Analysis and Modeling of Welding
Prof. Gandham Phanikumar
Department of Metallurgy and Material Science
Indian Institute of Technology, Madras

Lecture - 09
Analytical solutions to weld thermal field

Welcome to the lesson on Analytical Solutions to Thermal Field. This is a part of the MOOC on Analysis and Modeling of Welding offered by NPTEL - IIT, Madras.

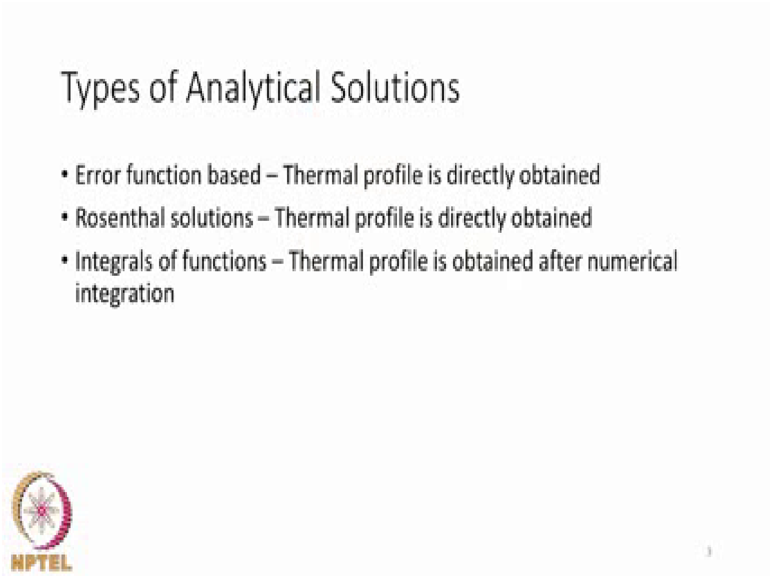
So, in this lesson, we will see what is the various analytical solutions that are available for thermal field during the fusion welding process.

(Refer Slide Time: 00:35)

So, there are some references I would like to bring your attention to the first reference by Abramowitz and Stegun, this is a book in which you would have access to all the different mathematical functions that you would need for this kind of an analysis. And the text book by Carslaw and Jaeger is the most important book for all kinds of analytical solutions to conduction in solids. And you would see that there are analytical solutions available or various geometries and initial and boundary conditions. And very often this is the book that has to be referring to when we have to bench mark any of our numerical solutions to analytical expressions of relevant geometries.

The third book I am referring to here is the book title Transport Phenomena and Materials Processing by Sindo Kou. As you would have noticed it is same author whose book on Welding Metallurgy is what is being prescribed as the text book for this course. And this second book by Sindo Kou on Transport Phenomena and Materials Processing has sections on various processes including welding. And you would see sample solutions for heat transfer, fluid flow and mass transfer for welding in this book. And these resources are referring to for all the information that is being given in the rest of the class.

(Refer Slide Time: 01:59)



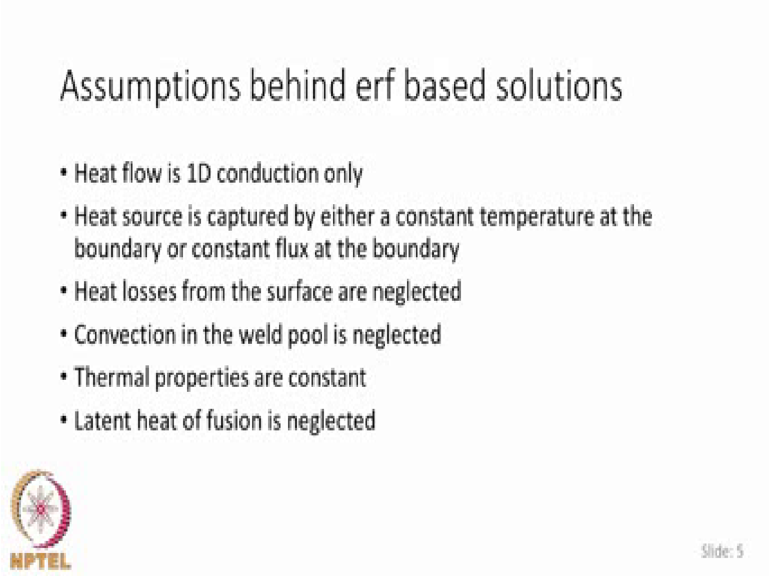
So, there are three types of analytical solutions that are generally possible in welding; one of the reasons that welding does not give it a possibility to have a large number of analytical solutions is because of the nature of the problem. We have a moving heat source which can have fairly complicated distribution of heat and we also have boundary conditions that could be a variety including conduction and radiation and different walls having different conditions.

Therefore, unlike the problem, for example, as a heat exchangers or pins, welding is the situation where the analytical solutions are very limited. And the three major categories of solutions that are available are listed here; error function based - Rosenthal solutions and integrals of functions that are listed as we see in the class.

(Refer Slide Time: 02:47)

So, the first solution given by the error function is motivated by the fact that error function is a solution to the conduction problem in one dimension. And this could then we used to see if such a one-dimensional solution is possible for welding as well. And one dimension solutions which are based on error function are limited by a number of assumptions which I have listed here.

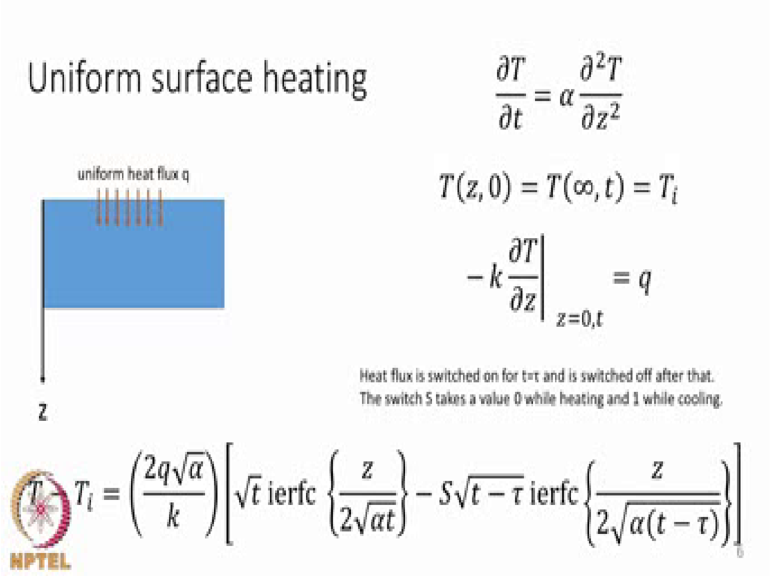
(Refer Slide Time: 03:12)



The first assumption is it is only met for 1D conduction, which generally is along the depth from the surface where the heat source is being applied. So, typically, this is used in situations where you want to only do laser hardening or surface hardening, and you want to look at the temperature as a function of depth. The heat source is basically either a constant temperature or a constant heat flux at the boundary, which is also the situation when you are doing hardening process or heat treatment process using welding kind of geometry.

The heat losses from the surface of the material are ignored; and also to note that the heat loss through conduction by other geometries, for example, in 2D or 3D geometry is not considered everything is suppose to be assumed as happening only in one dimension. The convection in any weld pool, if at all if it forms that is also being neglected. And all the properties that are relevant for the thermal transport are also taken as constant. Latent heat of fusion is also neglected, and this is mainly because the phase change itself has been neglected.

(Refer Slide Time: 04:21)

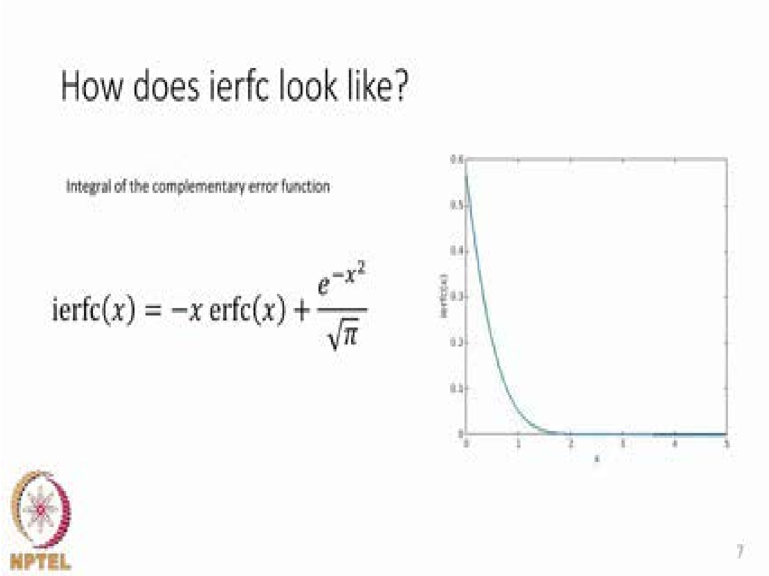


The problem definition is given here. We have a domain, which is given in the blue box; and the constant heat is given at the top, and the symbol q gives you what is the amount of heat flux that is being given from the top. The distance along which we are going to seek the solution of temperature field is given by downwards arrow z, and which means that we want to evaluate what would be the temperature as a function of z at any given amount of time.

And now you can see that the equation that is being solved is very simple; it is 1D heat conduction equation as given on the right hand side top. And the boundary conditions are given as the initial condition and the boundary condition is the ambient temperature or the pre-heat temperature of the base material. And the surface boundary condition is given has a heat flux condition balance of heat flux.

And you can see that the solution is written in two forms. The first part is the solution when we do not have a time variable subtracted with tau; and the tau is basically coming here as the amount of time for which the heat source is applied; and after the time, t is equal to tau the heat source has been switched off, which means that during say laser pulse heating, the time over which the pulse heating is applied is basically tau. And then we can see that the solution is given in terms of how much increase with respect to the pre-heat temperature we are seeing. And the second term is present only during the cooling part. And this can then be also plotted using some software such as mat lab to see how the plot looks like.

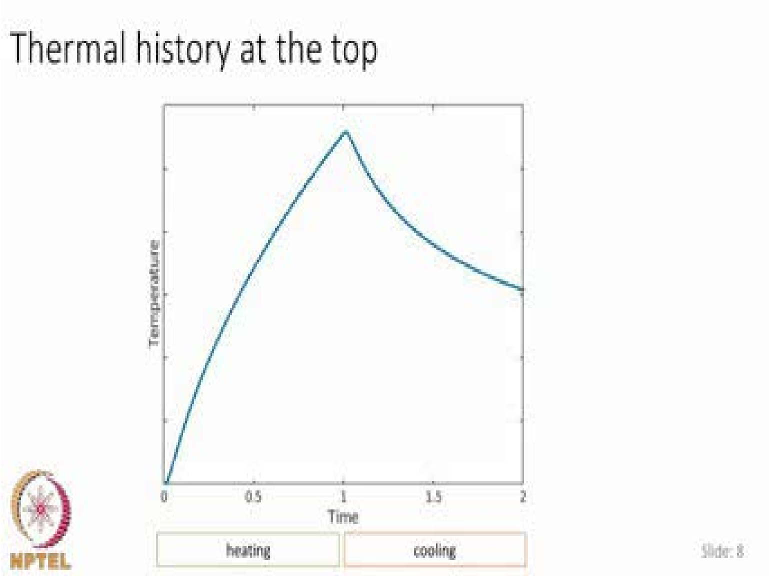
(Refer Slide Time: 06:03)



So, before we go we have to just see what is the function that we have used the ierfc function that is listed in the solution is basically the integral of the complementary error function.

Complementary error function is nothing but 1 minus error function; and the integral of it is simplified in the functional form as I have given you in the slide minus x into complementary error function plus e raise power of minus x square by root pi. Now, basically this solution is going as a smoothly varying function going downwards and reaching a flat value of 0 as you keep on increase the distance. Now this is very much suitable for the temperature distribution within a weldment, because we know that as time proceed the temperature would come down to the pre-heat temperature or the ambient temperature after sometime.

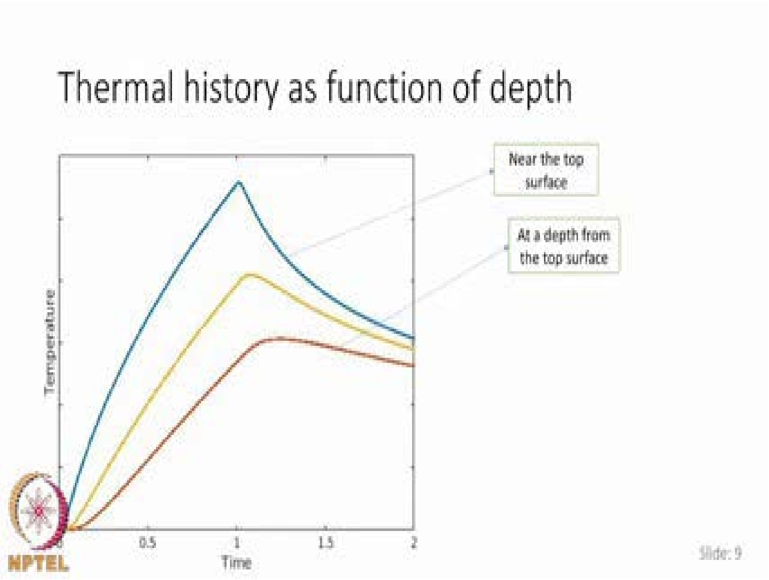
(Refer Slide Time: 06:57)



And how would the thermal history look like the temperature profile is given to you as a schematic; the y-axis where the temperature is plotted the values are not given, because they depend upon the amount of heat that we are giving. So, only the variation is being shown here. You can see that the temperature is increasing continuously, smoothly up to a particular maximum value during the heating process; and once the heat source has been switched off, then the cooling cycle starts and you can see the cooling rates are different from the heating rates.

And this kind of a profile can also be inspected at various depths, because we do have the solution available as a function of depth. And the plot I have shown you is at the very top of the material, which is being heated from the top using a uniform heat flux.

(Refer Slide Time: 07:48)



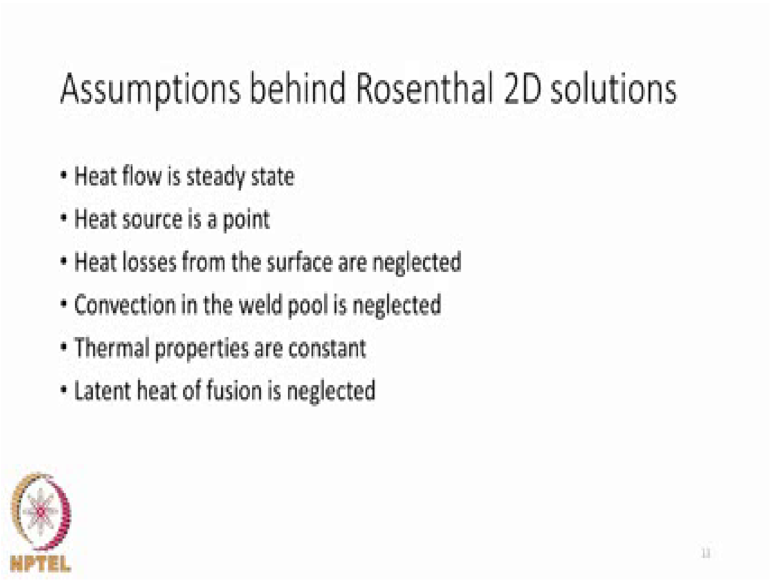
So, we can plot multiple thermal histories simultaneously. So, what I have shown here is a set of three plots. The first one at the top is the temperature history of a location at the very top surface of the material which is being heated. The second one is at a depth from the first point; and the third one is at high higher depth from the first point.

So, basically you can see that as you go down into the thickness of the material, the peak temperature that is achieved is lower, which is also obvious. And it is also clear from these plots, and you can also see that the variation of temperature is also can be determined. By a combination of such plots, we could also get what could be the gradient and cooling rate at any given location from the top surface for a 1D solution that we have right now here.

(Refer Slide Time: 08:41)

And these solutions are limited in their aspect. And if you want to go to higher dimensions, we then usually come to the classical work by Rosenthal; and he has given his solution in two modes 2D and 3D.

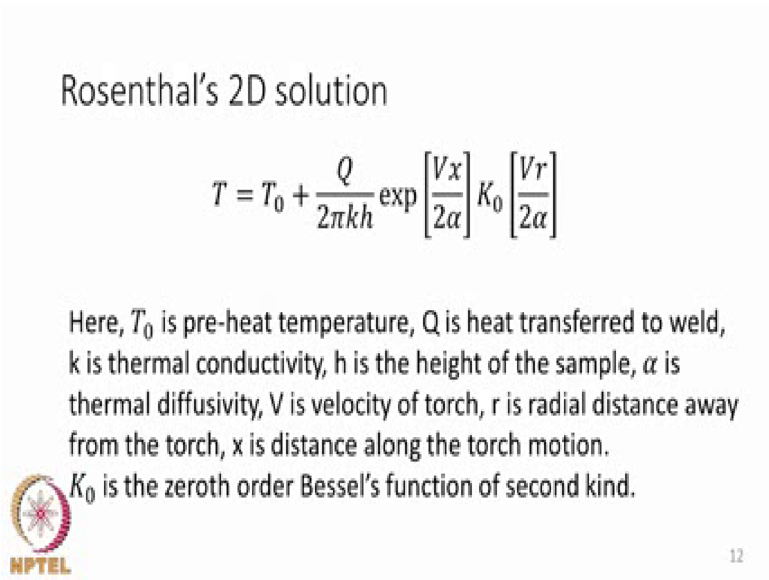
(Refer Slide Time: 08:55)



The assumptions that go behind the Rosenthal's solutions are listed here for a 2D solution; the assumptions are that the heat flow is actually in a steady state, which means that Rosenthal solution is to be applied when we have the heat source moving constantly over a long bead so that along the bead we can assume a steady state. And which also means that for capturing the temperature variations per pulsed heating etcetera, and then we will have to do some averaging before we can apply this kind of a solution. It is also assumed in a 2D solution of a Rosenthal that the heat source is taken as a point.

And the heat losses from the surface by convection and radiation are ignored; and if at all there is any liquid that is forming, then the convection in that liquid pool is also ignored. The thermal properties like the thermal conductivity or heat capacity are taken as constant values, which means the temperature variation of these parameters is not considered. And then the latent heat of fusion is also ignored in this particular solution. And we can see that essentially the liquid pool is defined only by the temperature correspond to the melting point, but otherwise no other physics regarding the phase change is considering in the Rosenthal solutions.

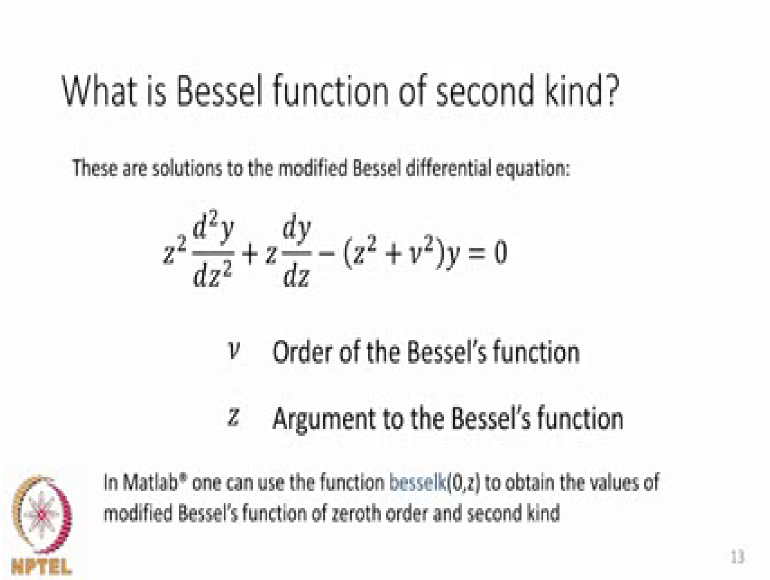
(Refer Slide Time: 10:14)



The 2D solution is written in this form here. We always see that the solution is written to indicate how much higher temperature is available beyond the pre-heat or base material temperature at ambient conditions, and that is because on the right hand side we have T 0 plus a function. There are two parts to this solution; the first part is an exponential function and then the second part is basically what is called as the zeroth order Bessel's function of the second kind. What are Bessel's functions, we will come to it in a moment.

And the various terms that are given in this expression are listed here; T 0 is a pre-heat temperature, Q is the amount of heat transferred to the weld; that means, that we have to also take into account the efficiency with which the heat is being transferred to the base material to arrive at the value of Q. K is the thermal conductivity, h is the height of the sample, which means that we are going to take it for a plate. So, what is the plate thicknesses that go in to the expression. Alpha is a thermal diffusivity, v is a velocity of the torch, and x is the distance along x-direction, which is also the distance along with the torch is moving. And r is given as a radial distance away from the center of the torch. And in the last part of the expression, we see K 0 that is what we have refer to as the zeroth order Bessel's function of second kind.

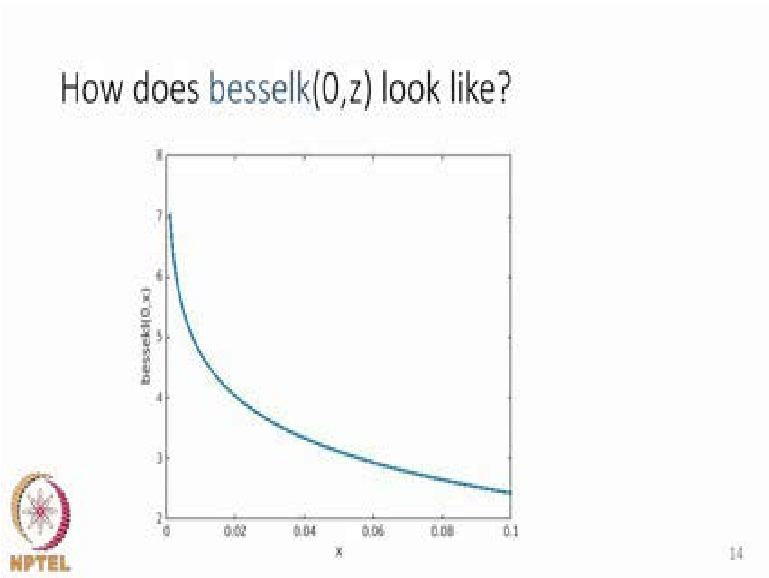
(Refer Slide Time: 11:43)



What are Bessel's functions I am just giving you a very simple explanation here. It is essentially solution to the differential equation that is written here. And we can see that there is a parameter in the differential equation nu, which will tell you what is the order of the Bessel's function and that is taken as 0 for the solution that is relevant for the welding situation here.

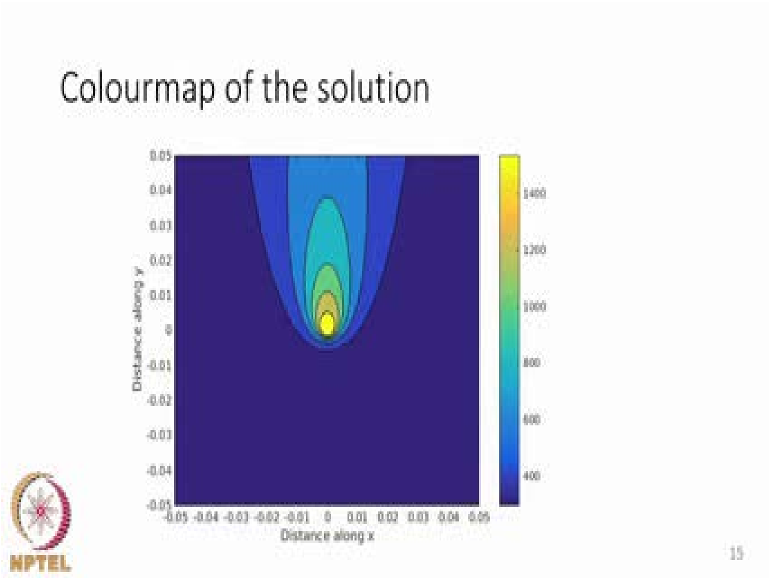
And z is the argument that we are giving to the Bessel's function; and the value of y which satisfies this equation is what we are taking as the Bessel's function that we need for the solution. And one can actually use a function called Bessel k which is available in matlab library to plot this particular Bessel's function as a numerical value to estimate what would be the solution.

(Refer Slide Time: 12:29)



And the nature of the variation of Bessel's function of a zeroth order second kind is shown here. It is also basically a smoothly decaying function going from a high value at low value of the argument to a very low value at high value of the argument.

(Refer Slide Time: 12:46)

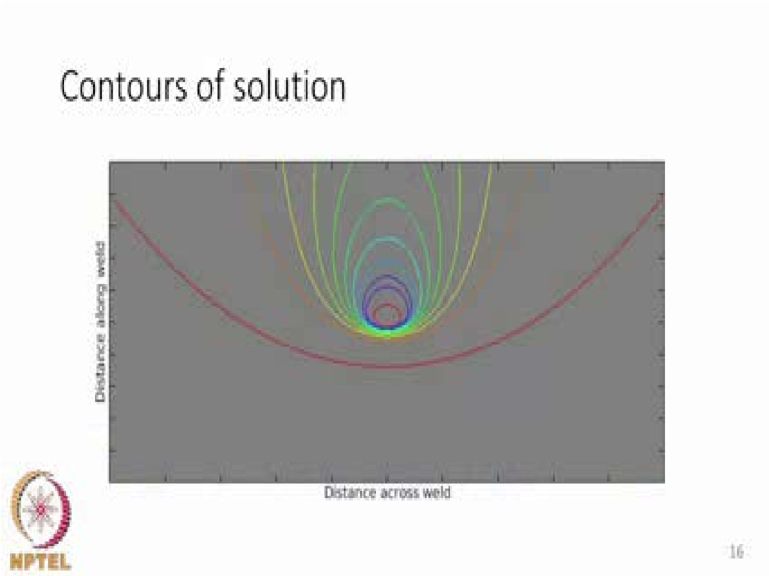


And we can actually then go ahead and calculate what would be the temperature at every location in the domain away from the center of the heat source and then depict that particular solution in the form of contours. So, I am showing you here, what is called the filled contour plot, which means that between contours, we have the color that are filled, so that you can see a visual appearance that is similar to how a real weld would look like. The color map that is used is shown in the right hand side, and you would see that at the center of the weld, you have a light color - yellow color showing you that is the molten part and then as you go away you can see how the temperature going.

Now you can see that a head of the weld pool, you have the contours coming very close. The distance between each contour, each pair of contours will give you an idea on the temperature gradient that is located there. And as the heat sourcing is moving in the situation vertically downwards direction, and then you can see that there is a trailing of the heat behind the torch.

And if you draw a horizontal line across the weld pool, then you can plot the temperature profile to see what would be the weld pool width; and you can also use the same plot to see what would be the heat affected zone width. And then you can draw the vertical line to also see what would be the temperature gradient along the weld directions. So, you can actually extract a number of parameters that are possible from this kind of a solution. You can actually depict this solution by other ways of plotting the data also, and these are available in software, such as matlab which you can explore at a separate location.

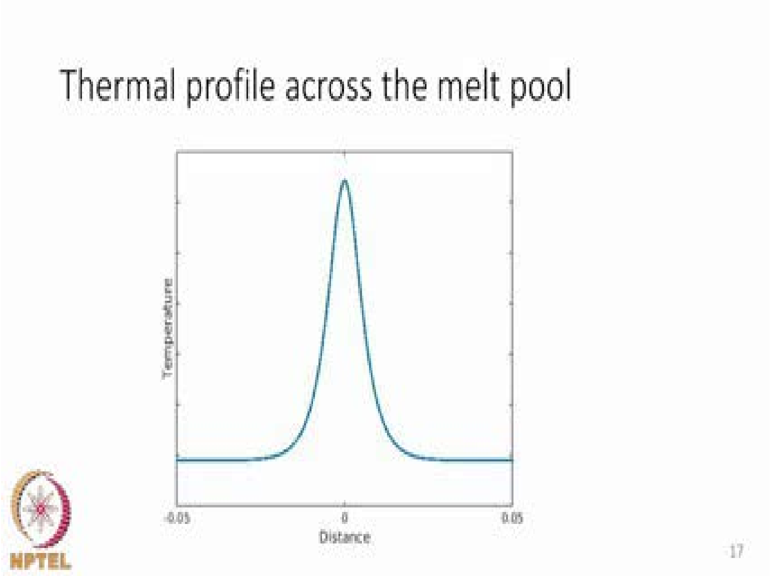
(Refer Slide Time: 14:27)



The contours can also be shown separately without filling, and I am showing you here. For example, it is very clear that the distance between the contours is showing you how much of temperature gradient is present at that particular location. By drawing two contours corresponding to the liquidus and solidus, you can see what would be the width of the partially melted zone; and by drawing contours between the solidus temperature and the temperature, when the microstructure is going to change significantly, you can find out what would be the width of the heat-affected zone.

So, you can see that the analytical solutions that are available from Rosenthal's approach can be used to estimate as a first order what would be the widths of various zones in the weldment.

(Refer Slide Time: 15:14)



The thermal profile across the weldment is plotted here. And you can see that this would be the schematic temperature profile we have been drawing every time. And you can see that this kind of a profile where this temperature is smoothly raising to a peak value at the center of the weld pool, and then decaying again back to the ambient temperature far away from the weld pool is possible to be taken as output from the Rosenthal the solution. The amount of time spent can then made using a similar plot except that the x-axis has to be time instead of distance.

(Refer Slide Time: 15:47)

Rosenthal 3D solution

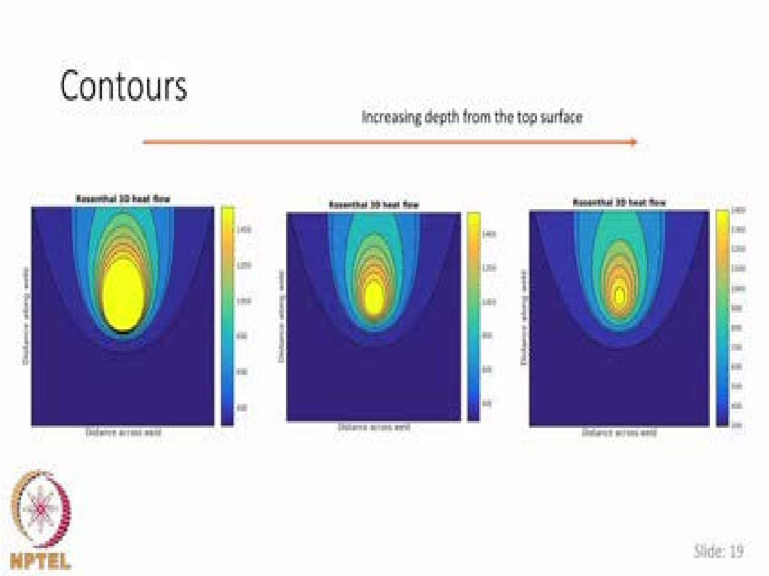
$$T = T_i + \frac{Q}{2\pi k r} \exp\left[\frac{V(x-r)}{2\alpha}\right]$$

$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

Now, the Rosenthal solutions are available also in three dimensions, which means that for a thicker plate one can also do a similar kind of a solution; and it is a simpler expression and only the exponential function is present. So, you can see that the solution has Q which is the same meaning as earlier, namely the amount of heat that is transferred to the material. k is the thermal conductivity.

R, in this case is not the two-dimensional radial distance away from the center of the weld pool, but actually it is the three-dimensional radial distance from the center of the heat source, which means that while evaluating you can also take into account the depth from the top surface to calculate the radial distance. And V is the velocity of the torch, alpha is the thermal diffusivity.

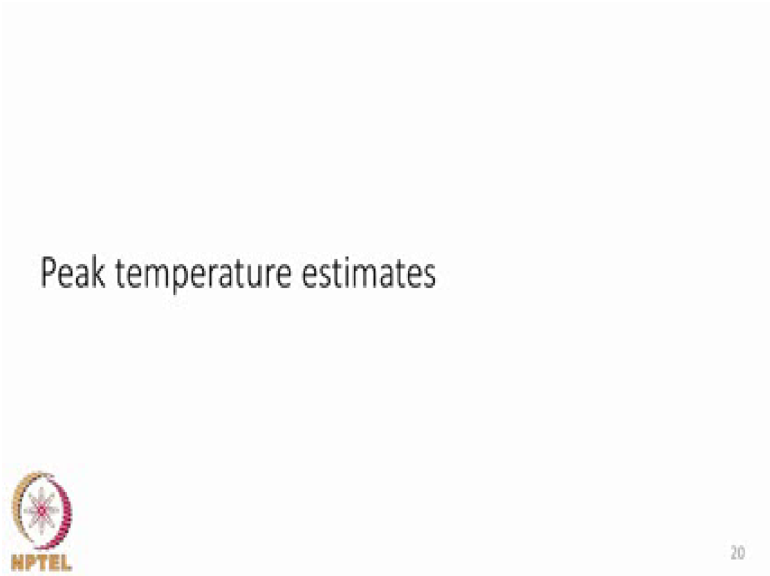
(Refer Slide Time: 16:36)



And you can then also plot the temperature profile at various depths. And what we are seeing here is a set of three plots. The first plot is taken at the very top of the weldment, which means that the amount of a material that is molten as is evident from the very first contour in the yellow color that is at the largest amount. And at a small depth from the surface, the second plot is showing you the temperature profile; and you can see that the yellow circle as a shrunk which means that as you go from the surface inwards in to the thickness of the weld then you can see that the weld pool as to shrink. And at a higher thickness from the surface of the weld pool, you can see that the weld pool as shrunk significantly as it is evident from the third plots.

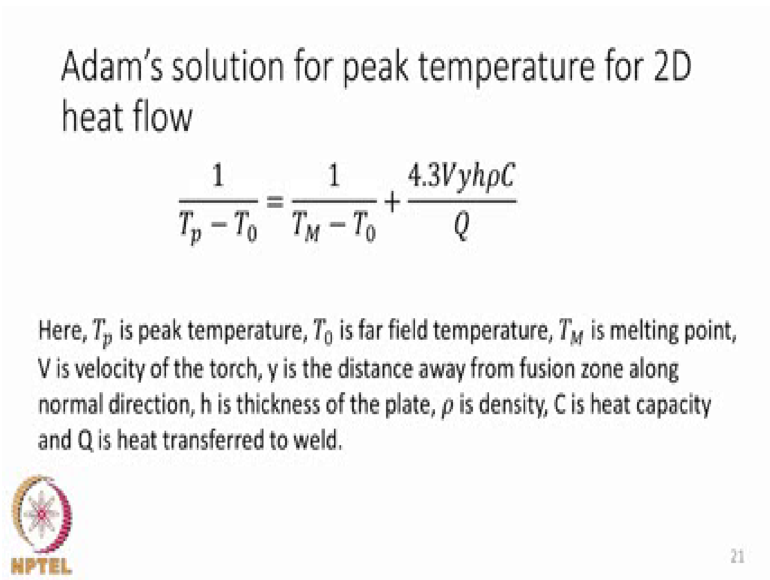
So, you can use z variable, and see how the temperature distribution is present in the entire thickness of the plate, and plot the temperature profiles in different manners. You could use contours in any of the directions that you want x y, y z or x z to see how the temperature contours are a lined up to see in what direction the heat is flowing and then confirm your understanding with the thermal model that is present.

(Refer Slide Time: 17:54)



So, apart from the Rosenthal solutions, we also have some analytical expressions that are available attributed to Adam's work to show you what would be the peak temperature that will be achieved on the top surface upon heating. And this is usually used to estimate the peak temperature under laser heating.

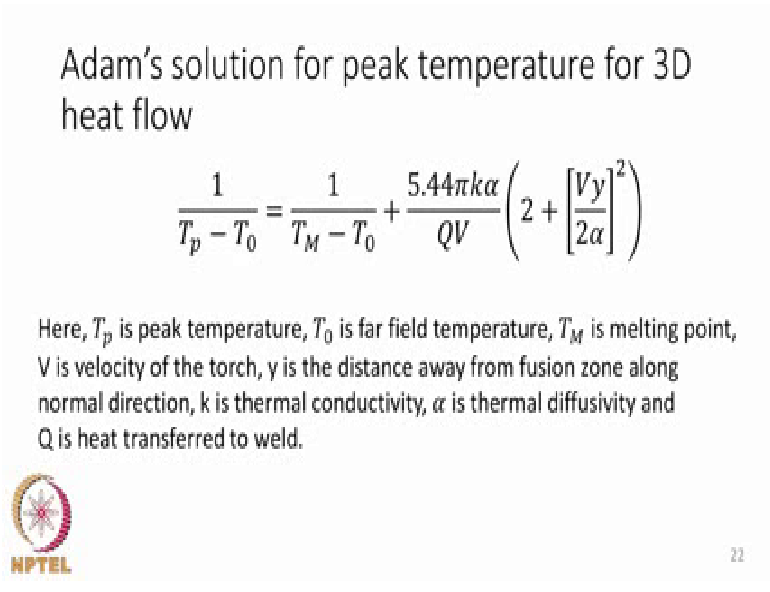
(Refer Slide Time: 18:11)



And these expressions are much simpler and to be used to essentially see a calibrative exercise to find out how to go about for the numerical solution later on. And you can see that the solutions are available for 2D and the peak temperature is then given as a function of the right hand side, where the melting point also is coming to picture, and the various parameters that we are using in the expression are listed here.

T M is the melting point; T 0 is a far field temperature; V is a velocity of the torch, y is the distance away from the fusion zone in the normal direction; h is a thickness of the plate. So, this is a solution for 2D, which means that in the depth direction, we do not have a information; and the thickness in the depth direction is what is being used as h. Rho is a density, C is the heat capacity, and Q is the amount of heat effectively transferred to the weld plate. So, using this expression, you can estimate the peak temperature, and which means that this expansion will not give you the thermal profile at a depth from the top surface.

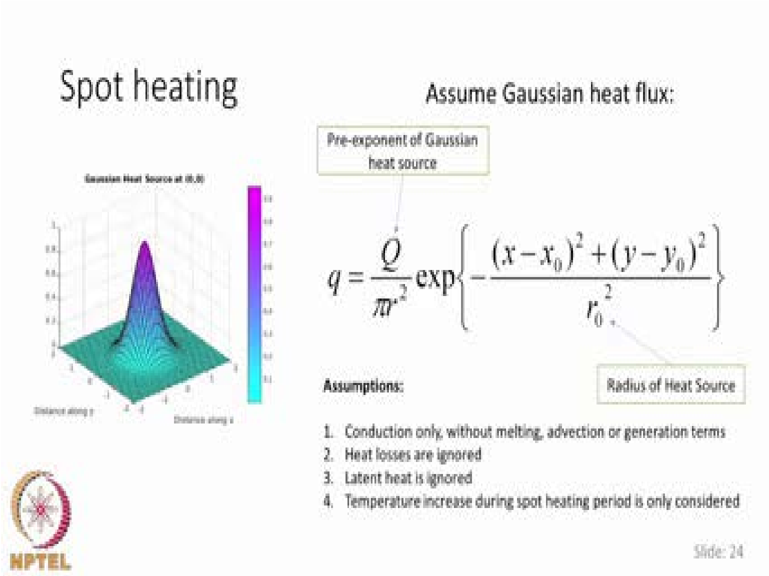
(Refer Slide Time: 19:14)



Such expression is also available for 3D; and you can see that in the 3D expression, you will have little more number of terms; and all the expressions are of the same meaning as in the previous slide. And you can then substitute the various quantities to obtain what would be the peak temperature for a 3D heat flow.

Please note that this is only giving you peak temperature, it will be on the surface except that the three-dimensional nature of the heat flow by conduction is being taken in to account. This does not give you the temperature profile at a depth from the top surface.

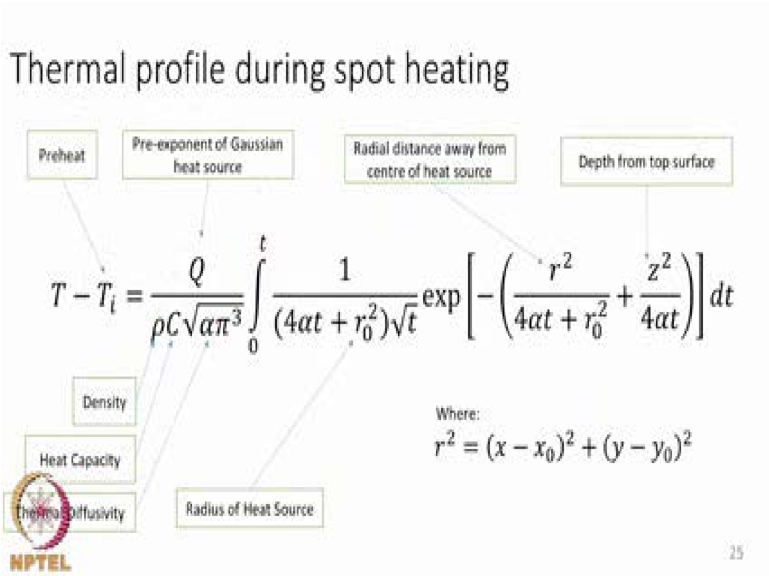
(Refer Slide Time: 19:55)



Now, there are also some other solutions that are available which are possible to be applied when the heat source is little detailed such as not a uniform heat source or a line heat source or a point heat source, but let us say a Gaussian heat source. So, naturally when the heat source is described elaborately then the solutions are not readily available to plug into a calculator. And you can see that they are available as in integral and such solutions I am giving you now in the next couple of slides.

Let us assume that the heat source is taken as a Gaussian heat flux and you can see that the Gaussian heat flux is described here, so which means that the surface condition for heart flow is given by a functional form like e raise of minus r square that kind of a form. And this solutions are under the assumptions as follows; only conduction is taken into account, no melting is taken into account, and melt full convection or any generation of heat are not taken. Heat losses by various modes on different walls of the weldment are ignored, and the latent heat of fusion is ignored, the temperature increase during the spot heating is only the sought solution.

(Refer Slide Time: 21:03)

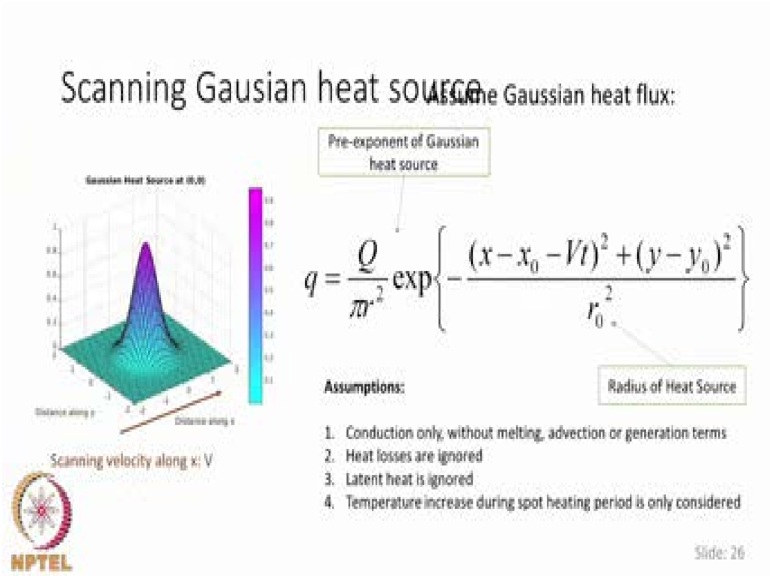


The thermal profile during spot heating that is when the heat source is Gaussian, but not moving then the temperature profile is given as follows. You can see that it is an integral of an exponential function. And you can see that the exponential function will have both x, y and z variables, which means that as a function of various distances away from the center of the heat source. In all the three directions, you can actually find out what will the temperature. By evaluating this integral at various distances and times, you can actually find out how would the temperature field evolve in the entire domain. And this would require the numerical integration, and one can use a language such as mat lab to achieve the solution in a numerical form.

All the terms are exposed here using boxes; you can see that Q is a pre exponent of the Gaussian heat source; it is not the same as the Q in the previous expressions. It is actually only the pre exponent of the Gaussian heat source, because here we are taking the heat source to be in a Gaussian form not as a line form or a point form. And r is a distance radically away from the center of the heat source; z is the depth from the top surface.

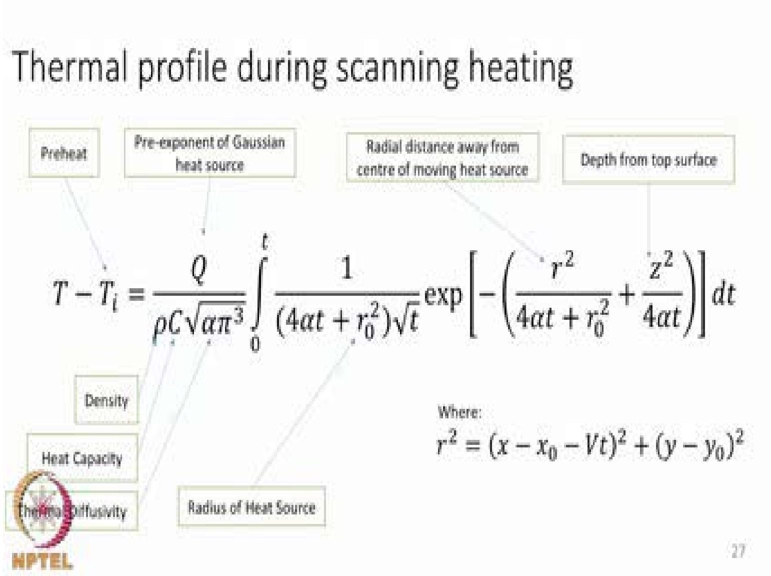
Rho, C is density and heat capacity respectively; alpha is a thermal diffusivity. r naught is basically the radius of the heat source; as you can see that radius of the heat source is going into the denominator in the exponential function, which means that it is one of the most sensitive parameters that to effect the temperature of the weld pool. And the radial distance is expanded in the form of the x and y distances; and x 0 and y 0, are the center of the heat source.

(Refer Slide Time: 22:43)



And the same thing can be also shown for a 3D heat source and the only difference is that the distance along the x will come continue to take form which is basically affected by the velocity of the torch that is why an extra variable V t is coming in to that part.

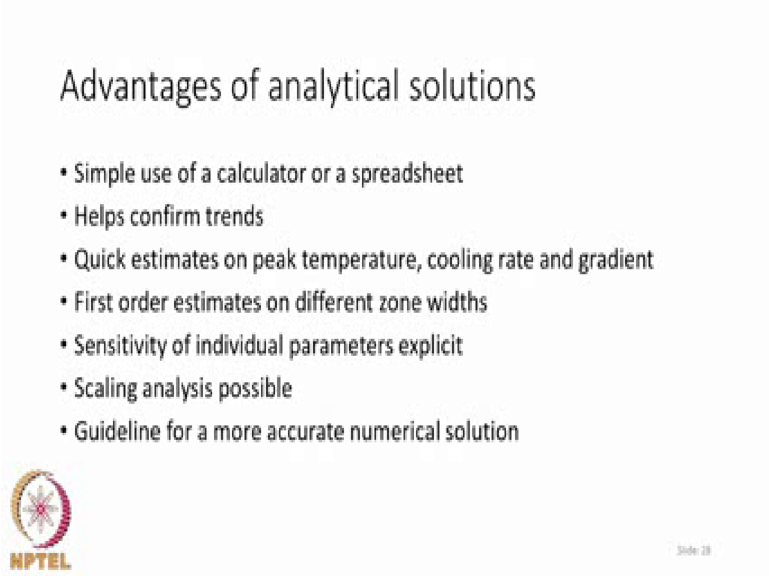
(Refer Slide Time: 23:01)



And how would the thermal profile look like, it would be a basically the same except for that the radial distance as to now take into account, the motion of the origin of the heat source along the x-direction, and V is the velocity of the torch along the x-direction, and t is the time elapsed. So, you can see that the origin of the heat source has changed by that much and that is only thing that will differ in the solution.

And so by plugging in the various, various values of the parameters, you can find out the heat profile at any x, y z location within the block for a moving it source which is Gaussian in nature. This is very nice because most of the heat sources such as in arc welding, the heat source can be described as a Gaussian. However, if the heat source is different from a Gaussian then we will not be able to use this kind of a solution.

(Refer Slide Time: 23:51)



So, as we have seen now a various number of analytical solutions that are available, we can actually appreciate what would be the advantages because of these solutions. And the first advantage is that you can use a calculator or a spreadsheet like an excel sheet or open office calc to estimate what it will be the temperature profile. And which means that without going to a programming or a numerical simulation, you can estimate the temperature profiles or the distances between two temperature contours or for example, to also find out the depth and width of the pool etcetera.

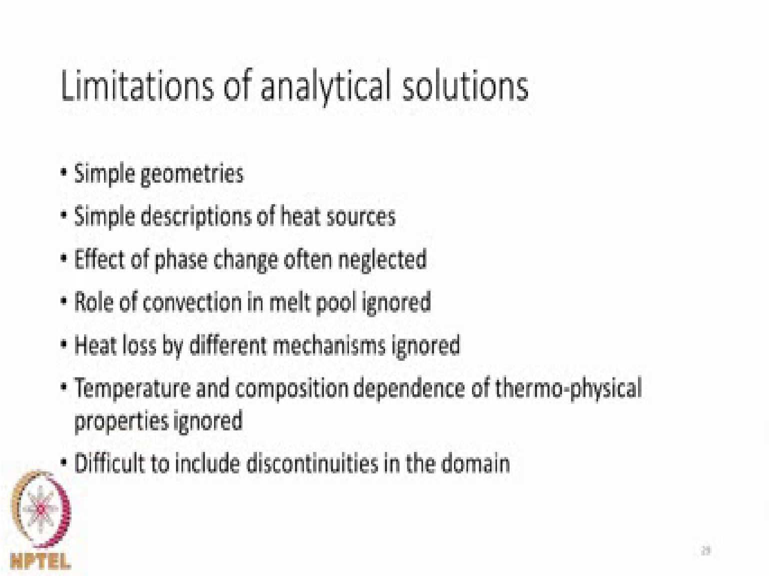
And very often, we want to confirm the trend of the numerical solution or the analytical solution as we change particular parameters. So, we could do that now quite easily, because the analytical form is available and so by differentiating with respect to the variable we want to change we can actually confirm the trend of the temperature profile variation. Now, we can also use these expressions to quickly estimate the peak temperature, cooling rate, gradient etcetera. I will illustrate to you one such set of problems we can do numerically at the end of this particular module. First order estimates on different zone widths can be made, which can be then corrected later on with a detailed stimulation.

The sensitivity of different parameters can be made and they were are all actually explicitly available in the expression itself, but then we can vary them to see how much the temperature would vary when a particular parameter is changed by let us say 5 percent or 10 percent. Scaling analysis is possible to estimate what would be the scale with which a particular parameter is changing as we change the entire expression. And this is actually going to act as a guideline for a more accurate numerical simulation, which means that numerical simulations when we take up we need to already have a rough idea of how the solution is going to look like.

And these analytical expressions will help you shall solve that. They also help you in forming something like a calibration exercise for the numerical solution itself. Not to withstanding all these advantages of analytical solutions, we also have limitations of these solutions some of the limitations I have listed here as you have seen already the solutions are for very limited set of geometries. You can see that they are basically for thin plates or thick plates in Cartesian geometry, and that is about it. So, if you are looking at for example, the thermal profile for fins on a cylinder shell, then normally these solutions may not be applicable.

And the heat sources are also falling into only a very small set of simple descriptions, for example, a point heat source, a line heat source or a Gaussian heat source. So, if you have for example, top hat or a downward kind of a heat source that is given then we will not be able to use these expressions.

(Refer Slide Time: 26:38)



And the effect of phase change as been completely neglected, and as we have seen in the formulation phase change requires a lot of information to come in or regarding the melting and the I will release of latent heat etcetera, and all thus has been ignored in the analytical expressions. And in situations, where the latent heat is going to play role, then we will need to correct the solutions by going to a numerical solution that can take into account such a release of latent heat.

Role of convection has been completely ignored. We have already seen that the shape of the weld pool is affected significantly by the convection that takes places in the melt pool and that has been ignored which means that again the analytical solutions are only give you a rough estimate of the thermal field, but accurate descriptions of how the weld pool shape will evolve as a function of the process conditions is only possible when we take into account the convection in the weld pool.

Heat loss by various mechanisms such as radiation and convection on the top or by excessive heat loss by a copper back up at the bottom of the plate, these are all completely ignored. Which means that when we want to solve a thermal profile solution by looking at the effect of copper backup setup with or without that for example, then these analytical expressions may not be of much use, one may I have to go the numerical solutions in some situations like that.

The temperature and composition dependence on various thermal physical properties has been neglected. As we know that most of the physical parameters are changed by the change of temperature; and if those are also significant then we will have to go to numerical solutions. And there are situations where in a real weld you may have discontinuities for example, you may have porosity, you may have a missed joint, and you may also have intentionally certain gaps given in the weld configuration, these kinds of situations are totally out of those scopes for analytical solutions. And to tackle them, you will have through then go for numerical simulations using a simulation package.

So, having said that then we will see how we can look at numerical simulation by telling you how they go about the solving the equations, and some simple numerical solutions to illustrate what is possible in a stimulation software. So, we look at them in a later lesson; and for now with these analytical solutions, we would say that we will close the analytical solutions to weld thermal field at this point.

Thank you.