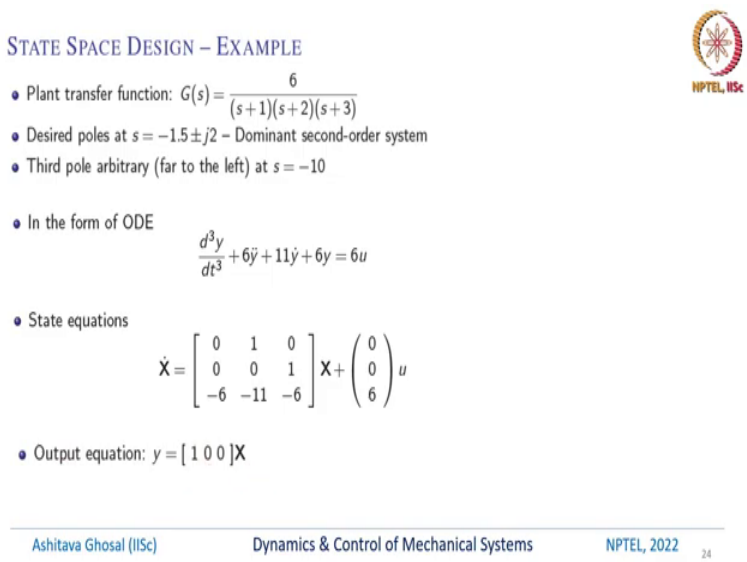

So, let us continue, we have this state space design. Let us take us another example in which the transfer function of the plant is 6 / (s + 1) (s + 2) (s + 3), we want the desired poles at s = –1.5 ± j 2. So, this is the dominant second order system that we want the controller to follow. The third pole can be arbitrary and let us choose it far to the left so, we have s = – 10.

So, the basic idea is that the effect of the third pole will die very quickly, and the controller and the system will behave with this dominant second order system. In the form of ODE, the differential equation is $d^3\, y/\, dt + 6\, \ddot{y} + 11\, \dot{y} + 6y = 6u$. In state equations, we can rewrite this as $\dot{X}$ = [1 0, 0 0 1, –6 –11 –6]**X** + (0 0 6)u - this we have seen earlier. And the output equation is we will assume that we are only going to measure one of the states, X_1.

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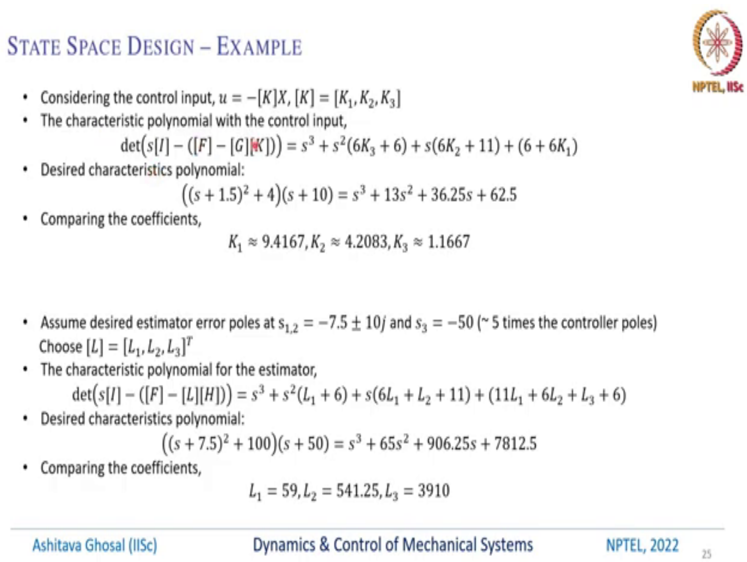

So, let us consider the control input which is u is $-[K]\mathbf{X}$, where $[K]$ is K_1, K_2, K_3. The characteristic polynomial with the control input is det $[s[I] - ([F] - [G]\ [K])]$ and we will get a cubic polynomial which is $s^3 + s^2\ (6K_3 + 6) + s\ (6K_2 + 11) + (6 + 6K_1)$. Let us look at the desired characteristic polynomial - remember we have chosen two dominant second order poles at –1.5 ±2 j. So, this will be $((s + 1.5)^2 + 4)\ (s + 10)$. The third pole is for at –10. If you expand this, you will get $s^3 + 13\ s^2 + 36.25s + 62.5$. Now, again we can compare term by term so, basically, $(6K_3 + 6)$ is 13. 6, $(6K_2 + 11)$ is 36.25, $(6 + 6K_1)$ is 62.5 and then we can solve these equations -- very simple three linear equations-- and then we will get K_1 as 9.4167, K_2 is very close to 4.2083 and K_3 is 1.1667.

We choose the estimator poles faster than the controller poles. The controller poles were –1.5 ± 2 j and s = –10. So, let us choose the estimated poles, as –7.5 ± 10 j and s_3 as –50. So, roughly five times faster than the controller poles. So, we can now obtain what is L_1, L_2, L_3 by looking at the characteristic polynomial for the estimator, which is det $[s[I] - ([F] - [L]\ [H])]$, which is $s^3 + s^2\ (L_1 + 6) + s\ (6L_1 + L_2 + 11) + (11\ L_1 + 6\ L_2 + L_3 + 6)$. So, again by comparing term by term basically, we can again solve for L_1, L_2, L_3. Why? Because what is the characteristic polynomial of the estimator, that we know, we have chosen the two poles at –7.5 ± 10 j and s_3 as –50. So then the characteristic polynomial is $s^3 + 65\ s^2 + 906.25s + 7812.5$. We can compare these two cubics and then we see that L_1 is 59, L_2 is 541.25, and L_3 is 3910. So, we can easily solve for the estimator gains L_1, L_2, L_3.

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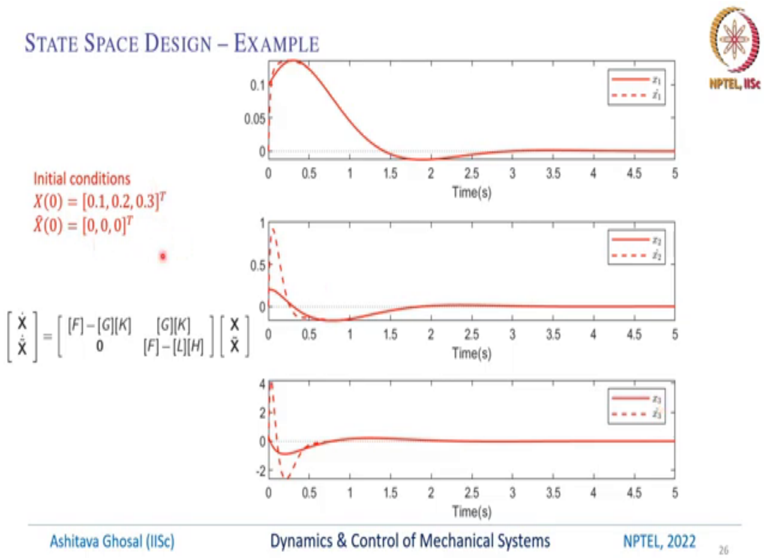

And we can again plot the states for this example we can choose **X** (0) as (0.1, 0.2, 0.3) and the $\hat{X}$ initial conditions are (0, 0, 0). And again we can solve these two differential equations. One is $\dot{X}$ = ([F] – [G][K]) **X**+ [G][K] $\tilde{X}$ and $\dot{\tilde{X}}$ = ([F] – [L][H]) $\tilde{X}$, so, we can solve these two differential equations. So, there are actually, six differential equations -- remember **X** is 3 x 1, $\tilde{X}$ will also be 3 x 1.

We can solve all of them with these initial conditions and then we can plot X_1 and $\hat{X}_1$, X_2 and $\hat{X}_2$, and X_3 and $\hat{X}_3$. What you can see is, after a very short time, less than 0.5 seconds, at about that time X_1 and $\hat{X}_1$ are very close to each other, X_2 approaches $\hat{X}_2$ or $\hat{X}_2$ approaches X_2, and $\hat{X}_3$ approaches X_3. So, even though the initial conditions are very different, the estimates match the states.

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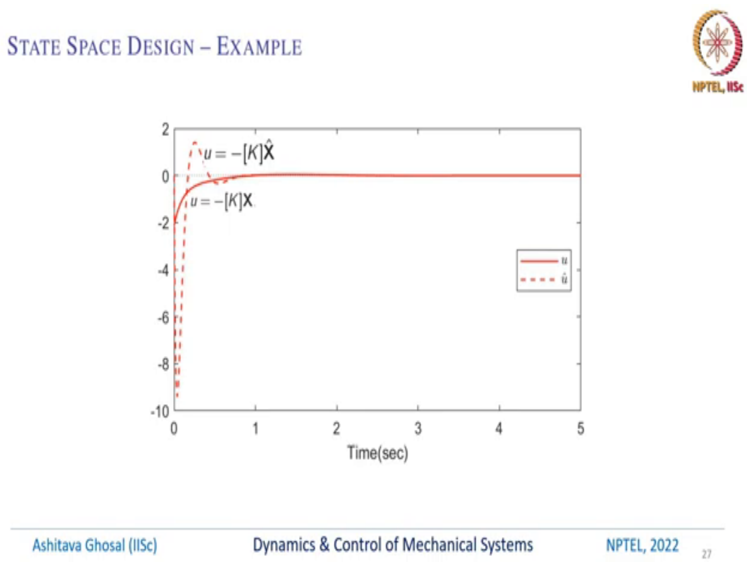

We can also plot u and $\hat{u}$ remember u is $-[K]$ $\mathbf{X}$ and $\hat{u}$ is $-[K]$ $\hat{X}$, and again you can see that initially $\hat{u}$ is more. So, although we are only measuring few things, so, we are gaining something but at the cost of more actuator input, at least initially. The same story as what happened for the inverted pendulum.

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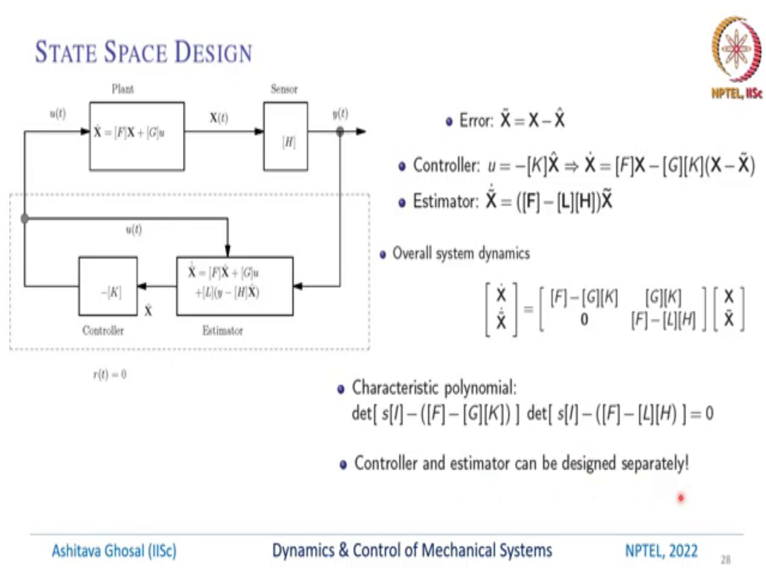

Continuing with the state space design, let us look at the block diagram, so, we have a plant which is given by $\dot{X} = [F]\mathbf{X} + [G]\ u$. There is an input $u\ (t)$ the output is $\mathbf{X}(t)$. There is a sensor, so, we get $y = [H]\ \mathbf{X}$. Then this y is fed back into this estimator because we are only using some measurements y. And then we also have u coming into this estimator and the output of the estimator is $\hat{X}$ and then we have $-[K]\,\hat{X}$ is the controller and that is what is u. So, this kind of block diagram explains where is the plant? Where is the sensor? Where is the estimator? And where is the controller? So, this can be represented mathematically using these symbols. So, first let us define variable $\tilde{X}$ which is $\mathbf{X} - \hat{X}$ the controller this $u = -[K]\,\hat{X}$, remember, $\hat{X}$ into $-[K]$, And then we have $\dot{X} = [F]\mathbf{X} - [G][K]$, and now, instead of writing $\hat{X}$, we can write $\mathbf{X} - \tilde{X}$. The estimator is get $\dot{\tilde{X}} = ([F] - [L][H])\,\tilde{X}$ - that we have seen. So, the overall system dynamics is given by $\dot{X} = ([F] - [G][K])\,\mathbf{X}$ and then the [G] [K] $\tilde{X}$ and $\dot{\tilde{X}} = ([F] - [L][H])\,\tilde{X}$. The characteristic polynomial of this overall system, containing the controller as well as the estimator, can be obtained by the determinant of this matrix. And that is det $[s\,[I] - ([F] - [G][K])]$ and it you can show that it the second term is again a determinant which is det $[s\,[I] - ([F] - [L][H])]$. So, the determinant of this whole system is a product of two determinants, and this is very important and very useful. What it is telling you is that the products of two determinants are zero, and hence we can design the controller using first, this part and then we can design the estimator using this part.

So, the controller and estimator can be designed separately. This is a very, very nice feature of the state space design.

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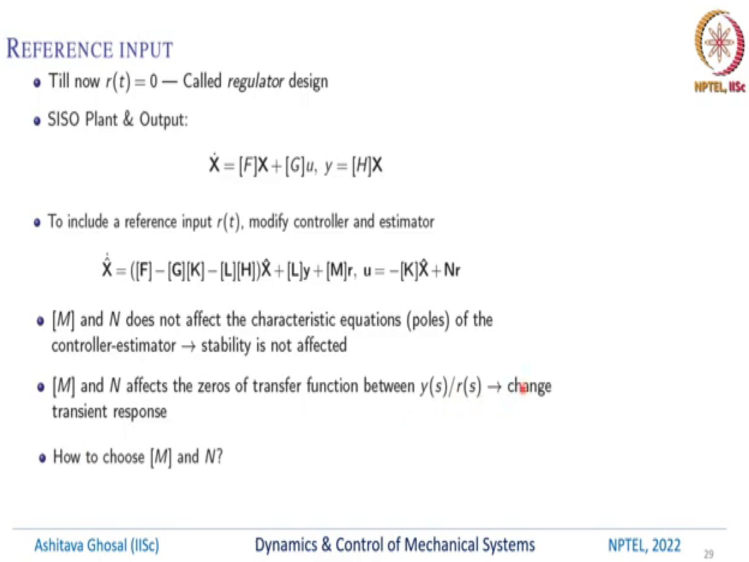

Till now, we have not considered the reference input till now, $r(t)$ was 0. This in control theory is called a regulator problem, so, we have designed a regulator with $r(t) = 0$. So, for a single input, single output plant and output we have $\dot{X} = [F]\mathbf{X} + [G]u$, we have seen this many times now and $y = [H]\mathbf{X}$. To include a reference input or $r(t)$ we need to modify the controller and the estimator.

One way is to rewrite the estimator equation as $\dot{\hat{X}} = ([F] - [G]\,[K] - [L]\,[H])\,\hat{X}$ and then we add this $[L]y$ and $[M]r$. Previously up till here it was the estimator equation but now we are adding this $[M]r$. We can also modify u which was $-[K]\,\hat{X} + N\,r$. So, this $[M]$ and N matrices or this matrix $[M]$ and N does not affect the characteristic equation or poles of the controller - estimator.

The stability is in no way affected when we choose some $[M]\,r$ and $N\,r$. So, the $[M]$ and N affects the zeros of the transfer function between y and r. So, they change the transient response. So, the question is how can we choose this $[M]$ and N?

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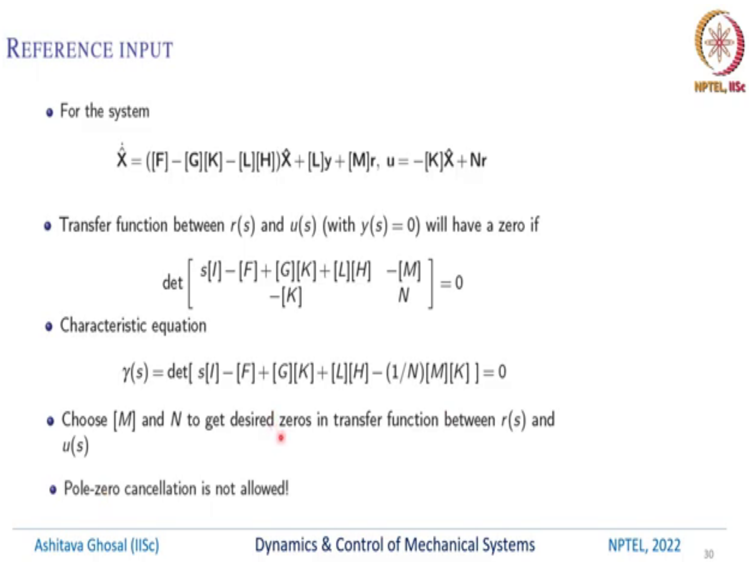

So, there are many ways of choosing $[M]$ and N. Let us go back and rewrite this $\hat{X}$ equation look at it again $\dot{\hat{X}} = ([F] - [G]\,[K] - [L]\,[H])\,\hat{X} + [L]y + [M]r$ and $u = -[K]\,\hat{X} + Nr$. So, the transfers function between $r(s)$ -- $r(s)$ and $u(s)$ with $y(s) = 0$ will have a 0 if determinant of this is equal to 0. So, transfer function of $r(s)/\,u\,(s)$ will have a 0 in the numerator because we want to change the zeros to obtain a good or a desired transient response.

The zero will occur if the determinant of this complicated thing is equal to 0. So, what is this? The first term is $(s[I] - [F] + [G][K] + [L][H])$ and this is $-[M]$ and this is $-[K]$ and this N equal to 0. So, the characteristic equation for this determinant is again we can find a polynomial $\gamma(s)$ which is det $[s[I] - [F] + [G][K] + [L][H] - (1/N)\,[M]\,[K]] = 0$. So, we can choose this $[M]$ and N to get the desired zeros in the transfer function between $r(s)$ and $u(s)$. One thing to be kept in mind is that we do not want any pole, zero cancellation. So, we do not want a zero which is sitting on top of an existing pole.

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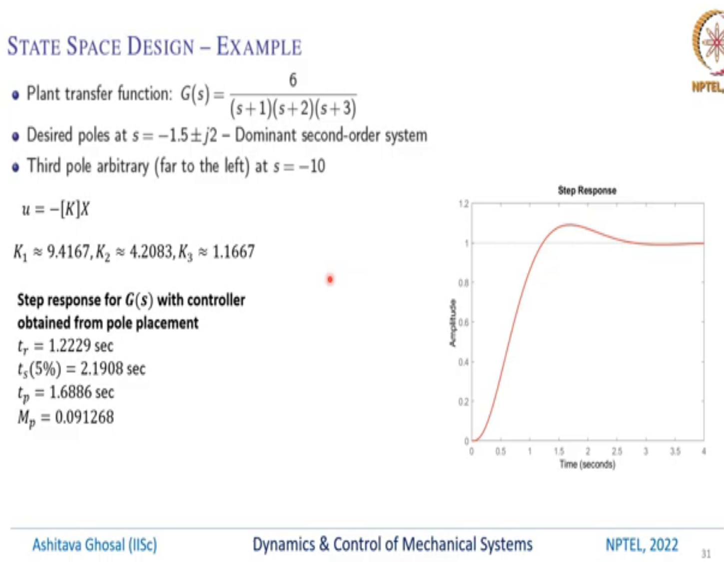

Let us look at an example, so, we have a plant transfer function which we have looked at before which is 6 / (s + 1) (s + 2) (s + 3). We want the dominant second order system with desired poles at s = –1.5 ± 2j. The third pole, as we had discussed earlier, could be far to the left, sort of arbitrary, let us choose it at s = –10. So, we have $u = -[K]\mathbf{X}$ we have done this earlier and we showed that K_1 is 9.4167, K_2 is 4.2083 and so on and K_3 is 1.1667.

We can give this system in Matlab and we can plot the step response for such a system with full state feedback and it looks like this. So, we have this step response this is the amplitude, and you can see that the rise time is 1.22 and so on. The settling time for 5% is 2.19. The peak time and the peak overshoot are given by 1.69 approximately an M_p is 0.09. But we had also designed the same system. We had looked at the same system and designed a controller based on compensators. Remember if you go back and see the results long time back, we had designed, we had taken the same system and we had designed a compensator for that to meet certain requirements.

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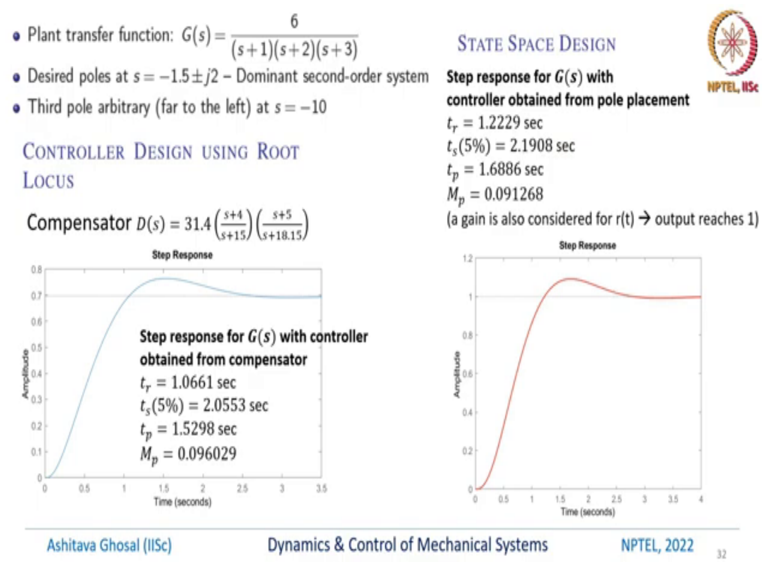

To again meet the same dominant, closed loop requirements. In that case again, we had the same system, the same desired poles, the same third pole at s = –10 and we did a controller design using root locus and a compensator. Please go back and see that the compensator which we designed was 31.4 (s + 4)/ (s + 15) and (s + 5) / (s + 18.15). These were designed such that the angle and the angle at that desired poles the angle was –180 degrees.

And again, we can find the step response of this system with this compensator and the step response looks like this. From the step response and again using Matlab Toolboxes, we can find what is the rise time? What is the settling time? What is the peak time? And what is the peak overshoot? Using state space design, we can also obtain what is the rise time, settling time, for the peak overshoot and so on.

The difference between these two designs are several -- one is in this design, the system settles at around 0.7, whereas here the system settles at around 0.1. So, there is a steady state error for this design, whereas there is no steady state error for this design. The other parameters like rise time and settling time and t_p and M_p peak overshoot are sort of similar. But you can see that the design using state space looks like better. And in fact, it is also easier so, there are lots of software tools which can be used to obtain this state space design, whereas when you want to do compensator design, it is a little bit more complex, and you have to do some trial and error. There are many more steps, whereas the state space design is becoming very, very popular and we can have

Control Toolbox in Matlab and various other software tools which can very quickly tell you what should be the controller gains and even estimator gains and so on.

So, in this example, we have used full state feedback, but we could have also continued and seen, what is the effect when you have an estimator. That is not done in this example, but it can be done.

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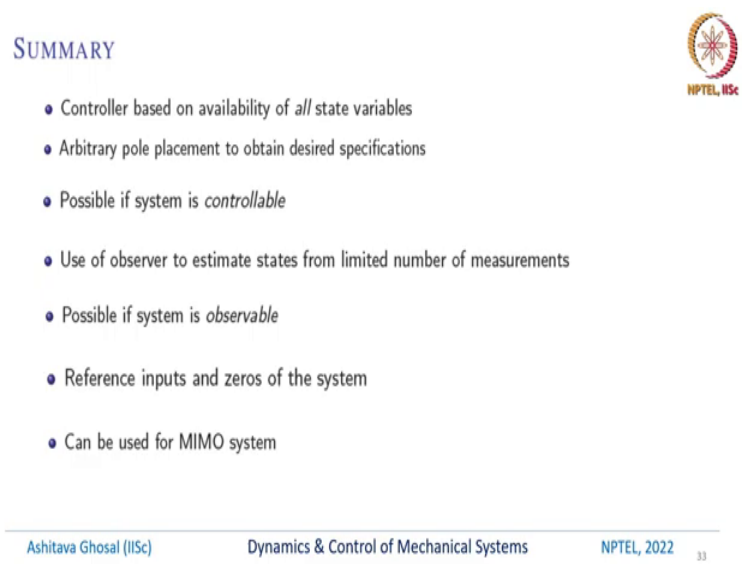

So, in summary, a controller based on availability of all state variables is very easy. So, we can easily find $u = -[K]X$ and we can design the $[K]$ or choose the $[K]$ such that it meets the desired closed loop polynomial just by comparison. We can achieve arbitrary pole placement to obtain desired specifications. Remember the desired closed loop polynomials could be anything. We can still find case such that the desired polynomial or the performance which is captured in the desired polynomial is achieved. And when is this possible, it is always possible if the system is controllable, so, the controllability matrix must be a full rank. We can also use observers to estimate the states, if you have limited number of measurements. We do not want to measure all the states, especially if the system is large when the number of state variables are large.

In that case we use an observer, and we can use these observers to estimate the states and we can design this estimator such that it very quickly follows $\hat{X}$ will follow X. And when is this possible? It is possible if the system is observable.

We can also introduce reference input, and this has to do with the zeros of the system. So, basically, when you introduce a reference input, the transient response changes. The poles of the transfer function or the poles of this system remain more or less the same. It does not affect the stability of the system but the reference input will change the nature of the transient response. Another very big advantage of state space design is that it can be used for MIMO system. So, although it has not been discussed in this course or in the in this lecture. We can easily find in many textbooks where the state space design is being used for multi-input multi-output systems. So, with that we come to an end of this week. In this week we have looked at three different ways of designing controllers -- PID controllers, root locus based controllers and also state space based designs.