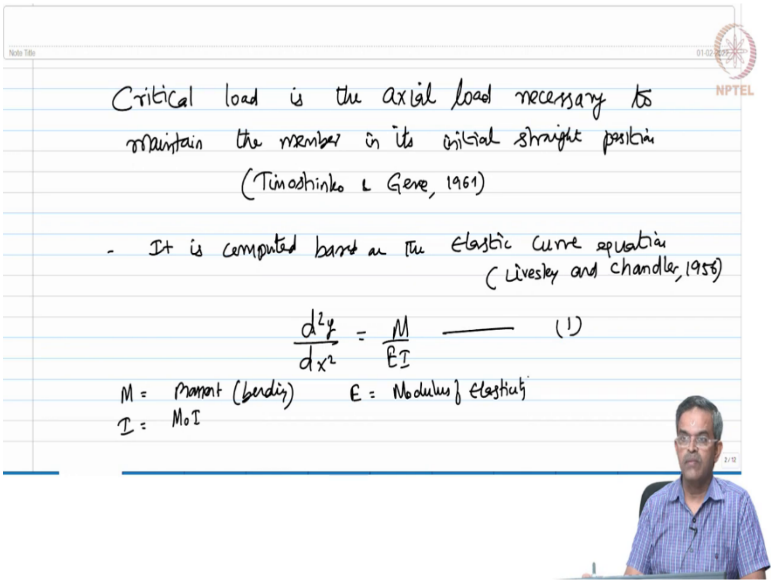
Advanced Design of Steel Structures

Dr. Srinivasan Chandrasekaran
Department of Ocean Engineering
Indian Institute of Technology, Madras

Lecture - 27
Euler's load

Welcome to the 27th lecture on Advanced Steel Design which is now going to focus on estimating Euler's critical load. So, in the last lecture we discussed about the conditions for stability. We defined stability, then we have also defined what is a critical load. Let us rewind that slightly and understand what do we mean by a critical load.

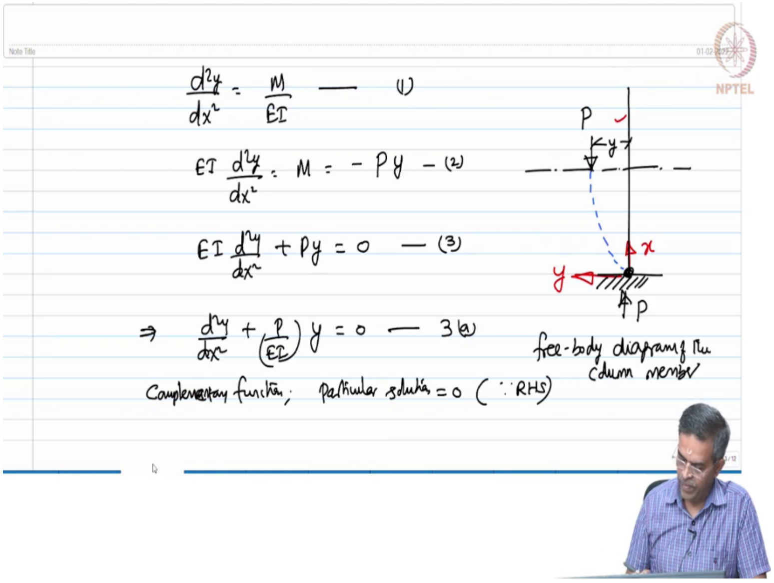
(Refer Slide Time: 01:08)



We can say the critical load is the axial load necessary to maintain the member in its initial straight position. It is a classical definition which is given by Timoshenko and Gere in 1961. The immediate question comes how do you estimate this critical load? Critical load is computed based on the elastic curve equation. So, the elastic curve equation is classical theory, we know that $\frac{d^2y}{dx^2} = \frac{M}{EI}$.

We call equation number 1, where M is the moment, to be very clear bending moment, I is the moment of inertia of the cross section and E is the modulus of elasticity of the material. And, y and x are defined according to this figure which I am going to draw.

(Refer Slide Time: 03:48)

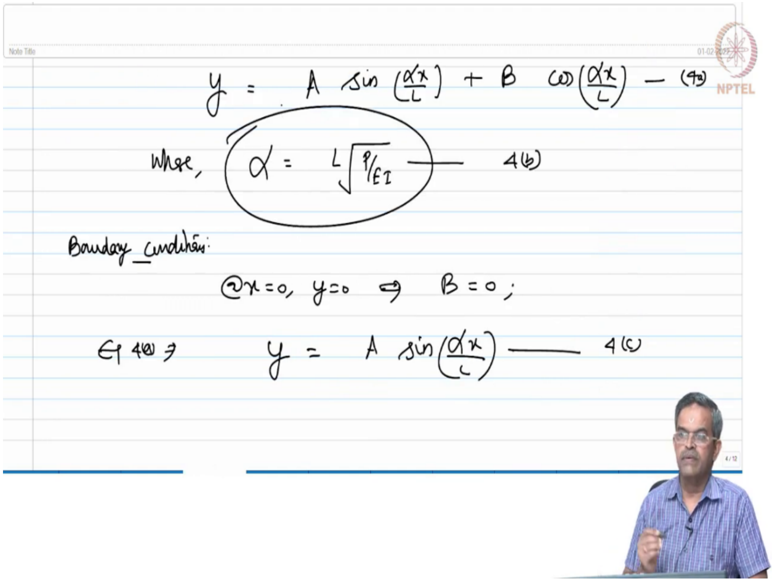


Let us say I have the initial portion of the column this way. We apply a load here, then at any section let us say this is my deflector profile and the load is now shifted here. The distance of this shift is y whereas, this is my origin and this becomes my x axis and of course, this is my y axis where I am measuring y in that axis. This is a free body diagram of the column member.

So, now we understand $\frac{d^2y}{dx^2} = \frac{M}{EI}$, let us go ahead. With reference to this figure, let us make the following statements $\frac{d^2y}{dx^2} = \frac{M}{EI}$which I can say, this is equation number 1 $EI\frac{d^2y}{dx^2} = M$ which in my case is going to $EI\frac{d^2y}{dx^2} = M$=-Py. Why negative? Because, this is going to open up the curvature, going to open up the curvature. So, $EI\frac{d^2y}{dx^2} + Py = 0$.

So, we can now say $\frac{d^2y}{dx^2} + \left(\frac{P}{EI}\right)y = 0$ is a classical second order ordinary differential equation, whose solution has got two components. The complementary function and the particular solution. Since, RHS is 0 particular solution will be 0 in this case.

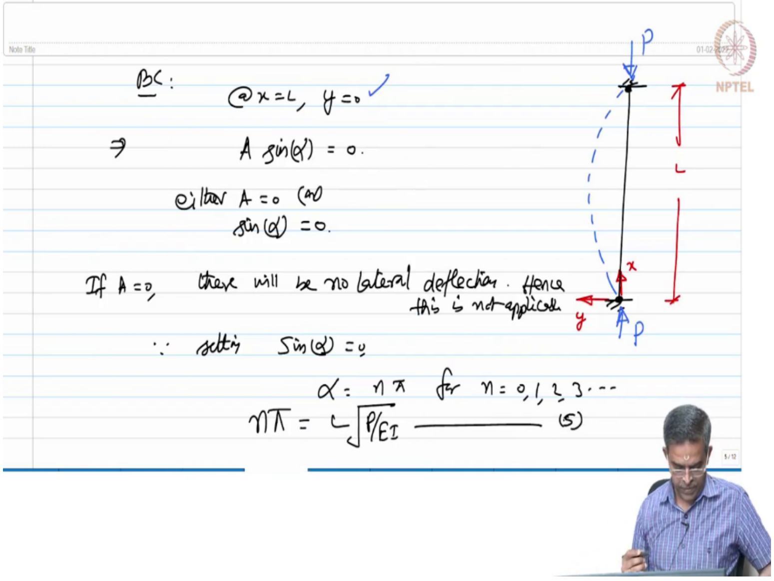
(Refer Slide Time: 06:51)



We can write the complementary solution like y is let us say some variable $y = A\sin sin\left(\frac{\alpha x}{L}\right) + B\ \cos cos\left(\frac{\alpha x}{L}\right)$; equation number 4. Where the above equation alpha $\alpha = L\sqrt{\frac{P}{EI}}$, we call this as 4a. Say equation 4 has got two unknowns A and B, we need to find out them. I can apply the standard boundary conditions and evaluate these constants A and B.

So, the boundary conditions at x equal 0, y is 0. See here at x equals 0 y is 0 which implies applying in this equation 4a, B becomes 0. Therefore, equation 4a is now written as $y = A\sin sin\left(\frac{\alpha x}{L}\right)$.

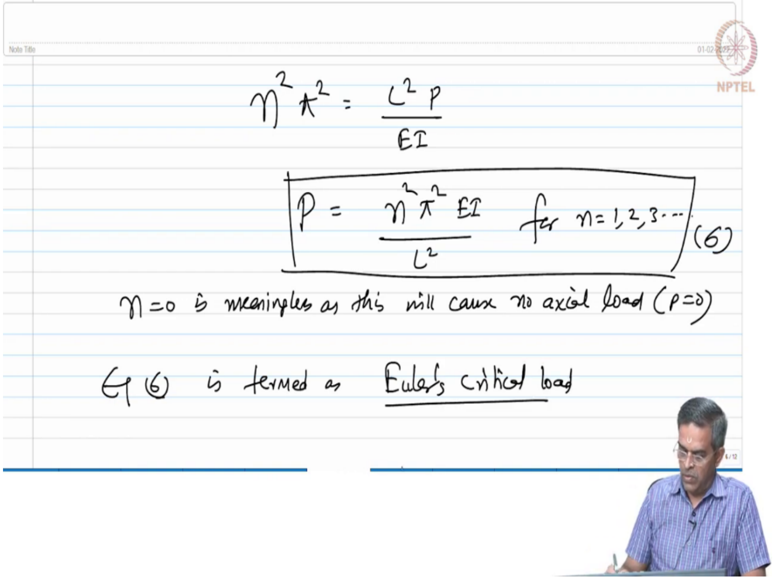
(Refer Slide Time: 08:40)



Let us supply next boundary condition. There is one more boundary condition at x equal to L, y will be again 0. I have a column both ends position restrain, x is measured from here and this is my y axis and the length of the column is L and the deflector profile of the column is this under the load P. So, at x is equal to L, y is again 0 which implies $A \sin sin\,(\alpha) = 0$.

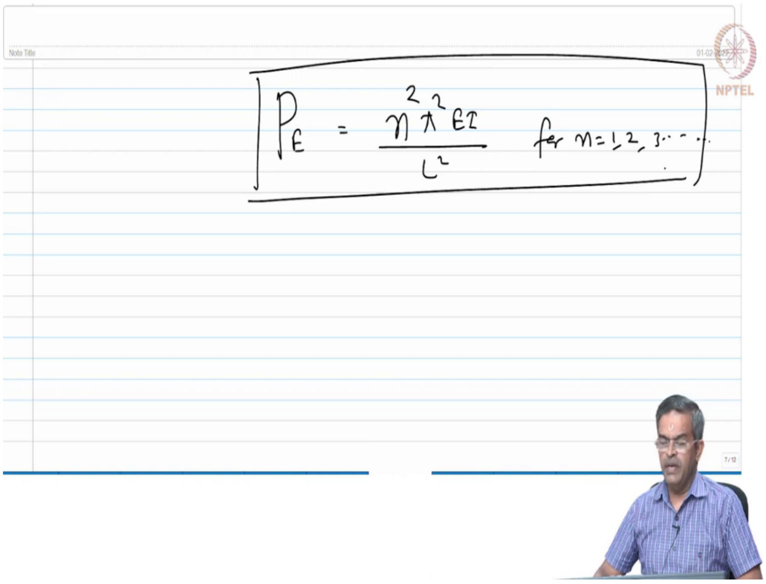
So, which means either A should be 0 or $\sin sin\,(\alpha)$ should be 0. If A equal 0 look at the original equation, there will be no lateral deflection, is it not. Hence, this is not applicable condition. Therefore, setting $\sin sin\,(\alpha) = 0$, we know alpha $\alpha = n\pi$ for n equal 0, 1, 2, 3 etcetera. So, now we can say we already know that alpha $\alpha = L\sqrt{\frac{P}{EI}}$. So, let us say $n\pi = L\sqrt{\frac{P}{EI}}$. Call this as equation number 5.

(Refer Slide Time: 11:18)



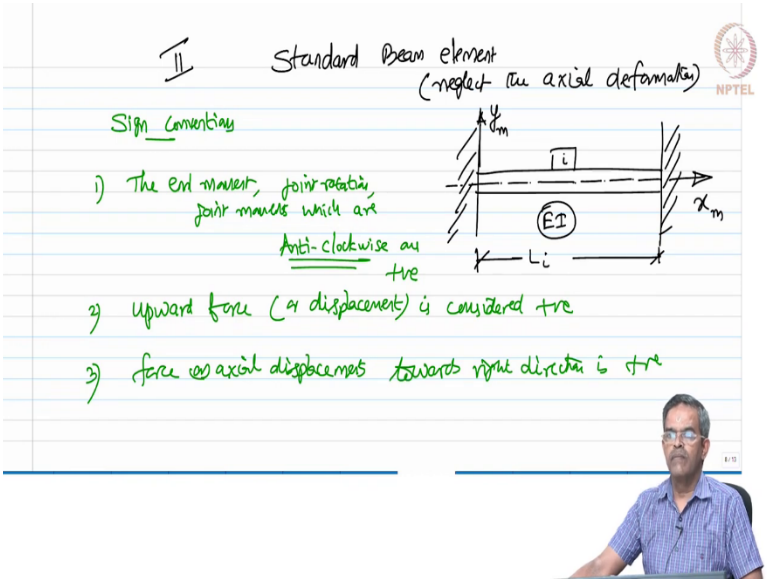
This says $n^2\pi^2 = \frac{L^2P}{EI}$ which says $P = \frac{n^2\pi^2EI}{L^2}$ for n equals 1, 2, 3 and so on. n equals 0 is meaningless as this will cause no load, that is P will become 0. So, when there is no axial load, the condition applicable is not effective. We call this equation as equation number 6. So, equation 6 is termed as Euler's critical.

(Refer Slide Time: 12:28)



So, we can now say $P_E = \frac{n^2 \pi^2 EI}{L^2}$ for n equals 1, 2, 3 etcetera. So, we have now estimated the Euler's load. Now, let us try to find out the stability functions considering a standard beam element.

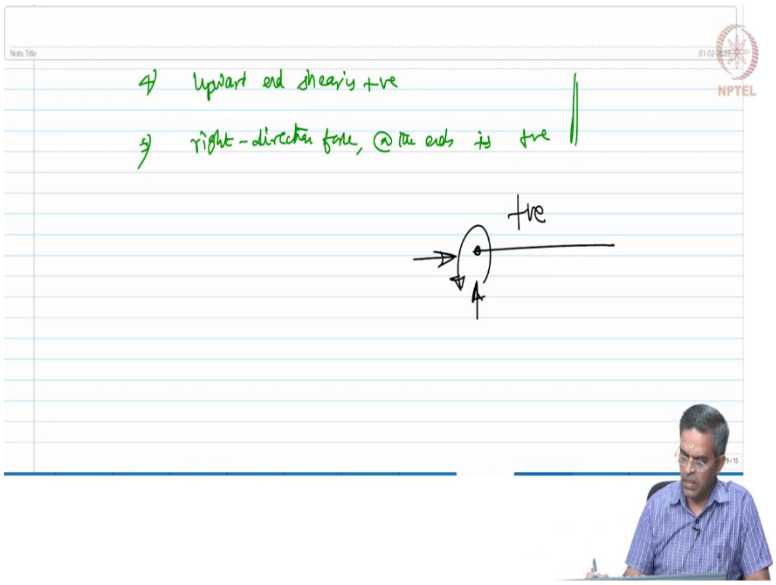
(Refer Slide Time: 13:06)



So, let us now consider a standard beam element and we neglect the axial deformation. Let us consider the standard beam element which is fixed at both the ends. The standard beam element. This is x axis, this is y axis, I put m indicating is for the member. We take a prismatic section so, EI is constant and this becomes my span of the member of the i^{th} member L_i. This is my i^{th}member. So, there are some sign conventions which we have to follow.

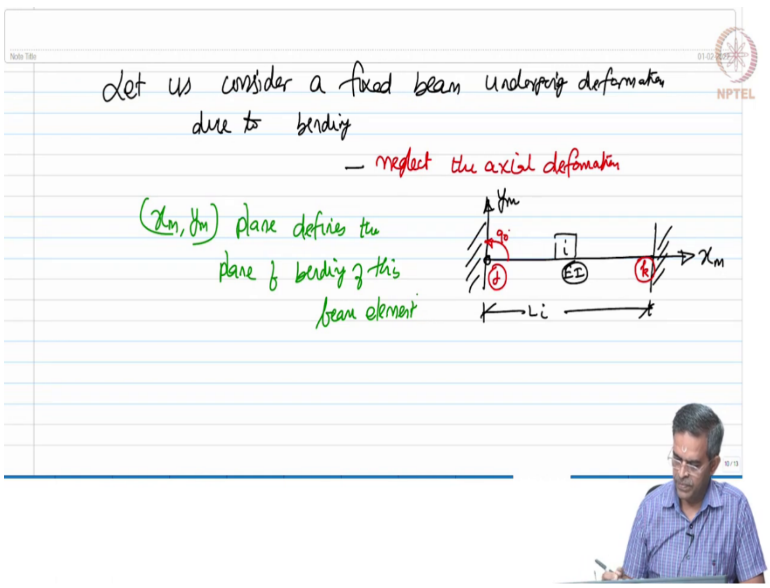
So, let us see what are the sign conventions. (1), the end moment, joint rotation and joint moments which are anti-clockwise or positive. The convention (2), upward force or displacement is considered positive. (3) Force or axial displacement towards right direction is considered positive.

(Refer Slide Time: 16:03)



(4), upward end shear is positive. (5), right direction force acting on the beam at the ends is positive. So, these are some sign conventions which we will be following. So, let us mark them. So, we say at any joint anti-clockwise, upward towards right or positive.

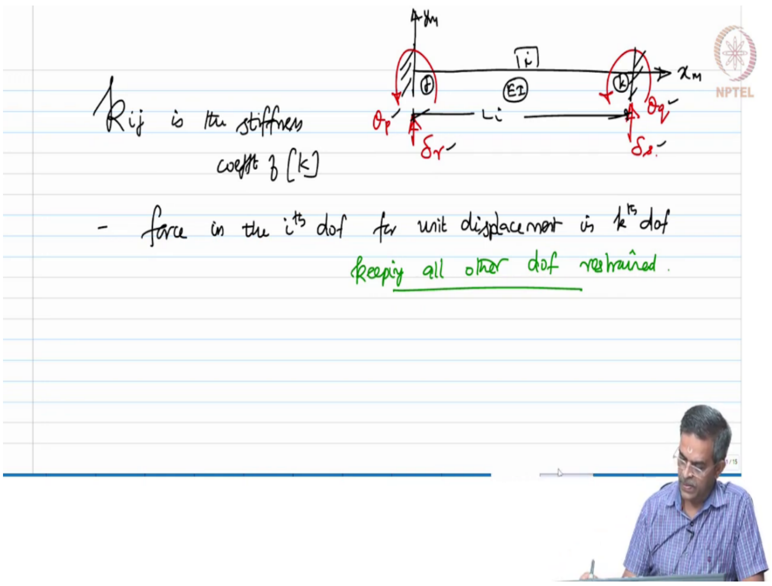
(Refer Slide Time: 16:54)



Now, let us consider a fixed beam undergoing deformation due to bending. We neglect the axial deformation in the member, that is deformation along the axial of the member is neglected. Now, the standard beam is what is shown here, the standard beam. Let us draw this figure again and mark it here. This is my origin, this is my x axis of the member, anticlockwise 90 degree is y axis, the i^{th} member. This member has got uniform cross section EI, span of the member is L_i and this is considered as a j^{th} end of the member. This is considered as a k^{th} end of the member.

So, there is a order by which we mark this x axis and y axis, identify the origin, mark the x direction or x axis along the length of the member, y axis is anticlockwise 90 to x axis and mark on j^{th} and k^{th} end of the member. And, very interestingly $x_m y_m$ plane defines the plane of bending of this beam element. Neglecting axial deformation; now, we have to identify the degrees of freedom for this beam. Let us mark them.

(Refer Slide Time: 19:40)

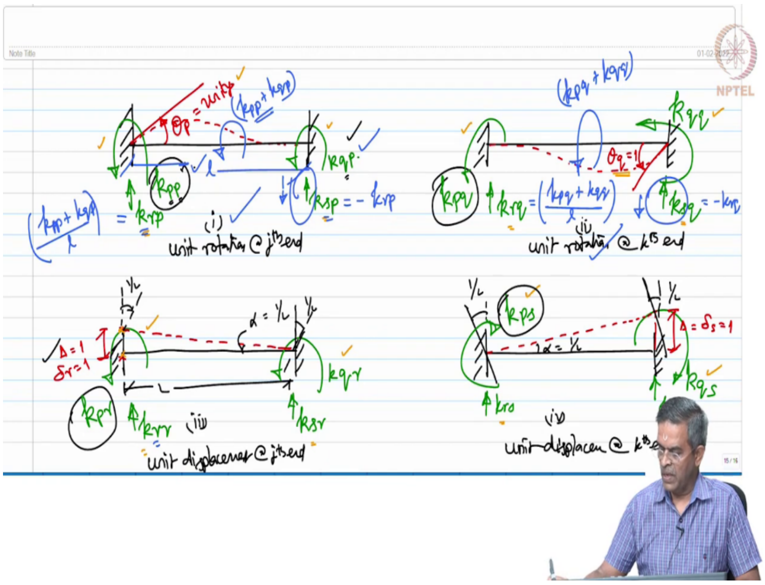


So, I am marking the beam again. This is my x_m, I know how to mark y_m. This is my j^{th} end, k^{th} end, prismatic cross section, length of the member is L_i and this member is actually i. So, let us mark rotation at j as θ_p, rotation at k as θ_q, vertical displacement at j along positive m is δ_r and vertical displacement at k^{th} along positive y_m is δ_s.

So, we are just marking these dimensions. I want to derive the stiffness matrix of this. So, to do that I have to apply unit rotation. So, let us quickly revise and understand what is stiffness coefficient; k_{ij} is the stiffness coefficient of the stiffness matrix k.

This is defined as force in the i^{th} degree of freedom for unit displacement in k^{th} degree of freedom, keeping all other degrees of freedom restrained. This is very important condition. So, I must give unit rotation at all the degrees of freedom 1 by 1 and try to find out the forces, that becomes a stiffness matrix. Let us draw those figures.

(Refer Slide Time: 21:54)



So, we will draw 4 figures. Let us mark the unit rotation at the j^{th} end and draw a tangent, measure the angle from the axis. I call this as θ_p and let the θ_p be unity. So, now this will invoke forces. I call that as k_{pp}, this as k_{qp} and this reaction as k_{rp} and this as k_{sp}. So, friends let us pay attention to the subscripts of this.

The first subscript indicates where do you measure, the force the second subscript indicates where do you give the unit displacement. Like look at this figure. This is p, this is q, r and s. So, I am measuring force at the p^{th} end. For giving displacement the p^{th} end so, it is k_{pp}. I am measuring force at the q^{th} end for displacement at the p^{th} end so, k_{qp}. Similarly, I get k_{rp} and k_{sp}. Let us call this figure number (i).

Let us go to figure number (ii), I want to give unit rotation at the k^{th} end. So, draw a tangent from the axis measure in the anti-clockwise direction call this as θ_q that is unity. So, the corresponding forces will be k_{qq} k_{pq} k_{rq} and k_{sq}. You will obviously, see the second subscript indicates the degree of freedom label where unit rotation is given. The first subscript indicates the degree of freedom, where the force is measured. I call this figure number (ii).

So, figure (i) and (ii) in fact, figure (i) refers to unit rotation at j^{th} end correct, figure (ii) refers to unit rotation at k^{th} end, is it not; can also give unit displacement at the j^{th} end. Let us say I want to move this end by unit displacement that is here. So, I want to give this displacement delta as unity which will be nothing, but δ_r is unity. So, now, I do not want to give displacement any other end.

Therefore, the displaced position will be this. It is a straight line. Let me rub this and redraw again, it is a straight line. Let me draw an axis normal to this. Let me call this angle as $\alpha = \frac{1}{L}$, because this is L and this is unity. Now, the angle between the normals will be equal. Therefore, friends this angle will also be $\frac{1}{L}$, same as this angle.

So, now this will invoke forces in the p^{th} degree, I call this as k_{pr}. In the q^{th} degree I call this as k_{qr}, in the r^{th} degree I call this as k_{rr} and this as k_{sr}. The second subscript refers to the place where we have given unit displacement. Similarly, let us say this unit displacement is figure (iii) at j^{th} end. Let us draw figure (iv) by giving unit displacement at the k^{th} end.

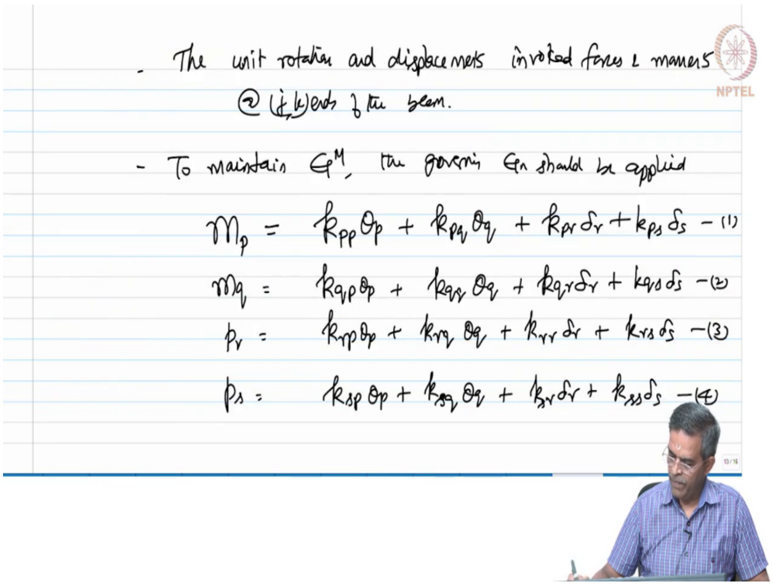
Let us do that. So, let us give unit displacement at the k^{th} end. This delta equals 1 which is actually equal to δ_s which is unity. And, this becomes my new axis of the member, let me draw normal to this. So, if I say this is $\alpha = \frac{1}{L}$ and we all agree that this angle will also be 1 by L as well as this angle. This has invoked forces as k_{qs} k_{ps} k_{rs} and k_{ss}. Friends you must wonder how am I marking the arrow directions of the moment and then the axial reactions.

So, please note here I have given an unit anticlockwise rotation to this. So, the moment is applied on the same direction, same direction is transferred here. Similarly, I have given anti-clockwise rotation at the k^{th} end, in the same direction moment is applied to cross this rotation, the same is transferred here. Whereas, in figure (iii) I have given upward displacement towards positive y at the j^{th} end.

So, I want to bring this position back to normal. So, this is calling it back anti-clockwise. Same is applied here to k_{qr}. When I come to figure (iv) k^{th} end has moved up. So, I want to bring it back. So, calling it back k_{qs} will now become clockwise. The same will be applied to ps and I am marking these reactions rp, sp etcetera similar to the degrees of freedom which is originally in the fixed beam.

So, now we have given unit displacements at j^{th} end, k^{th} end and unit rotations at the j^{th} end and k^{th} end, is it not. And, we have also marked the corresponding forces at both the ends j and k in all the 4 cases. Now, looking at this figure, we can write a statement.

(Refer Slide Time: 31:00)



The unit rotation and displacements invoked forces and moments at j and k^{th} ends of the beam. Now, to maintain equilibrium, the governing equation should be applied. What is that equation to be applied? I should say moment at p^{th} end should be some of force at p^{th} end due to θ_p, force at the p^{th} end due to θ_q, force at the p^{th} end due to δ_r and force at the p^{th} end due to δ_s, is it not. You can see here k_{pp} k_{pq} k_{pr} and k_{ps} or the net forces acting at the j^{th} end due to arbitrary displacements θ_{pq} δ_r and δ_s given in cycle, is it not.

$$m_p = k_{pp}\theta_p + k_{pq}\theta_q + k_{pr}\delta_r + k_{ps}\delta_s$$

$$m_q = k_{qp}\theta_p + k_{qq}\theta_q + k_{qr}\delta_r + k_{qs}\delta_s$$

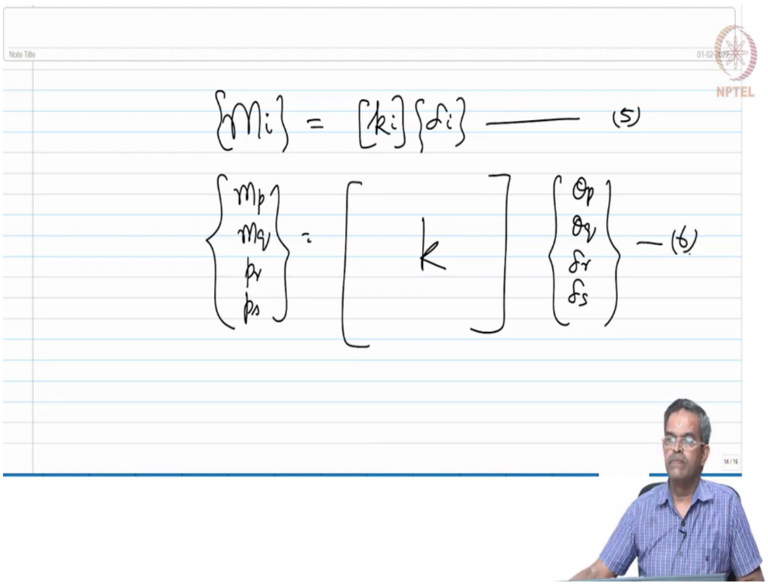
$$p_r = k_{rp}\theta_p + k_{rq}\theta_q + k_{rr}\delta_r + k_{rs}\delta_s$$

$$p_s = k_{sp}\theta_p + k_{sq}\theta_q + k_{sr}\delta_r + k_{ss}\delta_s$$

So, this equation should be valid. We call the equation number for example, we will continue with the new numbering. We call this equation number 1. Similarly, I can write the equation for m_q which is now at the q^{th} end. It should be the force at the q^{th} end because of θ_p, force at the q^{th} end because of θ_q, force at the q^{th} end because of δ_r and force at the q^{th} end because of δ_s. Then, can also find the reaction.

Let us call this as p r is the force at the q^{th} end, force due to θ_p, force at q^{th} end due to θ_q, force at the r^{th} end due to δ_r and s^{th} end due to δ_s. Similarly, p s will be force at the q^{th} end due to θ_p, force at the k^{th} end due to θ_q, force at the r^{th} end due to δ_r plus four force at the s^{th} end due to δ_s.

(Refer Slide Time: 34:40)

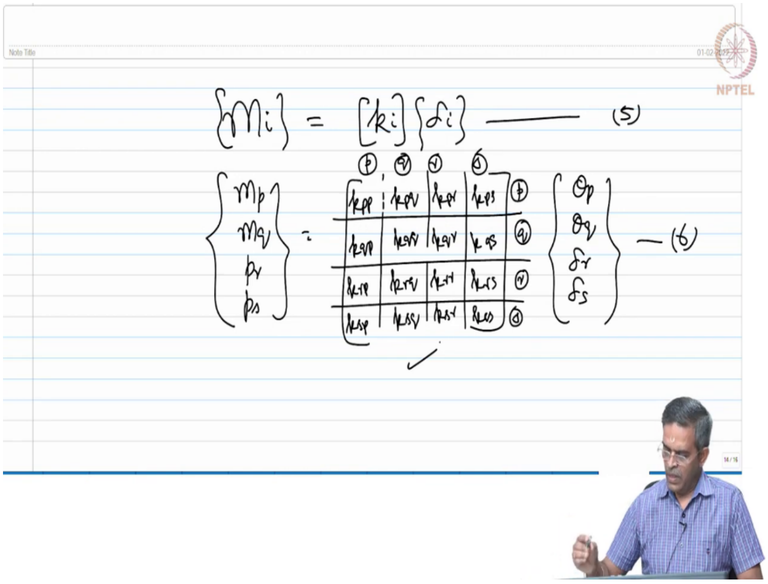


$$\{m_i\} = [k_i]\{\delta_i\}$$

$$\{m_p\, m_q\, p_r\, p_s\} = [k]\{\theta_p\, \theta_q\, \delta_r\, \delta_s\}$$

Now, we can write this in a matrix form. We know that
$\{m_i\} = [k_i]\{\delta_i\}$, $\{m_i\}$ is a vector, $[k_i]$ is a matrix, $\{\delta_i\}$is a vector. I call this equation number 5. So, where $\{m_i\}$ is $m_p\, m_q p_r$ *and* p_s , where $[k_i]$ is a full matrix and $\{\delta_i\}$ is $\theta_p\, \theta_q\, \delta_r$ and δ_s.

(Refer Slide Time: 35:28)



So, let us see what is this matrix which will be a 4 by 4 matrix. This will be the p^{th}, q^{th}, r^{th} and s^{th} columns; p^{th}, q^{th}, r^{th} and s^{th} rows. So, this is $k_{pp} k_{qp} k_{rp} k_{sp}$ it is a p^{th} column. So, $k_{pq} k_{qq} k_{rq} k_{sq}$ the q^{th} column, $k_{pr} k_{qr} k_{rr} k_{sr}$, the r^{th} column and $k_{ps} k_{qs} k_{rs} k_{ss}$, the s^{th} column.

$$\{m_i\} = [k_i]\{\delta_i\}$$

$$\{m_p\, m_q\, p_r\, p_s\} = [k_{pp}\, k_{pq}\, k_{pr}\, k_{ps}\, k_{qp}\, k_{qq}\, k_{qr}\, k_{qs}\, k_{rp}\, k_{rq}\, k_{rr}\, k_{rs}\, k_{sp}\, k_{sq}\, k_{sr}\, k_{ss}]\{\theta_p\, \theta_q\, \delta_r\, \delta_s\}$$

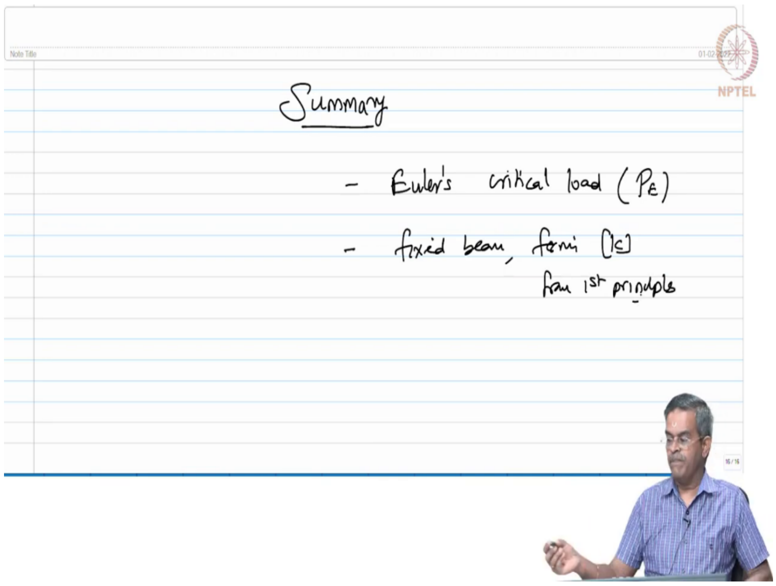
So, our job is to find out this matrix, which will be the elemental property based on the geometric and material characteristics of the beam member. Now, let us pick up the figures back again. So, I will copy this page, put it here. Now, friends for unit rotation given at the j^{th} end, I develop k_{pp} and k_{qp}. I can say this k_{rp} is actually equal to because the net moment now generated is k_{pp} plus k_{qp}.

So, k_{rp} can be simply said as k_{pp} plus k_{qp} by l, where this is my l and k_{sp} will be opposite to k_{rp}. So, I can write here, this is equal to minus of k_{rp}. So, this direction will be reversed because, this is now going to cause a couple which will counteract this moment. So, k_{rp} and k_{sp} depends on k_{pp} and k_{qp} only. If I know k_{pp} and k_{qp}, I can find the end reactions k_{rp} in k_{sp}. Similarly, let us go to this case (ii).

So, now, this is invoking an anti-clockwise moment of k_{pq} and k_{qq}, this will be counteracted by a couple which is k_{rq} and k_{sq}. So, k_{rq} will be now k_{pq} plus k_{qq} by l and k_{sq} with a minus of k_{rq}. Now, this will be get reversed. We have now expressed these reactions in figures (i) and (ii) as a function of the moments pq and qq. Similarly, I can also find or express these reactions as a sum of k_{pr} and k_{qr}.

Friends, please understand we are trying to express the reactions in terms of the end moments; we have done with figure (i), we have done with figure (ii). We will do the figure (iii) and figure (iv) plus the moment. We will do that. So, we will do it in the next lecture.

(Refer Slide Time: 40:16)



We will write a summary here. In this lecture, we learned how to estimate Euler's critical load which is called as $P_{Euler's}$. We have also started understanding the derivation for standard fixed beam and forming the stiffness matrix for this beam from 1st principles. We will continue in the next lecture and attend to this in the next lecture.

Thank you very much and have a good day. Bye.