Dynamics and Control of Mechanical Systems

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Lecture - 04
Linear and Angular Velocity of Rigid Body

Welcome to this NPTEL course on dynamics and control of mechanical systems. My name is Ashitava Ghoshal, I am a professor in the department of mechanical engineering also in the centre for product design and manufacturing and in the Robert Bosch centre for cyber physical systems, Indian institute of science Bangalore. In the last week we had looked at position and orientation of a rigid body and combined motion of a rigid body consisting of translation and rotation. In this week we will look at the velocity and acceleration of rigid body in 3D space.

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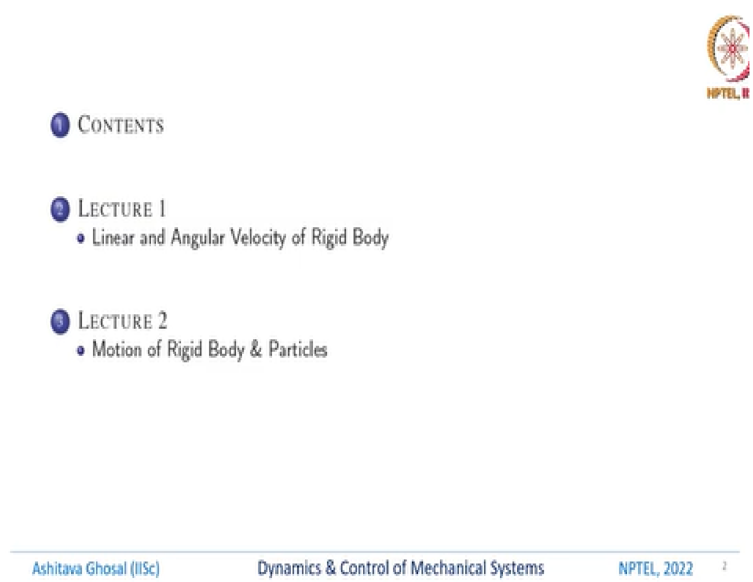

There will be two lectures in this week. The first lecture will be on linear and angular velocity of a rigid body and the second lecture would be the general motion of a rigid body and also particles where we look at the velocity and accelerations of the rigid body and the particles on a rigid body.

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So, the first lecture is on linear and angular velocity of a rigid body in 3D space.

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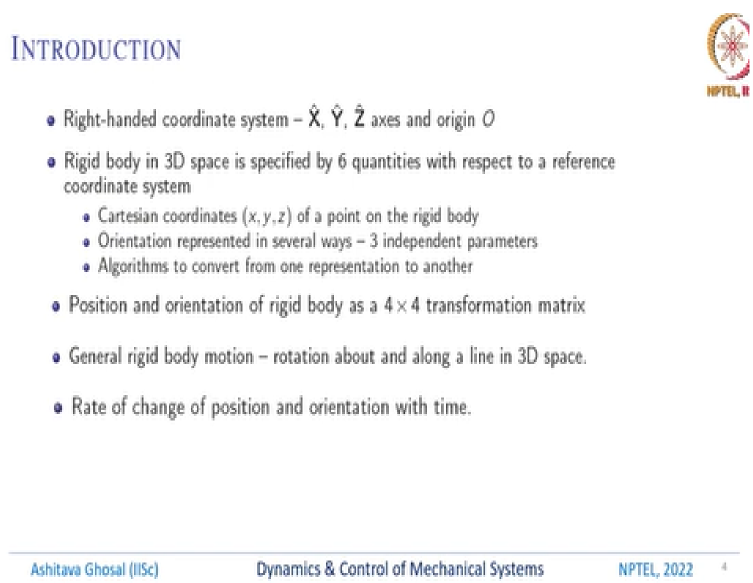

So, just to recaπtulate in last week we had looked at a right-handed coordinate system. So, basically any coordinate system will have an X axis a Y axis and a Z axis and an origin. The axes have unit vectors X hat, Y hat and Z hat. A rigid body in 3D space is specified by six quantities with respect to our reference coordinate system. So, they could be Cartesian coordinates x, y, z of a point on the rigid body.

The orientation of the rigid body can be represented in several ways. It always has three independent parameters and so we had looked at rotation matrices, angle axis form, Euler parameters, Euler angles and so on. And we developed algorithms to convert from one representation to the other and we finally looked at position. And orientation of a rigid body as a 4 by 4 transformation matrix where some part of this 4 by 4 matrix contained rotations of the rigid body.

And one column last column contain the translation of the rigid body. So, I showed you that the general rigid body motion in 3D space can be thought of as a rotation about and along a line in 3D space. In this week, we will look at rate of change of position and orientation with time.

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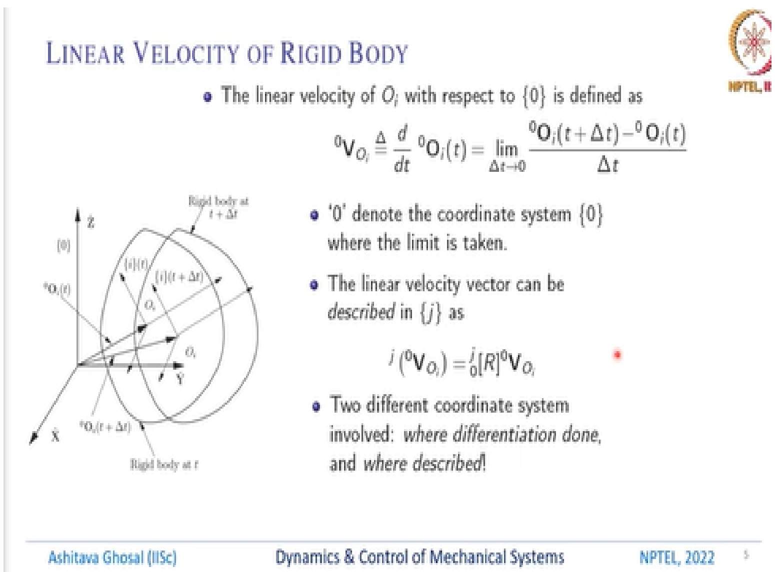

So, first let us look at the linear velocity of a rigid body. So, this figure here shows the rigid body in 3D space this odd shape thing and I have shown you a coordinate system which is $\hat{X}$, $\hat{Y}$, $\hat{Z}$ and this is the origin of the coordinate system. There is also another coordinate system which is attached to this rigid body and it is shown at two instance. So, this coordinate system is labelled as i.

So, this is at i (t + Δ t) so as you can see that the X Y and Z axis are translated in a parallel manner. So, if this were the X-axis this is the X-axis both are parallel to each other likewise the

Y and the Z-axis. The origin of the i th coordinate system is currently here and it goes to some other place. So, this is i and this is also O_i but at t + Δ t. So, as I said this is the rigid body at t and this is the rigid body at t + Δ t.

So, the linear velocity of O_i which is the point of interest basically the origin of the coordinate system which is fixed to the rigid body it can be defined using very basic notions of calculus. So, we see that the vector O_i at $t + \Delta t$ is subtract O_i at t then divided by Δ t and as the limit of Δ t goes to 0 we get the velocity of this origin of the coordinate system. It can be also denoted as $\frac{d}{dt}$ $0O_i(t)$ this is very basic calculus.

So, the 0 here denotes the coordinate system which is the reference coordinate system where the limit is taken. So, remember if you have two vectors and you are subtracting these two vectors both of them have to be in the same coordinate system otherwise it does not make sense. So, this vector O_i of t and O_i of $t + \Delta t$ are with respect to the zero-coordinate system the reference coordinate system.

So, the linear velocity vector can also be described in some other coordinate system. So, for example if there was a j coordinate system some other coordinate system let us say j and I want to describe this linear velocity vector in that j coordinate system. So, then what we can denote that vector in this $j(\,0V_i)$ but with respect to j and what it means mathematically is that we pre-multiply this linear velocity vector with a rotation matrix which is $0j[R]$.

So, basically 0 with respect to the j coordinate system. And again, if you go back and remember what we did with rotation matrices if you multiply these two basically, we transform it to another coordinate system which in this case is the j coordinate system. So, any linear velocity vector of a rigid body or a point on the rigid body can be associated with two different coordinate systems. This is a very useful thing to recognize and also very basic.

And it is also very we will see later on that it does should not cause any confusion. So, the two different coordinate systems are one in which the difference is taken or in which the derivative is taken and the other one is the coordinate system in which this velocity vector is described.

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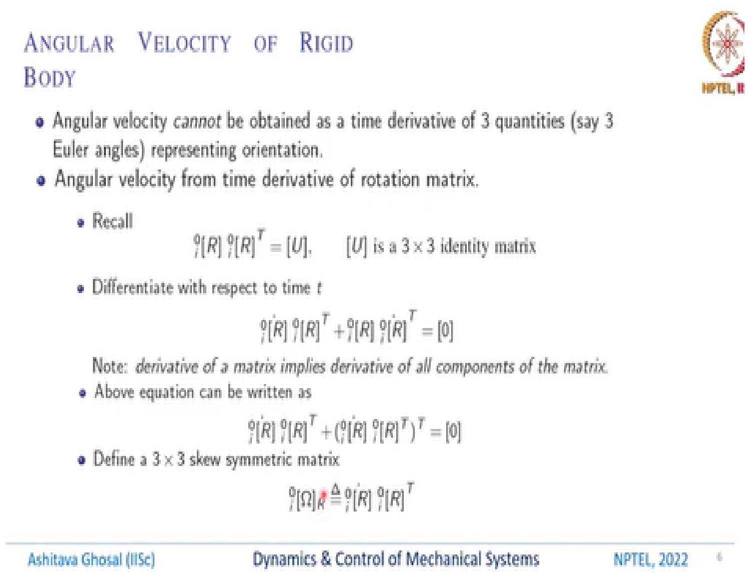

Let us continue and look at the angular velocity of a rigid body in 3D space. First thing to remember is angular velocity cannot be described as a time derivative of three quantities let us say three Euler angles. So, we can represent the orientation of a rigid body using three Euler angles let us say $\theta_1, \theta_2, \theta_3$. We cannot describe the angular velocity of this rigid body only in terms of $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$.

This is unlike the position velocity relationship. So, if the position vector is x, y, z the linear velocity is $\dot{x}$, $\dot{y}$, $\dot{z}$. So, angular velocity will be obtained from the time derivative of a rotation matrix. So, let us recall if you have a rotation matrix $0i[R]$ and if $i$post multiply this rotation matrix by the transpose of that same matrix, we get the identity matrix. So, this is same as saying that the inverse is same as the transpose.

So, if you have A into A inverse two matrices, we will get identity. In our case the rotation matrix inverse is same as the transpose so hence we get this. So, i coordinate system with respect to some reference coordinate system 0 in rotation matrix $0i[R]$ and $0i[R]^T$ is its multiplication will get the identity matrix. If we differentiate this relationship with respect to time so basically, we have to use chain rule.

So, you can see that the first term is so $[R].\ [R]^T$ and the second term will be $[R].\ [\dot{R}]^T$ and the whole of transpose. So, the right-hand side is identity so you will the derivative of that is 0. We can recall that the derivative of a matrix implies the derivative of all the components of the matrix. So, if you remember R contains R_{11}, R_{12}, R_{22} and with nine elements nine R ij's.

So, $\dot{R}$ means the time derivative of each one of those nine elements in the matrix. The above equation here which is $[\dot{R}].\ [R]^T + [R].\ [\dot{R}]^T = 0$ can be rewritten in this form. So, basically first term is the same but the second term is we are using the fact that A into B whole transpose is A transpose. So, I am going to rewrite this $[\dot{R}].\ [R]^T$ again the whole transpose.

So, R transpose R of the transpose of that will give me R and then this is $0i[\dot{R}]^T$ and the right-hand side is 0. So, now you can see that this total quantity which is $[R].\ [R]^T$ is a skew symmetric matrix. In linear algebra a skew symmetric matrix is one in which $[A] + [A]^T$ is 0. In the first module, we had looked at what is the skew symmetric matrix where the diagonal elements are 0 and the off-diagonal elements are arranged in a particular form.

And this is also another way of saying that this if you have a matrix A any matrix and then $[A] + [A]^T$ is 0 then A is a skew symmetric matrix. So, we are going to denote this $[R].\ [R]^T$ using this matrix Ω. We will continue to keep 0 and i because this is the skew symmetric matrix i with respect to 0 and this R here shows that we are starting from this kind of relationship which is $[R].\ [R]^T$ is identity.

Very soon we will see that we could also have started with $[R]^T.\ [R]$ which is identity then we would have got some other matrix. So, I have a skew symmetric matrix which I will call $0i[\Omega]$ and with the subscript R which is nothing but $[R].\ [R]^T$.

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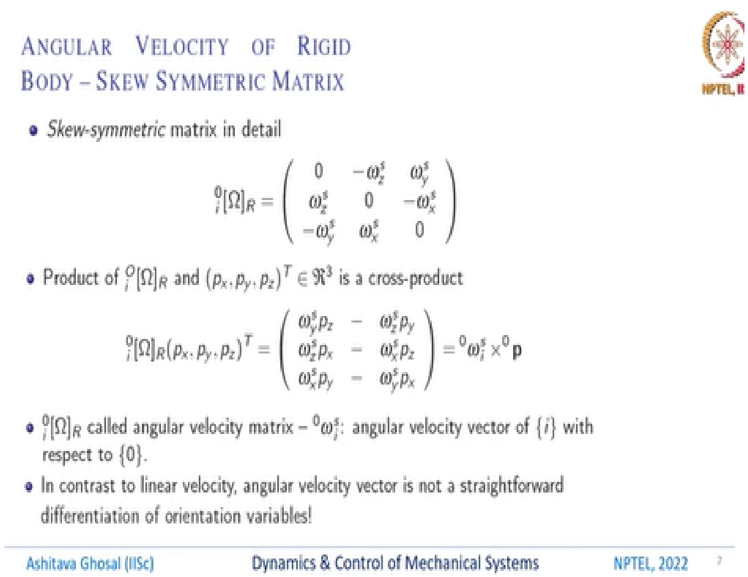

So, let us continue. So, we now I am going to show you that the angular velocity of a rigid body and this skew symmetric matrix are intimately related. So, the skew symmetric matrix $0i[\Omega]_R$ can be written in this general form of any skew symmetric matrix which is the diagonal elements are 0. So, this is that - z component, this is y component, this is - ω_x^s, this is by skew symmetric property, this should be + ω_z^s and this is - ω_y^s and this is + ω_x^s.

So, we will see later this s it has some significance, but it does not matter right now. So, we will come to what is s little while from now on. So, the product of the skew symmetric matrix $0i[R]$ into any vector as a has been mentioned earlier and one can verify this product is nothing but the cross product of this vector which is omega $0\omega_i^s \times 0p$. So, we are going to use this notation.

So, 0 here means the reference coordinate system, i means this is the ith rigid body, s here will stands for some space fixed angular velocity vector. $0p$ means this is a position of a point with respect to the zero coordinate system. So, important thing here is the skew symmetric matrix into some p x, p y, p z if you expand it, we will get this which is exactly the same as this cross product of two vectors, angular velocity vector and $0p$.

So, this quantity here is called the angular velocity of rigid body i with respect to the zero coordinate system. So, the matrix skew symmetric matrix is also sometimes called as the angular velocity matrix whereas this vector which is there are three components in the skew symmetric matrix which is $\omega_x^s \omega_y^s \omega_z^s$ that is called as the angular velocity vector of i with respect to 0.

So, important thing to realize is that in contrast to the linear velocity vector the angular velocity vector is not so straightforward. Linear velocity vector x, y, z is the position of a point on the rigid body we just take the derivative of that position vector which is $\dot{x},\ \dot{y},\ and\ \dot{z}$ and that is the linear velocity. In the case of angular velocity, we have to go through this skew symmetric matrix which is nothing but $[R].\quad [R]^T$. So, it is a little bit more complicated.

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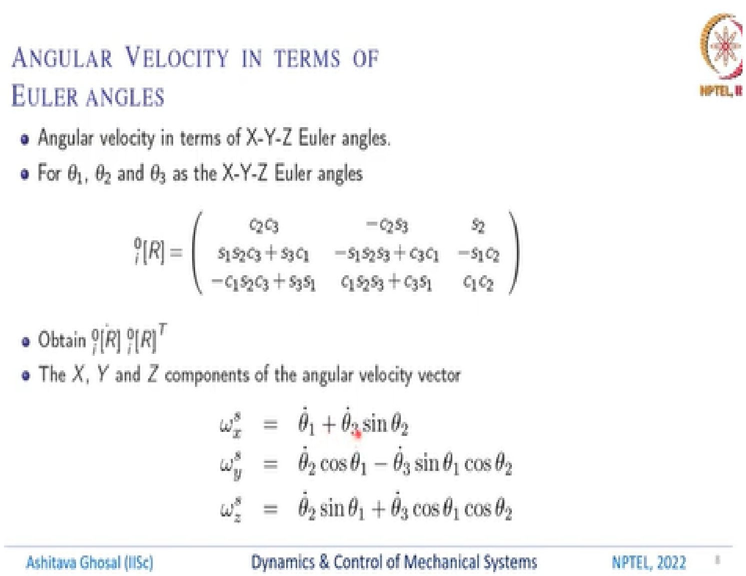

So, let us continue and we find the angular velocity vector in terms of Euler angles. So, we recall that the rotation matrix of a rigid body in terms of Euler angles can be given in variety of ways. So, if you have X, Y, Z Euler angles and rotation about x is θ_1, rotation about y is θ_2, rotation about z is θ_3 then the rotation matrix $0i[R]$ can be written in terms of cos and sin of these three angles.

So, we had discussed these Euler angles these are called X Y Z Euler angles because they are rotations about X axis, Y axis and Z axis and we can see that you will get terms like cos θ_2 cos θ_3 sin θ_2 cos θ_1 cos θ_2 and so on. So, it is a reasonably complicated expressions of cos and sin of θ_1, θ_2 and θ_3. We had done this in the first week if you have forgotten please go back and refresh.

We can now obtain $[R]$. $[R]^T$ of this matrix. So, what is the $\dot{r}_{11}$? We have to again use chain rule. So, cos θ_2 will be minus sin θ_2 into cos θ_3 + cos θ_2 into - sin θ_3. The first term should have $\dot{\theta}_2$, second term should have $\dot{\theta}_3$. So, the derivative of s_2 is cosine θ_2 into $\dot{\theta}_2$. So, remember when you take the derivative you have to use chain rule and also you have to introduce $\dot{\theta}_1$, $\dot{\theta}_2$ and $\dot{\theta}_3$.

So, the X, Y, Z components of the angular velocity vector can be obtained from the skew symmetric matrix and they look like this. So, the x component will have $\dot{\theta}_1$ + $\dot{\theta}_3$ into sin θ_2, y component is given by $\dot{\theta}_2$ into cos θ_1 - $\dot{\theta}_3$ into sin θ_1 cos θ_2 and the z component is $\dot{\theta}_2$ sin θ_1 + $\dot{\theta}_3$ cos θ_1 cos θ_2. So, I have skipped a few steps but this is very straightforward and routine.

You have to take element by element derivative. So, $\dot{r}_{11}$, $\dot{r}_{12}$, $\dot{r}_{13}$ and so on and use chain rule and introduce $\dot{\theta}_1$, $\dot{\theta}_2$ and $\dot{\theta}_3$. We will be left with also sin θ_2 cos θ_1 and so on. And then you multiply by R transpose and then you will get a skew symmetric matrix and you take out those terms which are the x component, y component and z component.

So, this is the way to obtain the angular velocity vector. If you are given the X, Y, Z Euler angles X, Y, Z means if you are given θ_1, θ_2 and θ_3 which are rotations about X, Y and Z either angles axis.

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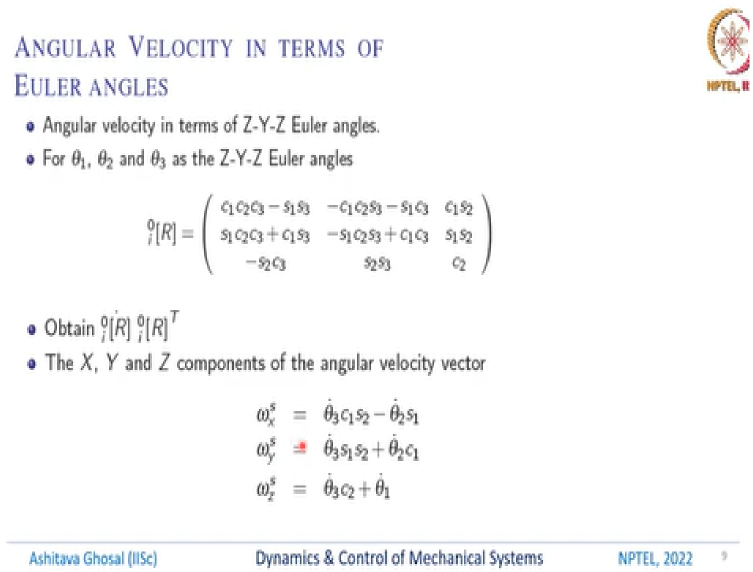

So, let us continue. If you want to find the angular velocity in terms of some other Euler angles. Remember Euler angles could be about three distinct axis which I showed you last one was bit about X, Y and Z. You can also have about two distinct axis which is let us say Z, Y and Z. And again, let us assume that rotation about first Z is θ_1, rotation about Y is θ_2, rotation about the last Z is θ_3.

Again, we can find the rotation matrix which is nothing but rotations about Z, Y and again Z multiply in that order simplify and you will get a rotation matrix which looks like this. So, here the term so r_{33} is c_2 r_{13} is c_1 s_2 again c_1 means cos θ_1, s_2 means sin θ_2 and so on. So, s_3 means sin θ_3. So, we can find this rotation matrix this was done again last week. So, please go back and refresh if you have forgotten.

Again, we can obtain this $[\dot{R}]. [R]^T$ $[\dot{R}]$ again means exactly the same thing. So, for example what is $\dot{r}_{33}$ it is minus sin θ_2 into $\dot{\theta}_2$ and we have to use chain rule for all others. So, as you can see you will have many terms. So, c_1, c_2 , c_3 so you have to use chain rule and you will get $\dot{\theta}_1$ then you will get $\dot{\theta}_2$ and you will have $\dot{\theta}_3$ and so on. And then you might post multiply by $[R]^T$ this $[\dot{R}]$ matrix.

And again, we can extract the X, Y and Z components of the angular velocity vector. The X component in this case is given by $\dot{\theta}_3$ into c_1 s_2 - $\dot{\theta}_2$ into s_1 and Y component is $\dot{\theta}_3$ into sin θ_1 into sin θ_2 + $\dot{\theta}_2$ into cos θ_1 and the Z component is $\dot{\theta}_3$ into cos θ_2 + $\dot{\theta}_1$. So, as you can see the X, Y and Z components of the angular velocity vector is different.

And that is because the we are using a different way of representing orientation. So, previously it was X, Y, Z Euler angles now we have Z, Y, Z Euler angles and naturally the rotation matrixes were different. So, the angular velocity vector would also be different.

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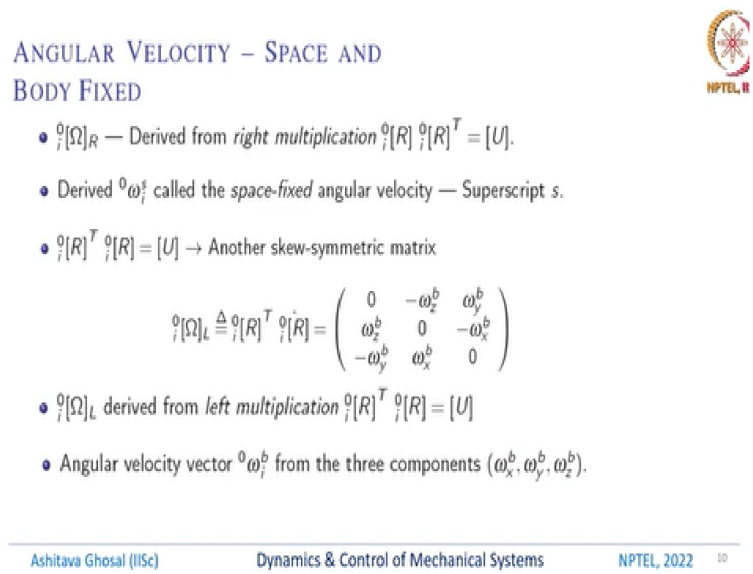

So, let us continue. So, there are two kinds of angular velocity vectors one is called space fixed and one is called body fixed. So, I showed you what is $0i[\Omega]_R$ and this R was derived from multiplying this rotation matrix to the right. So, this is like A into A transpose is identity or A into A inverse is identity. This is called right multiplication and that is where this caπtal R comes from.

This denotes the angular velocity obtained by multiplying the rotation matrix to the right. And the angular velocity vector which we obtained from $[R]$. $[R]^T$ is $0\omega_i^s$. So, this s here stands for space fixed angular velocity. So, this superscript s is from this right multiplication then the time derivative and then $[\dot{R}]. [R]^T$ and whatever the angular velocity vector we extract from the skew symmetric matrix this is called space fixed angular velocity.

We will see later that there is a nice interpretation of space fixed as and what will come right now. Another way to derive a skew symmetric matrix is to look at this pre-multiplication. So, I have a rotation matrix I can pre-multiply by R transpose and again I will get identity. So, again if I take the dot product sorry if I take the time derivative of this matrix into this matrix, I will get another skew symmetric matrix. If you think about it, it is not very hard.

So, this is called as the left multiplication because we are pre-multiplying the rotation matrix to the left. And the skew symmetric matrix that we will get from left multiplication and taking the time derivative of this equation is called $0i[\Omega]_L$ and that will be defined as $0i[R]^T$ into $0i[\dot{R}]$.

So, as you can see this can be remembered as $[R]^T [\dot{R}]$. previous one was $[\dot{R}]. [R]^T$. Now this is like product of two matrices but the order is reversed.

And we know matrix A into B is never almost never the same as B into A. So, the skew symmetric matrix that we will get this way which is $[R]^T [\dot{R}]$. is with a b superscript. So, again we will have 0 in the diagonal terms we will have minus ω_z^s with the b superscript, ω_y^b, minus ω_x^b and so on. So, we can again find a new skew symmetric matrix which will be different from the previous one.

Because remember the previous one was $[R]. [R]^T$ now it is $[R]^T [\dot{R}]$. So, we want to distinguish between these two angular velocity matrices which is one is $0i[\Omega]_L$ and the previous one was $0i[\Omega]_R$. So, $0i[\Omega]_L$ is derived from the left multiplication $[R]^T. [R]$ which is identity and the angular velocity vector which you obtain from the skew symmetric matrix from left multiplication they will be denoted as ω_x^b, ω_y^b, ω_z^b.

Previous one there was a superscript of s now there is a superscript of b. Just to distinguish the two ways which we derive the angular velocity vector.

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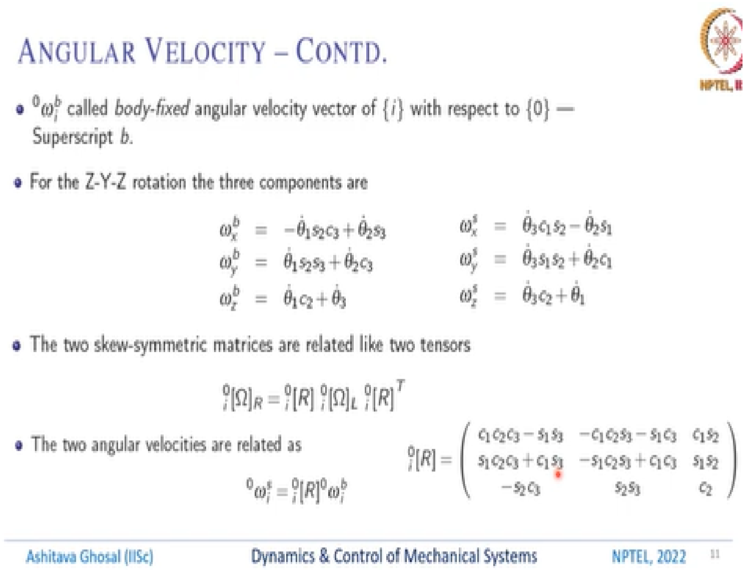

So, let us continue with the angular velocity vector. So, as I said we have an angular velocity vector with the superscript b. This is called the body fixed angular velocity vector of i with respect to 0. So, this came from the left multiplication and the skew symmetric matrix which was obtained from R transpose R. So, let us take an example, if you look at the angular velocity vector obtained from R transpose R and let us look at the Z, Y, Z Euler angles.

Then the ω_x^b is - $\dot\theta_1\ s_2 c_3 + \dot\theta_2\ s_3$, ω_y^s component is $\dot\theta_1\ s_2\ s_3 + \dot\theta_2\ \ c_3$ and the z component is $\dot\theta_1$. cos θ_2 + $\dot\theta_3$. So, as you can see the space fixed angular velocity vector again for Z, Y, Z Euler angles this was obtained from $[R]$. $[R]^T$ is identity. So, the x component of the space fixed is $\dot\theta_3 c_1 s_2$- $\dot\theta_2\ s_1$, ω_y^s is $\dot\theta_3\ s_1\ s_2$+ $\dot\theta_2\ c_1$ and ω_z^s this $\dot\theta_3\ c_2$ + $\dot\theta_1$.

So, as you can see there are very different. So, you need to be careful to understand which angular velocity vector you are talking about and we can go back and realize that one was obtained from left multiplication and one was obtained from the right multiplication. It turns out that these two skew symmetric matrices which is the left one and the right one are related like two tensors.

So, basically if you have a matrix a tensor in one coordinate system and a tensor in the other coordinate system. So, you can relate the right one as R into left into R transpose. So, this is similar to many other tensors which you might have seen in undergraduate. So, for example the stress tensor or for example the inertia matrix they transform according to these rules and the angular velocity matrices also transform according to this rule of tensors.

And the angular velocity vector also can be written as $0\omega_i^s$ is same as $0i[R]\,.\,0\omega_i^b$. So, basically what is happening is that we have this angular velocity vector which with the b superscript is related to the angular velocity vector with an s superscript by means of this rotation matrix. So, this will come in handy later on. The rotation matrix for this example just for recaπ tulation is shown here.

So, it has this c_2 here, $c_1\ s_2$ here, $s_1\ s_2$ here and so on. So, if I take this rotation matrix and put it here. And then pre multiply $0\omega_i^b$ with this rotation matrix I will get $0\omega_i^s$. So, x component, y and z of the space fixed angular velocity vector is related to the rotation matrix times the x, y, z angular velocity vector of the body fixed.

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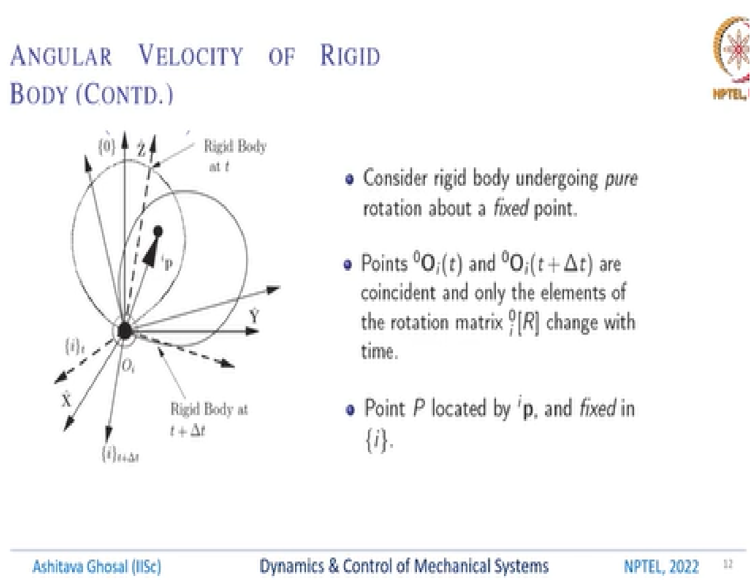

To get a little bit of more insight into these two different kinds of angular velocity vector, the space fixed and the body fixed let us look at this figure. This figure shows a rigid body, this is a rigid body at time t and this is the rigid body at time t + Δ t. We are only interested in the rotation. So, what we will see or look at is this rotation with one point fixed. So, the origin of the two coordinate systems X, Y, Z at 0 and X, Y, Z at some t and t + Δ t all of them are not changing.

So, 0 is the reference coordinate system the rigid body at t is given by i at t and the rigid body at i at t + Δ t is given by another coordinate system. So, this is the case or this in mechanics we say it is a rigid body undergoing pure rotation about a fixed point. So, one point in the rigid body is fixed and it is not translating. So, the points O_i at t and O_i (t + Δ t) are coincident and only the elements of the rotation matrix are change with time.

A point P can be located in the rigid body i by means of this vector. So, we will be using this vector which is i with a superscript i and a vector p.

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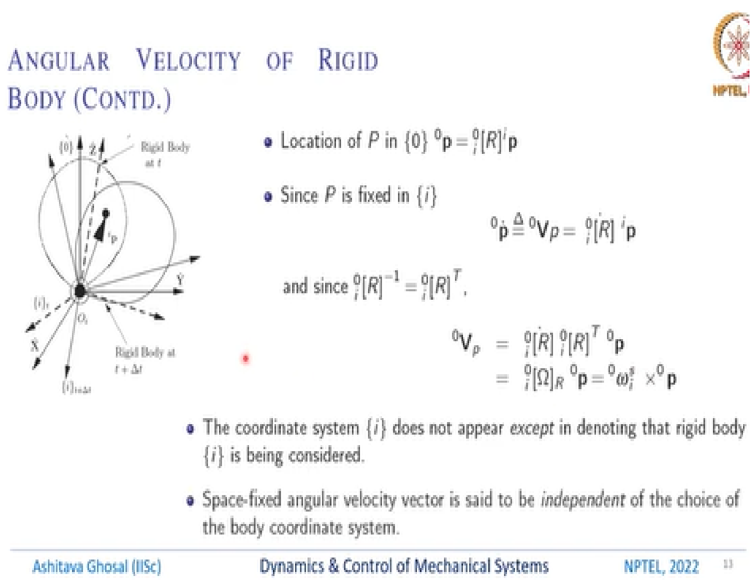

So, again the same πcture. So, the location of this point p in the ith coordinate system is given by this vector. The same vector can be written in the zero coordinate system or the reference fixed coordinate system as $0p$ is given by $0i[R]$ into ip. Again, it is very straightforward transformation of this position vector into the zero coordinates frames. So, since P is fixed in i, the derivative of this $0p$ i is same as $0p$ i and then we can write this as $0i[\dot{R}]$ into ip.

So, I am using the chain rule so derivative of the right hand side should have one term which is $0i[\dot{R}]$ into ip + $0i[R]$ into ip $\frac{d}{dt}$ of ip but P is fixed in i. So, $\frac{d}{dt}$ of ip will be 0 so hence we get this expression. So, now we are going to do a little bit of linear algebra and also little bit of manipulation. So, we know that the inverse is same as the transpose inverse of a rotation matrix is same as the transpose.

So, what we can do is we multiply this right hand side so this is $\dot{R}$ into ip so that can be written as $\dot{R}$ into $0i[R]^T$ $0p$. Because remember ip will be same as $0i[R]^T$ $0p$. So, transpose can be same as thought of as inverse. So, now let us look at this term. So, if you take a look at this term this is $[\dot{R}]$. $[R]^T$. So, this is the skew symmetric matrix coming from the right multiplication which is $0i[\Omega]_R$ and we are left with $0p$.

So, the linear velocity of this point in the zero coordinate system can be written as $0\omega_i^s$ cross $0p$. So, it is very similar to what we have learnt in undergraduate that the linear velocity of a point which is rotating about this is like R cross omega or omega cross R. So, we have this expression here for the linear velocity of this point which is rotating with one end fixed. So, if you look at this expression once more.

So, other than the coordinate system I which is denoting that we are interested in the rigid body i everywhere we have 0 here, this is 0 here, this is 0 here. So, the coordinate system i does not appear except in denoting the rigid body i that we are interested in the rigid body i. So, this is important that the linear velocity of a point which is of inner rigid body which is rotating with one point fixed is like omega cross R and that omega is the space fixed angular velocity vector.

So, since i is not appearing anywhere in this expression except to denote that we are interested in the rigid body i everything is with respect to 0, 0 and 0 here. So, the space fixed angular velocity vector is said to be independent of the choice of the body coordinate system. So, let us go over this a little bit. So, it does not really matter what is the body coordinate system because we are doing omega cross R and this omega is the space fixed angular velocity vector.

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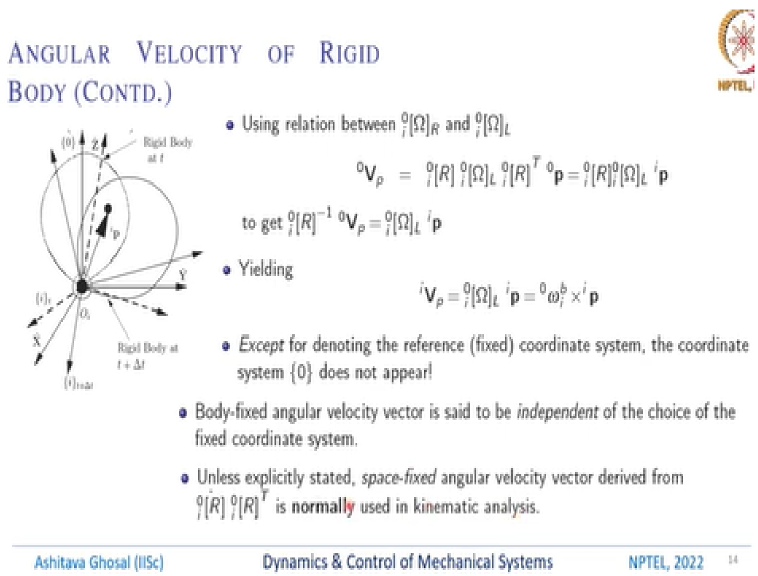

Let us continue. We again have this rigid body. There is a fixed or a reference coordinate system it is there is a ith coordinate system attached to the rigid body at time t and then this coordinate system goes at t + Δ t. This is the i at t + Δ t. So, now if you use the relationship between omega R and caπtal omega L which is remember it was transforming like a tensor. So, I can write the linear velocity of this point with respect to the zero coordinate system as $0i[R]$, $0i[\Omega]_L$, $0i[R]^T$, $0p$.

Because remember this is the ω^s space fixed angular velocity vector and the matrix associated with the space fixed triangular velocity vector was R here but then I am writing R as $0i[R]\ 0i[\Omega]_R\ \ 0i[R]^T$. So, let us continue. Now this part here can be written as ip. So, remember this is R inverse $0p$ so this is a vector now transformed to the ith coordinate system. So, we have one omega L here and rotation matrix and i, the point in the i coordinate system.

Now we can pre-multiply both sides of this and this with $0i[R]^{-1}$. So, $0i[R]^{-1}$ into $0V_p$ will be left with $0i[\Omega]_L$ into ip. So, R into R inverse is identity. So, if you do this, this quantity here if you think a little bit is the linear velocity of the point p expressed in the ith coordinate system. So, $0i[R]^{-1}$ into $0V_p$ is the same as i 0 into $0V_p$. So, this again 0 0 will cancel out and we will be left with iV_p.

And this is equal to the left angular velocity matrix into ip. So, remember this is a skew symmetric matrix. So, this is nothing but the body fixed angular velocity vector cross ip. So, we have this point p in the ith coordinate system with respect to the ith coordinate system. If you pre multiply by the angular velocity vector b with the superscript b I will get the linear velocity of the point p in described with respect to the ith coordinate system.

So, again if you think a little bit except for the fact that the 0 is showing you that we are discussing a fixed reference coordinate system, it does not appear anywhere else. So, the body fixed angular velocity vector is said to be independent of the choice of the fixed coordinate system. This is reason why one is called body fixed because it does not matter what is the in choice of the reference coordinate system or the zero-coordinate system in the case of space fixed.

It does not depend on what is the choice of the coordinate system attached to the body. So, unless specified we most of the time we will be using space fixed angular velocity vector. And just to recollect the space fixed angular velocity vector is derived from $\dot{[R]}.\ [R]^T$.

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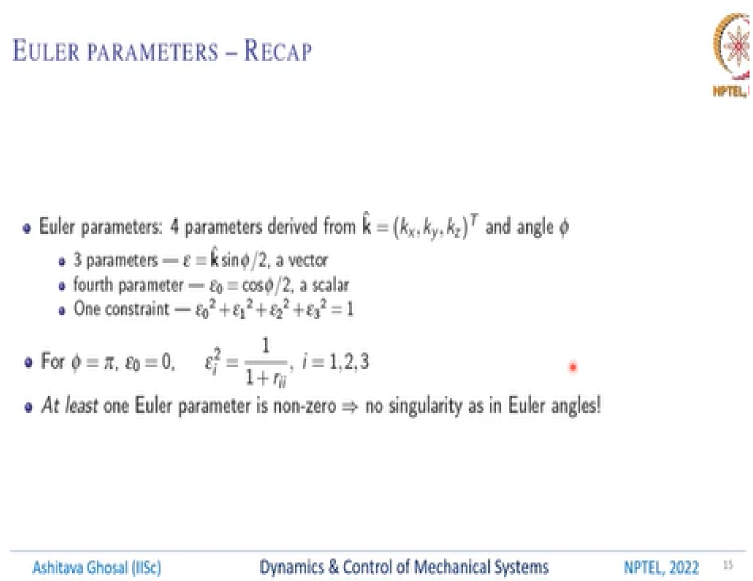

Let us go back and see how we can look at this angular velocity vector in terms of some other representation of orientation. So, one of the representation of orientation was this Euler parameters and if you recall the Euler parameters consisted of four parameters. They are basically the axis above k which is k_x k_y k_z it is a unit vector and then there is this angle which is the rotation about that axis. And the Euler parameters three of them were nothing but this k vector into sin ϕ /2. So, this is a vector and the fourth parameter was the scalar which is ϵ_0 which is cos ϕ /2 and there is a constraint. So, there are three parameters here ϵ_1, ϵ_2, ϵ_3 and there is a fourth one ϵ_0 and the constraint is $\epsilon_0^2 + \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 = 1$.

So, for ϕ equals π or ϵ_0 is 0. So, cos of π is π /2 because ϕ is π so this becomes π /2 cos of π /2 is 0. So, but ϵ_i^2 is given by this $1 + r_{ii}$, r_{13}+ $r_{22}$and r_{33}. So, they were the diagonal terms in the rotation matrix. So, what it means is even though ϕ is π so cos π /2 is 0 all the Euler parameters are non-zero. So, there is no singularity in Euler as in Euler angles.

Singularity means that not it is not I cannot determine all the Euler parameters. In the case of Euler angles if there was a singularity, we saw that we could only do θ_1 plus minus θ_3. So, two of the angles cannot be determined uniquely. So, we chose theta one as something and θ_3 was 0. So, please go back and refresh the algorithm to determine Euler angles given some rotation matrix.

So, for certain values of θ_2 we saw that it was singular. In the case of Euler parameters singularity does not happen because not all the epsilon eyes become 0.

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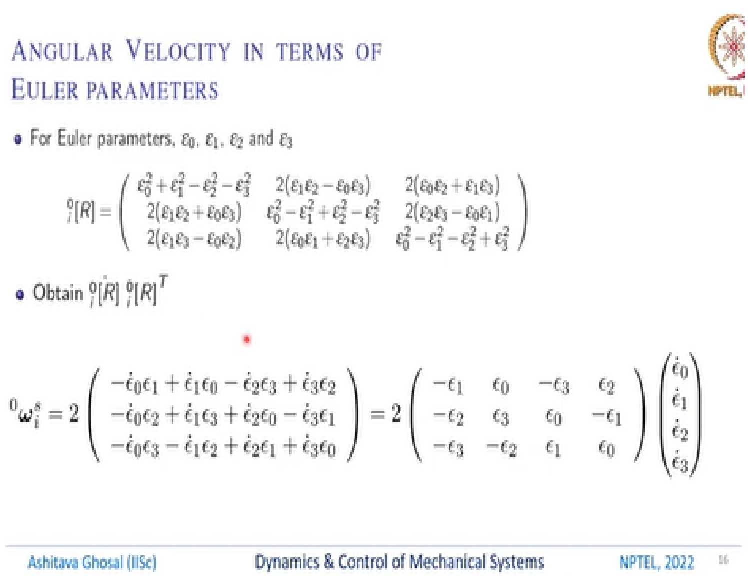

So, let us see if we can derive the angular velocity in terms of the Euler parameters. So, it turns out that for Euler parameters $\epsilon_0\ \epsilon_1, \epsilon_2\ and\ \epsilon_3$ the rotation matrix is given by this. Again, this was derived this is just to recapitulate. So, the r_{11} term is this $\epsilon_0^2 + \epsilon_1^2 - \epsilon_2^2 - \epsilon_3^2$ and so on. So, these are some terms which we obtained from what is k and ϕ and then k and ϕ was related to $r_{ij's}$.

We can now obtain $\dot{[R]}.\ [R]^T$ this is the space fixed angular velocity vector and we can take the derivative of each one of these terms epsilon. So, the derivative of first one will be 2 $\epsilon_0\ \dot\epsilon_0$ and so on. We have to use the chain rule and if you take those derivatives and reorganize and you can see that the space fixed angular velocity vector is given in this form. So, it is 2 into - $\dot\epsilon_0$ into ϵ_1 + $\dot\epsilon_1\ \epsilon_0 - \dot\epsilon_2\ \epsilon_3 + \dot\epsilon_3\ \epsilon_2$.

So, this comes from applying chain rule multiplying the taking the derivatives and reorganizing so two will come out. This can further be written all this vector with three you know column vector can be written as a matrix. So, this is a 3 by 4 matrix into 4 by 1 vector derivative of ϵ_0, ϵ_1, ϵ_2 and ϵ_3. So, the space fixed angular velocity vector which is nice to see that it can be obtained as ϵ_0, ϵ_1, ϵ_2 in some matrix into the time derivative of those all four Euler parameters.

So, note this is not a square matrix. So, this is not a 4 by 4 into 4 by 1, this is a 3 by 4 into 4 by 1. Why? Because the angular velocity vector is a 3 by 1 vector, it has three elements.

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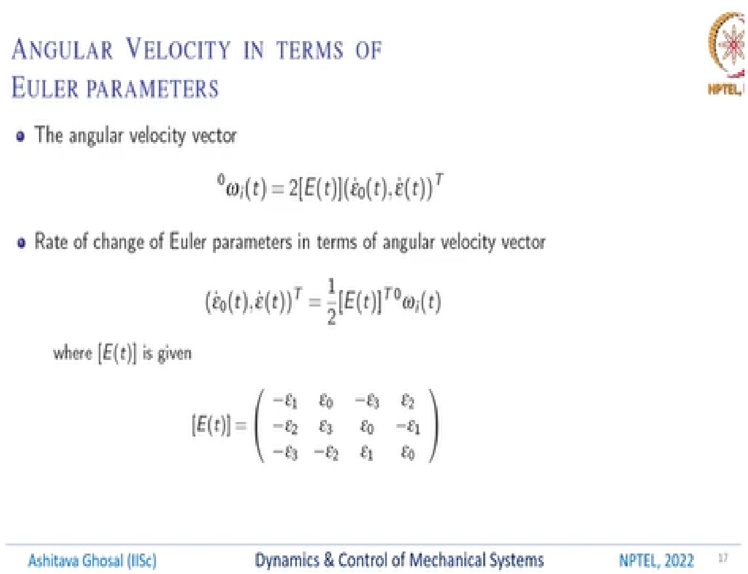

We can also obtain the reverse. So, we can also show that the angular velocity vector if it is given $\dot{\omega}_i(t)$, I can obtain that time derivatives of ϵ_0, ϵ_1, ϵ_2, ϵ_3 and epsilon and so on. So, this is a 4 by 1 vector of time derivative of the Euler parameters. It is given as $1/2[E(t)]^T\, 0\omega_i(t)$. So, these two expressions tell you that if I give you that rate of change of the Euler parameters, I can find the angular velocity. If you give me the angular velocity, I can find the rate of change of the Euler parameters. So, there is no inverse going on because this E matrix is a 3 by 4 matrix. So, E matrix from the previous slide is a 3 by 4 matrix so inverse is not possible. But we do not need to obtain the inverse. In one case you have two E t, in other case it is a $1/2[E(t)]^T$. So, it is a very nice way of showing how the Euler parameters and their time derivatives are related to the angular velocity vector.

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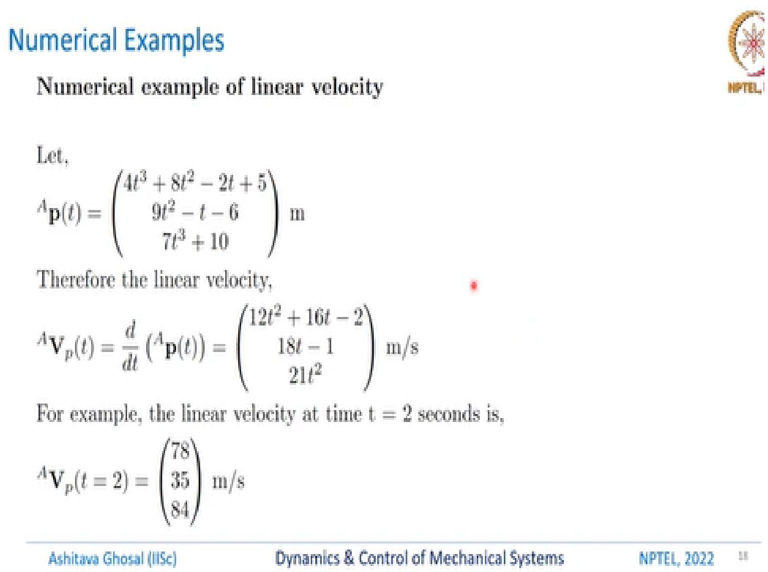

So, let us continue and look at simple some numerical examples and these numbers are chosen arbitrarily just to show that we can calculate numerically the linear velocity, the angular velocity and also various other quantities. So, if the position vector in the a coordinate system as a function of time is chosen arbitrarily as $4t^3 + 8t^2 - 2t + 5$ this is the x component, y component is $9t^2 - t - 6$, the z component is $7t^3 + 10$.

So, the derivative of this will give you the linear velocity. So, this is $\frac{d}{dt}$ of this vector will give you the linear velocity of the point. And just by very simple calculus we can show that the linear velocity is given by $12t^2 + 16t - 2$ and so on. So, this is $18t - 1$ and this is $21t^2$. So, if you want to evaluate the linear velocity at any time so let us say t = 2 you can just substitute t = 2 and we will get these numbers.

The basic idea here is that if you give me the position vector as a function of time, I can find out the linear velocity very easily just by taking derivatives of each of the components.

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Numerical Examples

Numerical example of space-fixed angular velocity

From the previous slides we know that for Z-Y-Z space-fixed Euler angles,

$$\omega^s = \begin{pmatrix} \dot{\theta}_3 \cos\theta_1 \sin\theta_2 - \dot{\theta}_2 \sin\theta_1 \\ \dot{\theta}_3 \sin\theta_1 \sin\theta_2 + \dot{\theta}_2 \cos\theta_1 \\ \dot{\theta}_3 \cos\theta_2 + \dot{\theta}_1 \end{pmatrix}$$

Let,

$$\theta_1 = 5t^2 - 4 \text{ rads}$$
$$\theta_2 = -10t \text{ rads}$$
$$\theta_3 = 7t^3 + 9t - 1 \text{ rads}$$

Therefore the space-fixed angular velocity at time t = 3 seconds is,

$$\omega^s(t = 3) = \begin{pmatrix} -194.7397 \\ -21.1580 \\ 60.5418 \end{pmatrix} \text{ rads/s}$$

Whereas if you want to find the angular velocity vector then we have to cannot take simply the derivative of the components of $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$. So, for example in the Z, Y, Z space fixed Euler angles if I take the derivatives, I will get this, and the ω^s is given by these functions of $\dot{\theta}_3 \cos\theta_1 \sin\theta_2$ and so on. And if I choose θ_1 as $5t^2 - 4$ again chosen arbitrarily θ_2 as - 10t any function of time.

You have to substitute back all these things here at to find out what is the angular velocity vector. So, at t = 3 we find out what is θ_1 then we find out what is $\dot{\theta}_1$. How do I find out $\dot{\theta}_1$ we have to take the derivative of this so this will be 10t. And then substitute back all of this back here to get the space fixed angular velocity at t equals_3 given by this. So, the basic idea here is to show that the angular velocity is a much more complex thing. It is not very simply the time derivative of the position vector.

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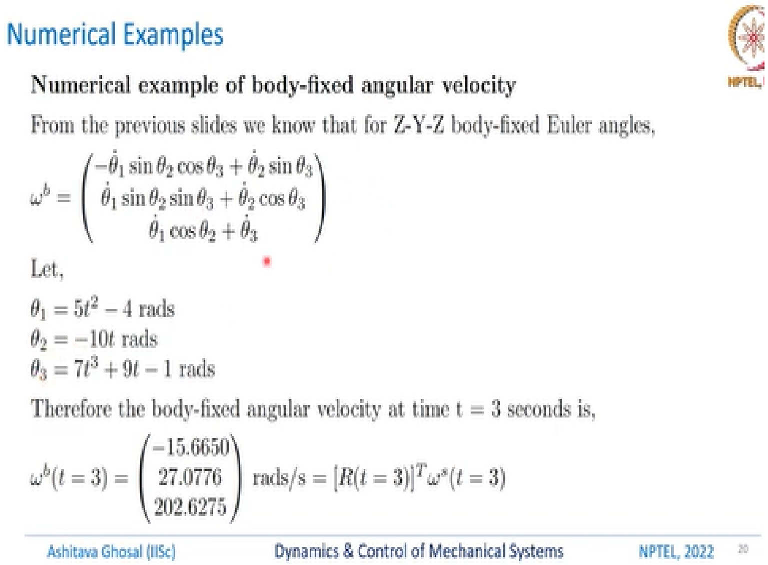

We can also find ω^b with the superscript b for Z Y Z the angular velocity with superscript b is given by - $\dot{\theta}_1$ sin θ_2 cos θ_3 $\dot{\theta}_2$ sin θ_3 and so on. The z component is $\dot{\theta}_1$ cos θ_2 + $\dot{\theta}_3$ this has been derived earlier. And then we can substitute again the same θ_1 as a function of time, θ_2 as a function of time, θ_3 as a function of time and we can substitute back θ_1, θ_2, θ_3 here.

And $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$ here and again for t = 3 we see that the angular velocity vector has completely different components. So, I wanted to show you that if you choose the body fixed angular velocity vector with the superscript b then it is very much different from the space fixed angular velocity vector with the superscript s. And then you can also find the rotation matrix and if you multiply $[R]^T \omega^s$ you will get back this ω^b.

In the previous slide I had showed you what is ω^s and in the slide before that I showed you ω^s is R ω^s. So, if you go back and sub compute ω^b and ω^s using these formulas and then do this matrix multiplication before $[R]^T \omega^s$ you will get ω^b. Just to tell you that numerically also it matches with whatever we have derived the analytical expressions for ω^s and ω^b.

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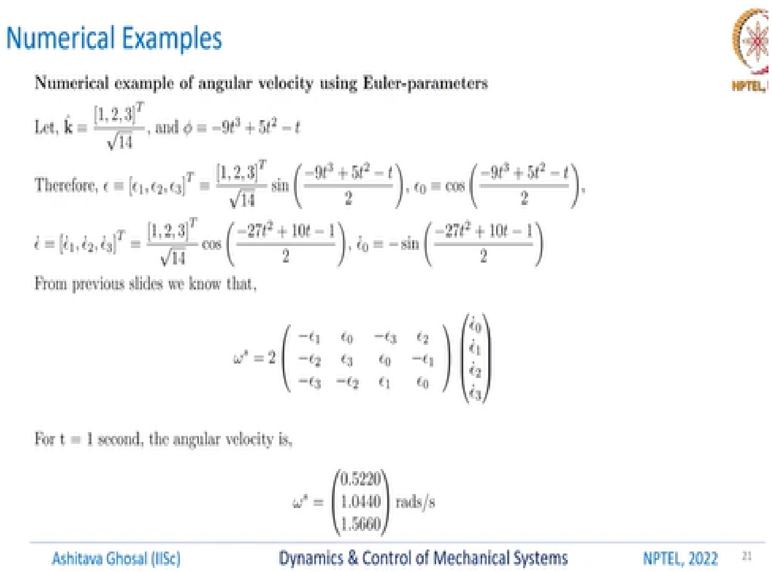

The same thing we can show with Euler parameters. So, if the k is given by 1, 2, 3 square root of 14 so basically it is a unit vector and this ϕ is some again randomly chosen - $9t^3 + 5t^2$ - t. I can find out what is the Euler parameters so ϵ_1, ϵ_2, ϵ_3. The vector part is given by this expression and ϵ_0 is cosine of this so k and this is sin ϕ /2 and ϵ_0 is cos ϕ /2.

$\dot{\epsilon}$ I can take the time derivative of this and obtain this $\dot{\epsilon}_0$ is given by this. And from the previous slide we know ω^s is some matrix 3 by 4 matrix of all the 4 Euler parameters into $\dot{\epsilon}_0$ $\dot{\epsilon}_1$ $\dot{\epsilon}_2$ $\dot{\epsilon}_3$ and the angular velocity for this example this numerically chosen k and ϕ is given by 0.522, 1.040 and 1.566. So, this is a space fixed angular velocity vector for k and ϕ chosen randomly.

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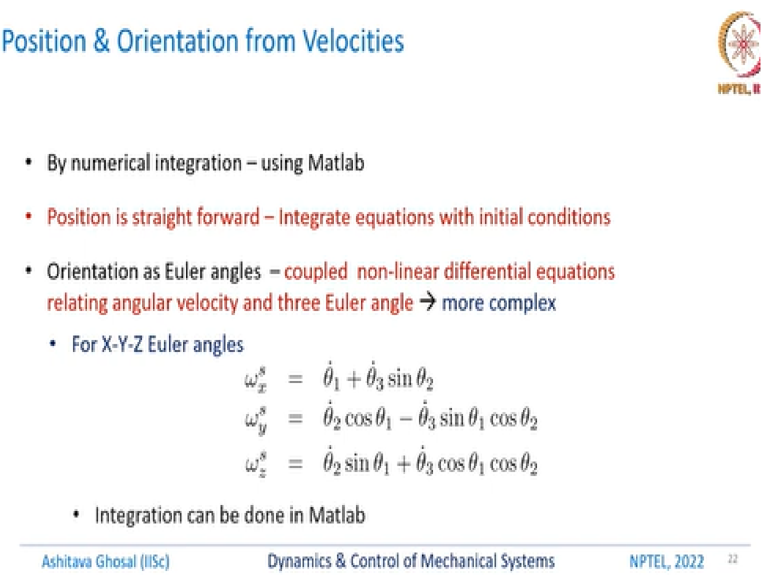

So, the next question which is of interest is suppose you are given position. So, you are given velocities both linear and angular velocities can we obtain the position and orientation of a rigid body. So, clearly, we have to numerically integrate the velocities because integration of velocities will give the position vector. Similarly, somehow integration of the angular velocity will give you the orientation.

So, position is very straightforward, we have say V which is the linear velocity which is nothing but $\dot{x}$, $\dot{y}$, $\dot{z}$. So, we can just integrate and with initial conditions, so position is very straight forward. Orientation from angular velocity is slightly more complex. Why? Because we have coupled non-linear differential equations which relate the angular velocity and the three Euler angles.

So, as an example if you remember this for X, Y, Z Euler angles the angular velocity x component is theta one dot into $\dot{\theta}_3$ into sin θ_2. The y component is $\dot{\theta}_2$ cos θ_1- $\dot{\theta}_3$ sin θ_1 cos θ_2 and the z component is $\dot{\theta}_2$ sin θ_1 $\dot{\theta}_3$ cos θ_1 cos θ_2. So, what is the problem? We are given the left-hand side. So, I am given let us say some 1 radian per second here, 0.2 radians per second here and let us say - 0.03 radians per second as the z component.

The goal is to find out θ_1, θ_2, θ_3 which are the X, Y, Z Euler angles. So, in order to find out θ_1, θ_2, θ_3 given the left-hand side I have to integrate these three equations and as you can see these three equations are coupled and non-linear. So, we need to go and use some MATLAB or some other tool to integrate.

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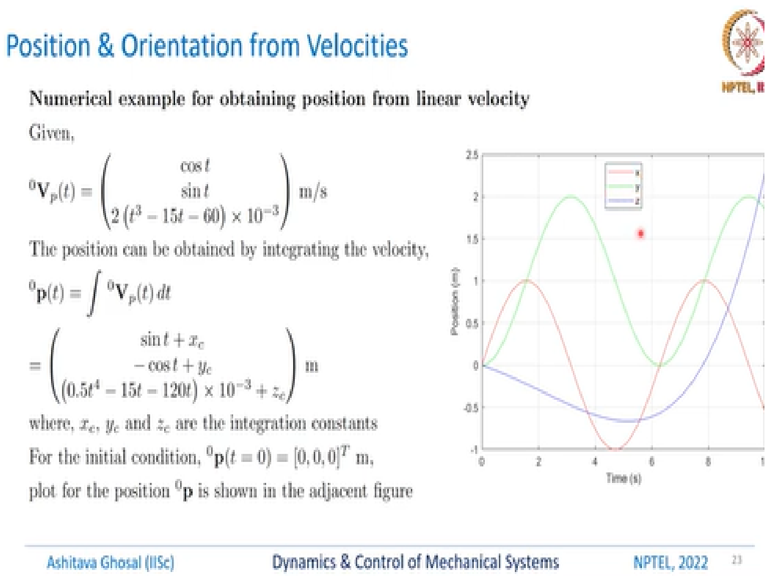

So, again let us take some simple numerical examples. So, the linear velocity is given as cos t sin t and this - 2 into t^3 - 15t - 60 into 10 - 3 meters per second. So, this is the x component, this is y component and this is the z component so again randomly chosen. So, how do I find the position of the rigid body? If the linear velocity of the point is given you just integrate. And if you integrate you can see integration of cos t is sin t + some constant, sin t is - cos t + some constant and integration of this is given by this.

So, this x_c, y_c, z_c are the integration constants. So, if I assume that its initial conditions are zero then we can obtain the position vector given this velocity vector and this is shown in the adjacent figure. So, I can integrate this, and I can plot the position x, y and z, x is the red line, y is the green line, z is the blue line and with time I can plot these expressions. So, this is sin t you can recognize now that x is sin t.

So, the red line looks like a sin t, the green line is - cos t so it looks like this whereas the z component is some complicated or you know polynomial in t.

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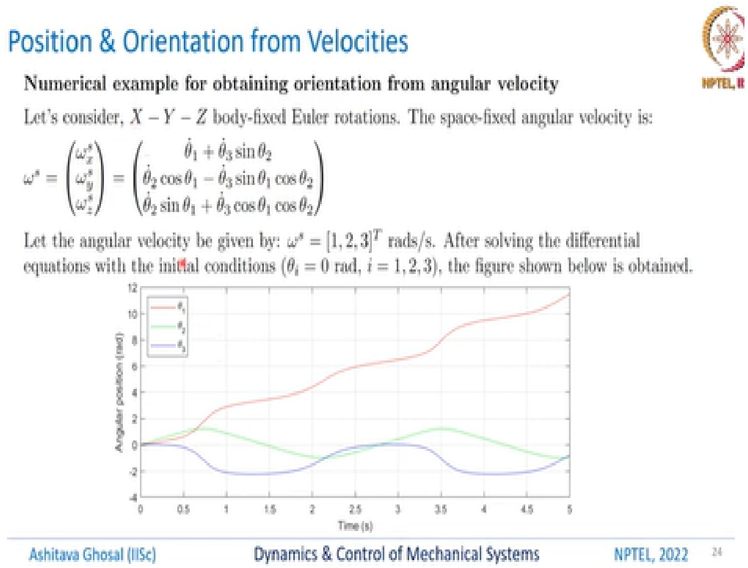

So, let us continue. I want to find the angular velocity, I am given the angular velocity vector I want to find the X, Y, Z Euler rotations. So, I want to find out θ_1, θ_2, θ_3 which are the rotations about x, y and z. So, once I know θ_1, θ_2, θ_3 body fixed Euler rotations I can find the rotation matrix x for θ_1, y for θ_2, z for θ_3 multiply the matrices in that order and I can get back the direction cosines in the rotation matrix.

So, as an example let us assume that the space fixed angular velocity vector is given as [1, 2, 3]. So, it is a vector x component is 1, y component is 2, z component is 3 again you know arbitrarily chosen. And we need so the left-hand side is given we need to integrate these three equations. And we have to assume some initial condition let us assume that the initial conditions are all zero.

So, the TA has made a nice video of various ways to use different you know ways to in MATLAB to integrate. So, there are routines which are available for integrating differential equations and we can use those routines to solve this problem of integrating three nonlinear equations three coupled non-linear equations and it looks like this. So, θ_1 as if you plot as a function of time it looks like this.

It is constantly increasing this is in radians and this is in time. θ_2 goes like this θ_3 goes like this. So, we know at any instant of time what is θ_1, θ_2 and θ_3 let us say at t = 2 we can find out these and then we can find the rotation matrix corresponding to θ_1. Let us say this is something like 4 radians, I can go back and so x rotation is 1 0 0, first column is 0 1 0 0, first row is 1 0 0 and then cos θ_1- sin θ_1 sin θ_1 cos θ_1.

So, these are the simple rotations which we had discussed last week. So, I can find out the simple rotation about X, simple rotation about Y, simple rotation about Z and then multiply the matrices in the order which you do which is X, Y and Z and then you can find the equivalent rotation matrix and the equivalent rotation matrix has all the direction cosines. So, basically it tells you how the moved coordinate system with respect to the original reference coordinate system.

So, we know what the orientation of this rigid body with respect is to the zero coordinate system. So, the model of the story is that for orientation we have to solve a set of coupled non-linear differential equation. These are ordinary differential equations whereas in the case of position we just integrate separately each one of them. The x component integration V x will give you x, V y will give you y, V z will give you z.

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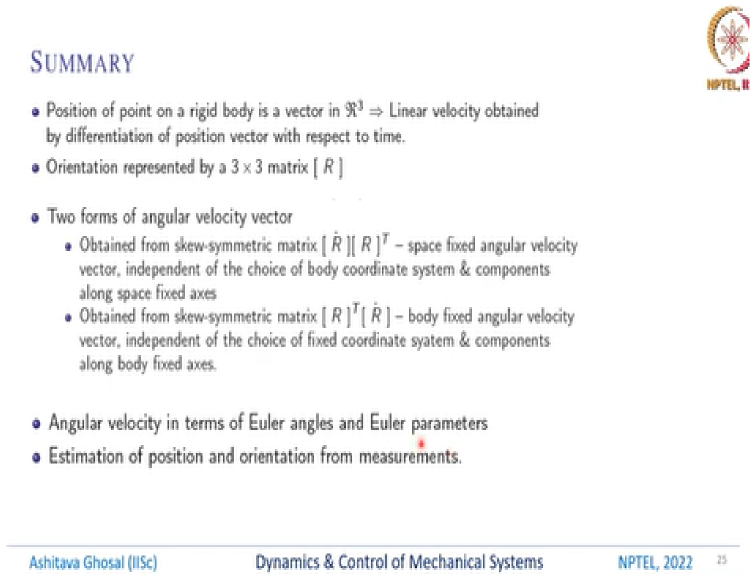

So, in summary the position vector of a point on a rigid body is a vector in 3D space. The linear velocity can be obtained by simply differentiating the position vector with respect to time. The orientation of a rigid body is represented by a 3 by 3 rotation matrix R which contains all these direction cosines $r_{ij}'s$. From here we get two different forms of angular velocity vector. So, one is a skew symmetric matrix which is obtained as $[\dot{R}]\ [R]^T$.

This is the angular velocity vector from this skew symmetric matrix is called the space fixed angular velocity vector. This is independent of the choice of the body coordinate system and components along space fixed axis. We can also obtain a skew symmetric matrix which is $[R]^T\ [\dot{R}]$ and the angular velocity vector obtained from this skew symmetric matrix is called the body fixed angular velocity vector.

This is independent of the choice of the fixed coordinate system and components along body fixed axis. The angular velocity vector from these two are related by a rotation matrix and we can also obtain the angular velocity vector in terms of Euler angles and Euler parameters if it is required. We can also estimate the position and orientation of a rigid body from the measurements of the angular velocity vector add or the linear velocity so basically by integration.