Advanced Design of Steel Structures
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Lecture - 29
Stability functions - 1

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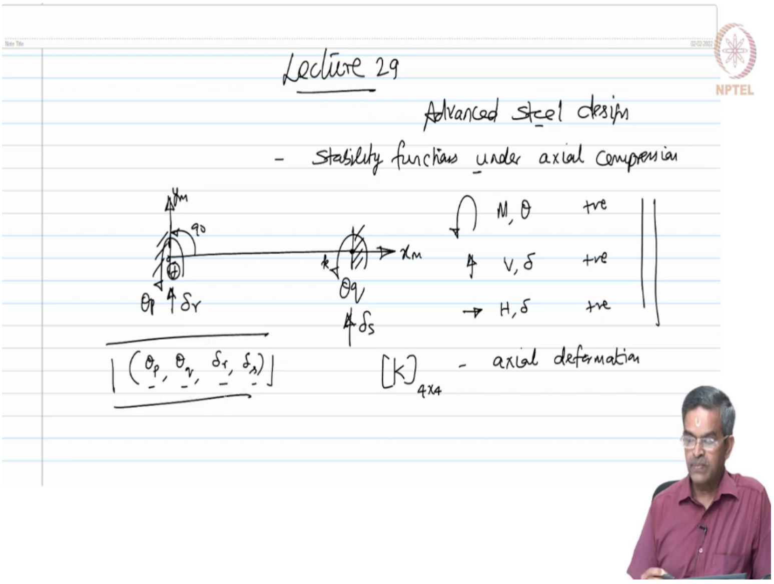

Friends, welcome to the 29th Lecture on Advanced Steel Design course, in this lecture we are going to learn how to derive the Stability Functions under axial compression. Friends, in the last lecture we discussed about the derivation of a stiffness method using stiffness method of a standard fixed beam and we understood that a standard fixed beam has got 4 kinematic degrees of freedom as marked on the screen.

We have used a specific sign convention that anticlockwise moments and rotations are positive, displacements upward shear and displacements upward are positive. Similarly, along x axis displacement and forces will be positive, we have used this n convention and we have derived a stiffness matrix of 4 by 4; obviously, we neglected axial deformation.

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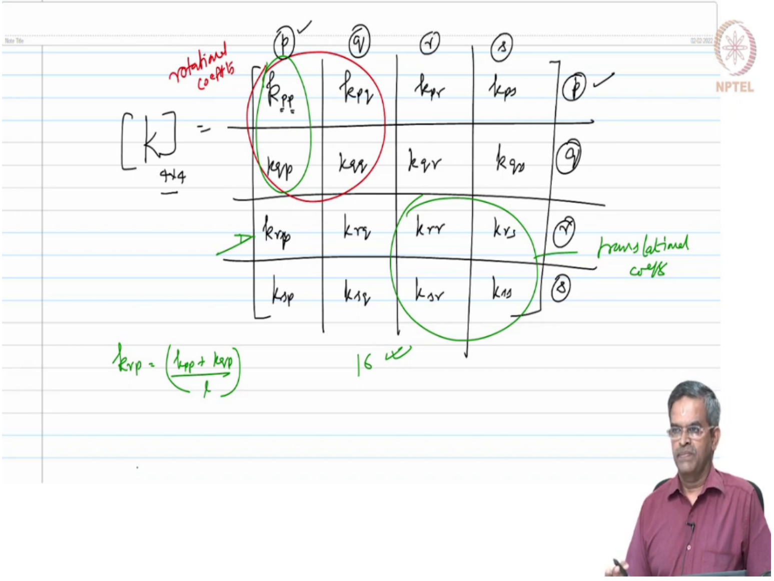

$$[K] = \left[k_{pp}\ k_{pq}\ k_{pr}\ k_{ps}\ k_{qp}\ k_{qq}\ k_{qr}\ k_{qs}\ k_{rp}\ k_{rq}\ k_{rr}\ k_{rs}\ k_{sp}\ k_{sq}\ k_{sr}\ k_{ss} \right]$$

And we derive the stiffness matrix as you see on the screen, we have the labels p q r s and p q r s these labels have a specific order. Please see this order rotation, if this is my j^{th} end of the member this is my k^{th} end of the member along the length of the member is my x axis y axis is anticlockwise 90 to x axis.

Therefore, the degrees of freedom are labelled in such a manner rotation at j^{th} end, rotation at k^{th} end, displacement along positive y at j^{th} end and displacement along positive y at k^{th} end. So, p q r s this is the order.

So, we have written this order here and we know this is going to be k_{pp}, k represents the element of this matrix, $k_{pp}, k_{pq}, k_{pr}, k_{ps}, k_{qp}, k_{qq}, k_{qr}, k_{qs}, k_{rp}, k_{rq}, k_{rr}, k_{rs}, k_{sp}, k_{sq}, k_{sr}, k_{ss}$. While writing this subscripts of these elements of the stiffness matrix the first subscript refers the row and second subscript refers the column, that is a standard practice what we do in representing the equations or mathematical formulae in a matrix form.

So, this is a 4 by 4 matrix, yesterday we learnt that these 4 coefficients are named as rotation coefficients and these 4 coefficients are termed as translational coefficients. And we derived this the remaining coefficients as a function of this is not. For example, I am just revising k_{rp} is sum of these two by l. So, I can say $k_{rp} = \left(\frac{k_{pp}+k_{qp}}{l}\right)$and so on and so forth.

So, we have got all the 16 coefficients learnt and understood for a standard fixed beam neglecting axial deformation is it not? Having said this, we will follow the same analogy, but now we will derive the stability functions under axial compression, it means I am going to apply an axial compressive force to the same fixed beam module and derive the matrix in the same order as we have derived earlier.

Let us see how we are going to do that.

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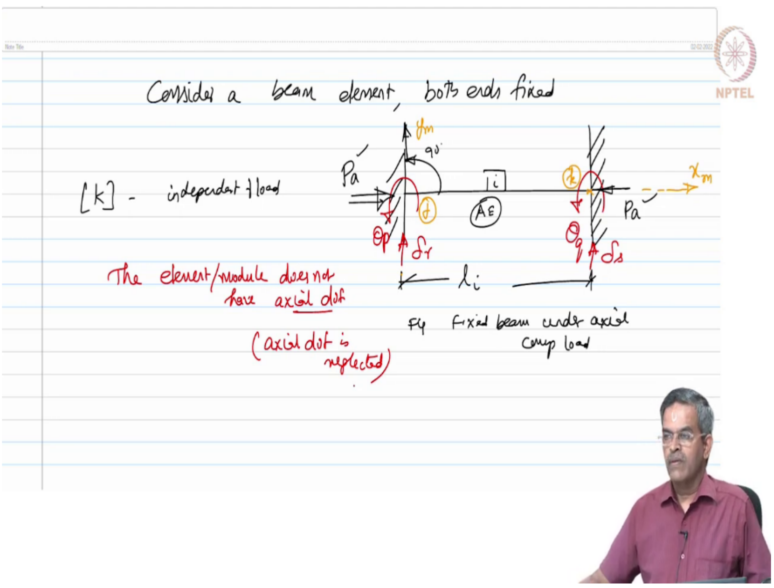

So, now, I am going to say consider a beam element both ends fixed. Let us consider the element this is my element and let us mark the axis we know this is my x axis, my y axis measured anticlockwise ninety and this A and E is considered to be constant along the length of the member it is a prismatic section.

So, let us mark the degrees of freedom for this they are all restrained let us mark them in red colour. So, this is θ_p, this is θ_q and this we know is δ_r and this we know is δ_s.

In addition to this I am going to apply an axial force p_a at both the ends. Let me mark this x_m slightly in a different colour. So, that it does not get confused this is y_m let me also mark this

y_m here this is x_m, m represents the member and this is my force which is axial. So, there is no confusion in this and the length of the member is l_i let say is an i^{th} member, it is an i^{th} member and we know that this is my j^{th} end, this is my k^{th} end that is how we have marked.

Now, this is the figure of a fixed beam under axial compressive load. Friends, please understand when you derive the stiffness matrix there was no load applied, stiffness matrix is of course, independent of any load, but now in this case I am going to derive the stability function. Therefore, we already learnt stability is that function or that capacity or that ability of the structure to perform its intended function up to the critical load for which it is designed.

So, I am applying the axial load which is in compression, I am going to compress this beam from either ends. Please note very carefully here; the element or the module does not have axial degree of freedom, the axial degree of freedom is neglect. Please note that I have only 4 degrees of freedom p q r s. Similarly, what we had in the fixing there is no change in that.

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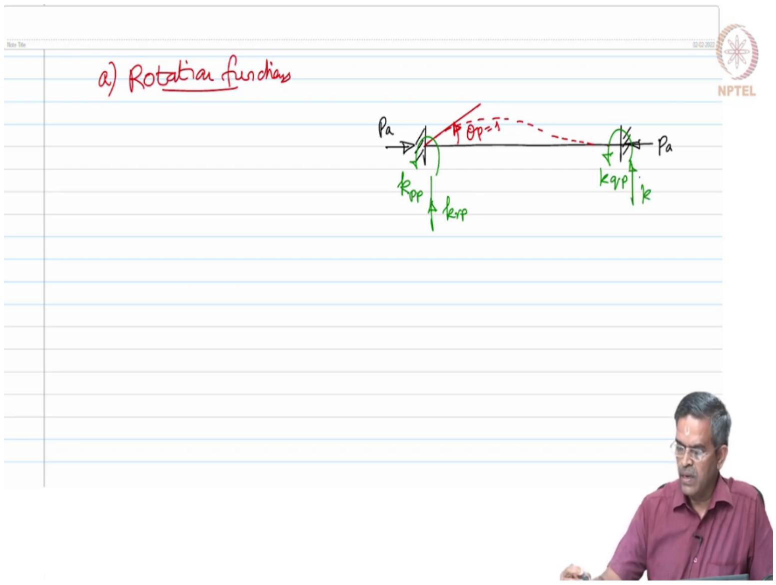

Having said this let us now derive what is called rotation functions, as earlier we derived what is called rotational coefficients, now I am talking about stability function therefore, I have rotation functions.

So, now to get this rotation function I must apply a unit rotation let us see that, I am taking the beam fix both the ends the beam is now subjected to and unit rotation θ_p. So, it is the same algorithm what we did in the beam earlier that is why I explained the beam analysis first.

Now, let us mark the axial force present in the section, is it not? This was new in the earlier beam analysis this was not there, let us also mark the degrees of freedom, let us say. I will not mark the degrees I will mark the forces. So, what I am going to say this is my force k_{pp}, this is my force k_{qp} and this is k_{rp} and this is k. Let us write this k_{qp} here.

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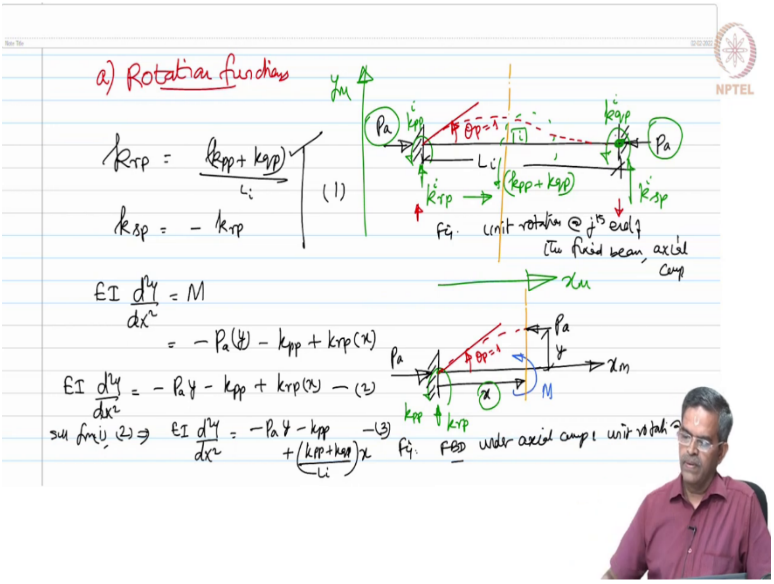

Let us write this k_{pp} here, let us mark exactly this here so, k_{rp} and k_{sp}. What are these k_{rp} k_{sp} etcetera? They are all forces when the system is subjected to unit displacement. So, they are actually similar to stiffness coefficients can also put a superscript i, saying that this is for the i^{th} member, but that is redundant when it will it may cause additional confusion. So, I am avoiding that and of course, we know this is the length of the member L_i.

$$k_{rp} = \frac{(k_{pp}+k_{qp})}{L_i}$$

$$k_{sp} = -k_{rp}$$

$$EI\frac{d^2y}{dx^2} = M$$

$$=- P_a(y) - k_{pp} + k_{rp}(x)$$

$$EI\frac{d^2y}{dx^2} =- P_a y - k_{pp} + k_{rp}(x)$$

$$EI\frac{d^2y}{dx^2} =- P_a y - k_{pp} + \left(\frac{k_{pp}+k_{qp}}{L_i}\right)x$$

So, now this figure indicates unit rotation at j^{th} end of the fixed beam, subjected to axial combustion correct. So, now, when you do this, this is start developing moments and shear at both ends of the member both j^{th} and k^{th} ends. So, now, this has an unbalanced moment anticlockwise which will be k_{pp} plus k_{qp}.

So, can I now say k_{rp} will be actually equal to this unbalanced moment by l or l I, because there is going to form a couple when I say this is a couple. So, this will be upward this will be downward. So, can I say now k_{sp} is same as k_{rp}, but with a negative sign. So, I will call this equation as 1.

I hope there is no confusion at this stage, is it not same identical analysis what we discussed for the fixed beam, I am just following the same algorithm. So, it is easy for us to do it. Now, what I do I draw a free body diagram under the influence of this by cutting a section somewhere here let us see and draw free body diagram let me do that.

I draw a free body diagram fixed beam one end fixed I have applied a rotation an donut engine and this is unity it is subjected to an axial force P_a it is having end moment k_{pp} because free body diagram is a true representation of all the internal external forces acting on the segment considered. So, this is going to be k_{rp}.

Now, at this stage I have a balance force P_a to counteract this let us say that is at a distance y from the axis of the member and it also has a moment which is yeah, I should say this figure represents free body diagram under axial compression and unit rotation at j^{th} end.

Now, we also have the classical differential equation to represent this which we discussed. So, I will copy this figure or write it I write it here itself. So, now, I say with reference to this figure free body diagram I can say $EI\frac{d^2y}{dx^2}$ is yeah and this

$M =- P_a(y) - k_{pp} + k_{rp}(x)$. I am considering a section at a distance x from here.

$$EI\frac{d^2y}{dx^2} =- P_a y - k_{pp} + k_{rp}(x)$$

$$EI\frac{d^2y}{dx^2} =- P_a y - k_{pp} + \left(\frac{k_{pp}+k_{qp}}{L_i}\right)x$$

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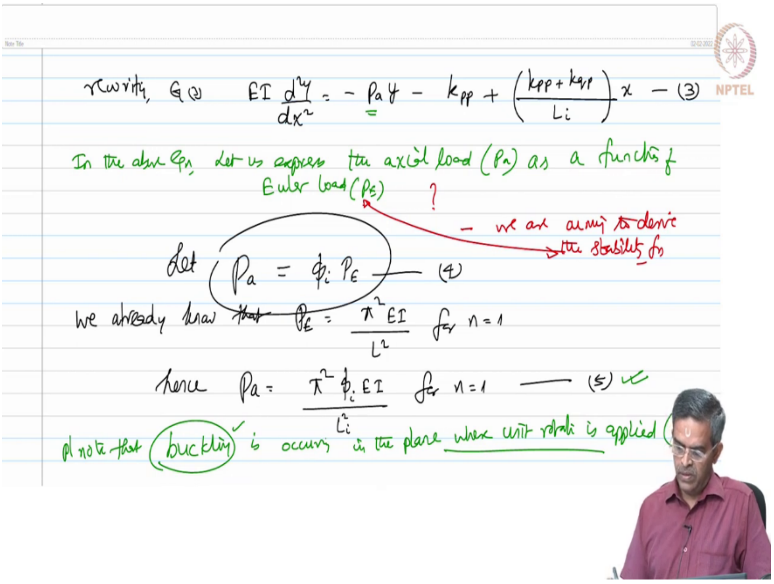

$$P_a = \phi_i P_E$$

$$P_E = \frac{\pi^2 EI}{L^2}\ for\ n = 1$$

$$P_a = \frac{\pi^2 \Phi_i EI}{L_i^2}\ for\ n = 1$$

In the above equation, let us express the axial load that is P_a as a function of Euler load P_E, you may ask me a question why are you doing this? Because answer is we are aiming to derive the stability function, stability function is related to Euler load. So, therefore, let P_a be expressed as some function of P_E call this equation number 4.

We already know that $P_E = \frac{\pi^2 EI}{L^2}$ for n equals 1, hence P_a will be $\frac{\pi^2 \Phi_i EI}{L_i}$ for n equals 1 can I have this as equation number 5. So, friends, it is a very important observation you want to make here. Please note that, buckling is occurring in the plane where unit rotation is applied, you may ask me a question, suddenly how I have introduced the term buckling? Friends, look at this figure this is a beam or a column element subjected to axial compression.

Under axial compression column will buckle and that is one of the failure mode, we already discussed that in the previous lecture in detail is not it. So, I will not no more call this as bending this is not happening because of any lateral load or gravity load, the profile what is shown on the screen is happening purely because of buckling and it is happening on the same plane where unit rotation is given, is that.

Then you may ask me a question, sir, why we are estimating buckling when rotation is given? That is very good; we are estimating that load which is going to cause failure. So, we are expressing that load of P_a as a function of Euler's load, if that load P_a exceeds Euler's load it will fail and that failure will be called as buckling failure, that is what Euler stated, is it not? So, I am using the same logic I am expressing it.

So, therefore, I am calling this as buckling and let us remember that this is happening on the same plane where unit rotation is applied. What is this plane to be very clear; it is x_m y_m plane, is it not see here, it is this is x_m and this is y_m that is the board, is it not? It is happening in the same plane. Therefore, let us modify equation 3 in this understanding, because we have substituted, we have got P_a in equation 3. P_a is expressed in a different form in equation 5. So, let us substitute equation 5 and update equation 3.

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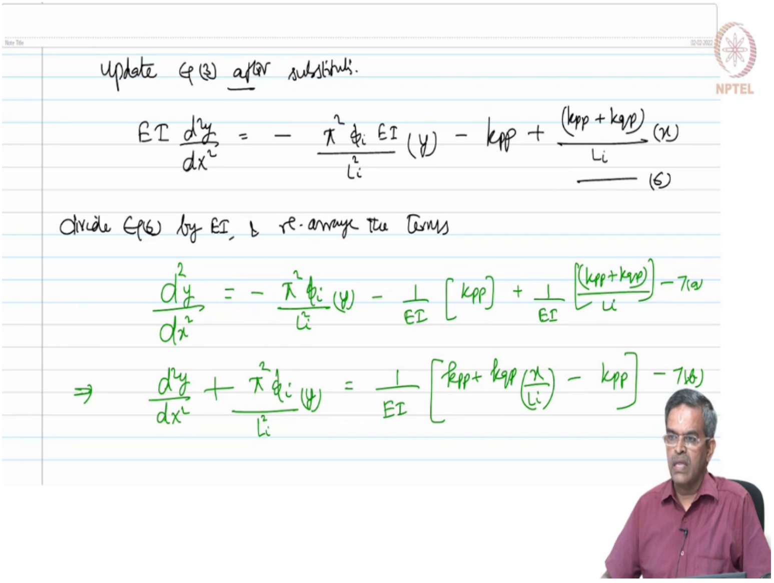

$$EI\frac{d^2y}{dx^2} = -\frac{\pi^2\Phi_i EI}{L_i^2}(y) - k_{pp} + \left(\frac{k_{pp}+k_{qp}}{L_i}\right)(x)$$

$$\frac{d^2y}{dx^2} = -\frac{\pi^2\Phi_i}{L_i^2}(y) - \frac{1}{EI}k_{pp} + \frac{1}{EI}\left(\frac{k_{pp}+k_{qp}}{L_i}\right)x$$

$$\frac{d^2y}{dx^2} + \frac{\pi^2\Phi_i}{L_i^2}(y) = \frac{1}{EI}\left[k_{pp} + k_{qp}\left(\frac{x}{L_i}\right) - k_{pp}\right]$$

So, let us update equation 3 after substitution. So, that becomes $EI\frac{d^2y}{dx^2} = -\frac{\pi^2\Phi_i EI}{L_i^2}(y) - k_{pp} + \left(\frac{k_{pp}+k_{qp}}{L_i}\right)(x)$,we call this equation as 6.

Now, I will divide this equation 6 by E I and rearrange the terms, divide equation 6 by E I and rearrange the terms let us do that. So, if you do that I will get this equation now which is $\frac{d^2y}{dx^2} = -\frac{\pi^2\Phi_i}{L_i^2}(y) - \frac{1}{EI}k_{pp} + \frac{1}{EI}\left(\frac{k_{pp}+k_{qp}}{L_i}\right)x$.

Let me rearrange it further $\frac{d^2y}{dx^2} + \frac{\pi^2\Phi_i}{L_i^2}(y) = \frac{1}{EI}\left[k_{pp} + k_{qp}\left(\frac{x}{L_i}\right) - k_{pp}\right]$ we will call this as equation 7 a and this as 7 b.

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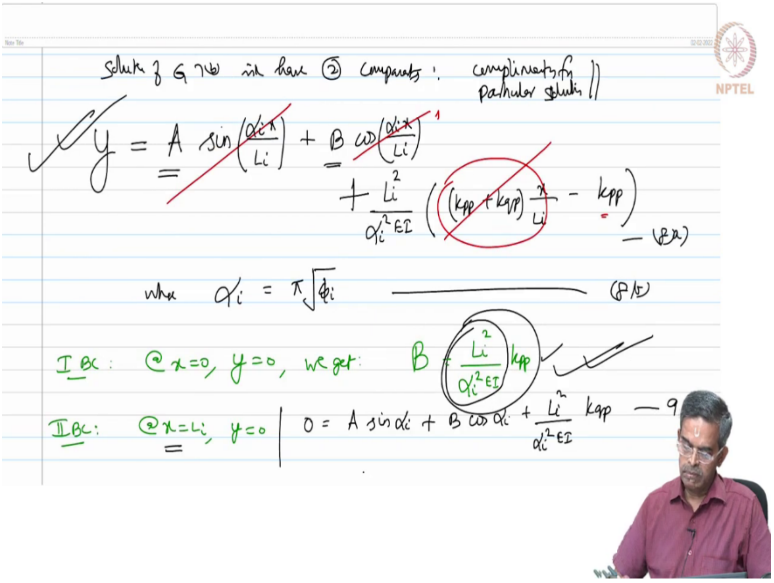

$$y = A \ \sin sin \left(\frac{\alpha_i x}{L_i}\right) + B \ \cos cos \left(\frac{\alpha_i x}{L_i}\right) + \frac{L_i^2}{\alpha_i^2} \left(\left(k_{pp} + k_{qp}\right)\frac{x}{L_i} - k_{pp}\right)$$

$$\alpha_i = \pi\sqrt{\phi_i}$$

$$I)BC: @x = 0, y = 0, \ we \ get \ B = \frac{L_i^2}{\alpha_i^2 EI} k_{qp}$$

$$II)BC: @x = L_i, y = 0, \ we \ get \ 0 = A \ \sin sin \ \alpha_i + B \cos cos \ \alpha_i + \frac{L_i^2}{\alpha_i^2 EI} k_{qp}$$

Now, look at this equation 7 b it is a second order differential equation where right hand side is not equal to 0. So, this will have two solutions one is a complementary function are there is a particular integral. So, the solution of equation 7 b will have two components.

Namely; complementary function and particular solution is a standard procedure which you will have, described well in many mathematic books on differential equations. So, we will not cover that discussion at this moment in this lecture, but I request you to go through additionally some material to learn this.

So, therefore, now the solution y will be equal to $A \ \sin sin \left(\frac{\alpha_i x}{L_i}\right) + B \ \cos cos \left(\frac{\alpha_i x}{L_i}\right)$ this is the complementary function then the particular solution comes $+\frac{L_i^2}{\alpha_i^2}\left(\left(k_{pp} + k_{qp}\right)\frac{x}{L_i} - k_{pp}\right)$, we call equation number 8 the standard solution which can be easily understood from ordinary differential equation textbooks.

We will also say where α_i which is introduced here in the above equation is $\pi\sqrt{\phi_i}$, where ϕ_i already is here we have derived we have defined that. Here is actually the number or the ratio between the axial force supply and the Euler's load that is what we have said in equation 4.

We will call this as 8 a for example, this is 8 b for our understanding. Now, this equation 8 a has got 2 differential equation coefficients that is a and b we need to find so; obviously, we need to apply the boundary conditions. Let us apply the boundary condition first boundary condition at x equals 0, y is 0, that is what it is? See, this figure at x equals 0, y 0, is it not, that is this point.

So, if you apply that condition, we get B because this term go away this term becomes unity, this term goes away and you get this which is negative. So, can I say B is going to be equal to $\frac{L_i^2}{\alpha_i^2 EI} k_{qp}$, can I say that. Let us apply the second boundary condition, what is that condition, at x is equal to l i again y is 0 see the original beam at x is equal to l that is here, again y is 0, is it not?

So, let us apply that condition here. So, if you apply that condition, we will get 0 equals $A \sin sin\, \alpha_i + B \cos cos\, \alpha_i + \frac{L_i^2}{\alpha_i^2 EI} k_{qp}$, because L_i and x_i gets cancelled because x is equal to l is a condition this is what I get. We already have the value for B, let us substitute that here.

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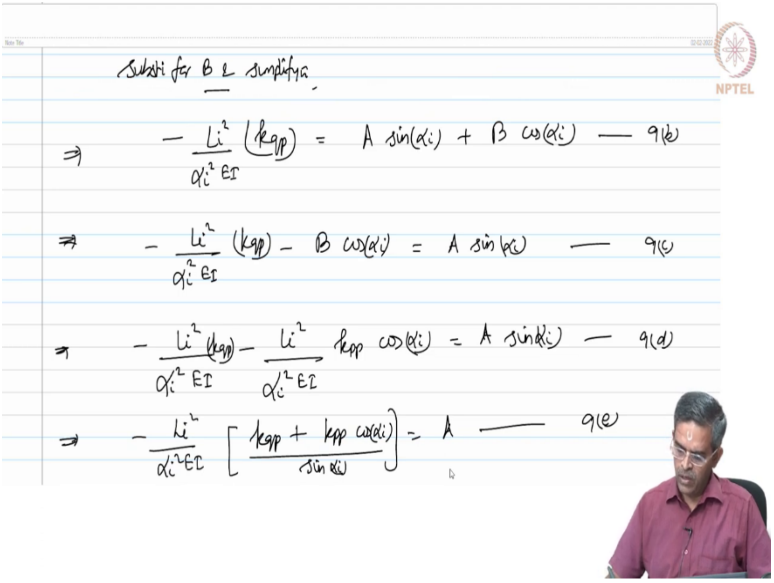

$$-\frac{L_i^2}{\alpha_i^2 EI}\left(k_{qp}\right) = A\ \sin sin\left(\alpha_i\right)\ + B\cos cos\left(\alpha_i\right)$$

$$-\frac{L_i^2}{\alpha_i^2 EI}\left(k_{qp}\right) - B\cos cos\left(\alpha_i\right)\ = A\ \sin sin\left(\alpha_i\right)$$

$$-\frac{L_i^2}{\alpha_i^2 EI}\left(k_{qp}\right) - \frac{L_i^2}{\alpha_i^2 EI}\cos k_{pp} cos\left(\alpha_i\right)\ = A\ \sin sin\left(\alpha_i\right)$$

$$-\frac{L_i^2}{\alpha_i^2 EI}\left[\frac{k_{qp}+\cos k_{pp} cos\left(\alpha_i\right)}{\sin sin\left(\alpha_i\right)}\right] = A$$

Now, substituting for B and simplifying, $-\frac{L_i^2}{\alpha_i^2 EI}\left(k_{qp}\right) = A\ \sin sin\left(\alpha_i\right)\ + B\cos cos\left(\alpha_i\right)$ .Let us do this. So, $-\frac{L_i^2}{\alpha_i^2 EI}\left(k_{qp}\right) - B\cos cos\left(\alpha_i\right)\ = A\ \sin sin\left(\alpha_i\right)$. we will name these equations we will call this as 9 a, this is 9 b, this is 9 c, this is 9 d. So, now I can say $-\frac{L_i^2}{\alpha_i^2 EI}\left(k_{qp}\right) - \frac{L_i^2}{\alpha_i^2 EI}\cos k_{pp} cos\left(\alpha_i\right)\ = A\ \sin sin\left(\alpha_i\right)$, can I say that? $A = -\frac{L_i^2}{\alpha_i^2 EI}\left[\frac{k_{qp}+\cos k_{pp} cos\left(\alpha_i\right)}{\sin sin\left(\alpha_i\right)}\right]$;

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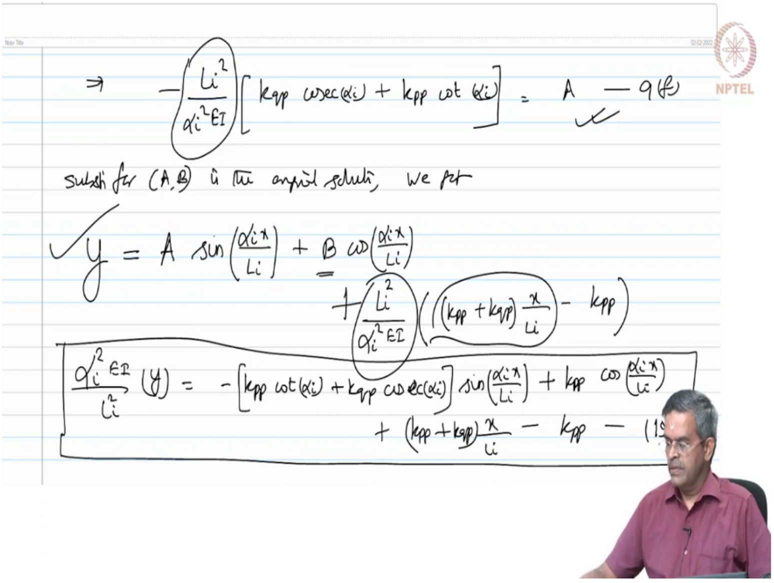

$$-\frac{L_i^2}{\alpha_i^2 EI}\left[k_{qp}\csc csc\,(\alpha_i) + k_{pp}\cot cot\,(\alpha_i)\right] = A$$

$$y = A\sin sin\left(\frac{\alpha_i x}{L_i}\right) + B\cos cos\left(\frac{\alpha_i x}{L_i}\right) + \frac{L_i^2}{\alpha_i^2 EI}\left(\left(k_{pp} + k_{qp}\right)\frac{x}{L_i} - k_{pp}\right)$$

$$\frac{\alpha_i^2 EI}{L_i^2}(y) = -\left[k_{pp}\cot cot\,(\alpha_i) + k_{qp}\csc csc\,(\alpha_i)\right]\sin sin\left(\frac{\alpha_i x}{L_i}\right) + k_{pp}\cos cos\left(\frac{\alpha_i x}{L_i}\right) + \left(\left(k_{pp} + k_{qp}\right)\frac{x}{L_i} -\right.$$

So, which means $-\frac{L_i^2}{\alpha_i^2 EI}\left[k_{qp}\csc csc\,(\alpha_i) + k_{pp}\cot cot\,(\alpha_i)\right] = A$. So, let me call this as 9 f. So, I have the value for A I have the value for B also see B is available here, A is available here. So, substituting for A and B in the original solution we get, what do we get? Please check that, what do we get? Where is the original solution? This is the original solution I have, this is the original solution I have. So, let us rearrange the terms, let us I think you can even copy this let us try to copy this.

Let us remove this marking you know the number also which is equal to α_i because you know A has a multiplier α_i^2, if you look at B it also has a multiplier and this term also has a multiplier. So, if I keep all this common out and take it to the left hand side this will become $y = A\sin sin\left(\frac{\alpha_i x}{L_i}\right) + B\cos cos\left(\frac{\alpha_i x}{L_i}\right) + \frac{L_i^2}{\alpha_i^2 EI}\left(\left(k_{pp} + k_{qp}\right)\frac{x}{L_i} - k_{pp}\right)$.

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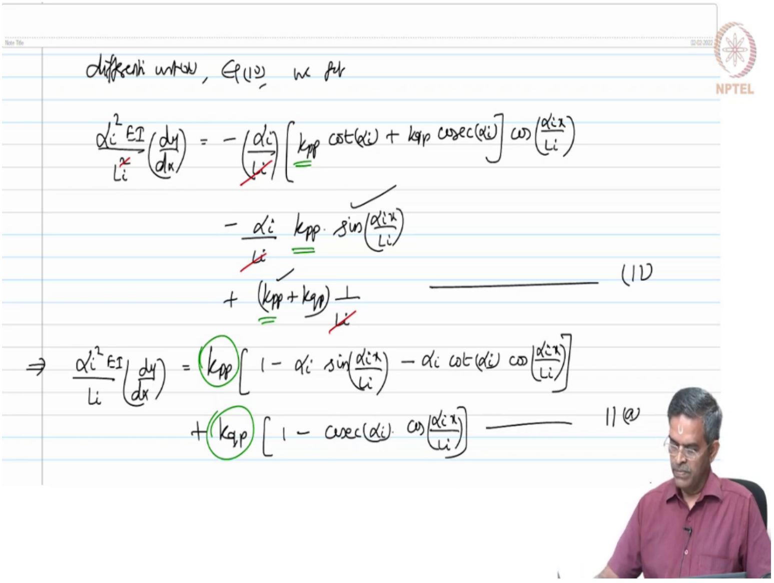

$$\frac{\alpha_i^2 EI}{L_i^2}\left(\frac{dy}{dx}\right) = -\frac{\alpha_i}{L_i}\left[k_{pp}\cot cot\,(\alpha_i) + k_{qp}\csc csc\,(\alpha_i)\right] - \frac{\alpha_i}{L_i}k_{pp}\sin sin\left(\frac{\alpha_i x}{L_i}\right) + \left(\left(k_{pp} + k_{qp}\right)\frac{1}{L_i}\right)$$

$$\frac{\alpha_i^2 EI}{L_i^2}\left(\frac{dy}{dx}\right) = k_{pp}\left[1 - \alpha_i \sin sin\left(\frac{\alpha_i x}{L_i}\right) - \alpha_i \cot cot\,(\alpha_i)\,\cos cos\left(\frac{\alpha_i x}{L_i}\right)\right] + k_{qp}\left[1 - \csc csc\,(\alpha_i)\,\cos cos\left(\right.\right.$$

Differentiating with respect to x equation 10 we get equation 11.

So, we have 2 equations now for the displacement and for the slope $\left(\frac{dy}{dx}\right)$is slope is it not. So, let us rearrange this term this equation in a closed form which will now become $\frac{\alpha_i^2 EI}{L_i^2}\left(\frac{dy}{dx}\right)$ will be equal to because there are many k_{pp} terms let us group them all.

So, k_{pp} term here, there is a k_{pp} term here, let us group them which will become k_{pp} times of $\left[1 - \alpha_i \sin sin\left(\frac{\alpha_i x}{L_i}\right) - \alpha_i \cot cot\,(\alpha_i)\,\cos cos\left(\frac{\alpha_i x}{L_i}\right)\right]$.

Let us talk about k_{qp} of$\left[1 - \csc csc\,(\alpha_i)\,\cos cos\left(\frac{\alpha_i x}{L_i}\right)\right]$, you may wonder why we are arranging the equation 11 in this order. Please understand, I want to obtain k_{pp} and k_{qp} ultimately that is how we already derived the rotation coefficients for the fixed beam when we do the stiffness analysis, is it not? So, you want to group it in that order, we are trying to do that let us call this as equation 11 a because equation 11 and 11 a are only rearranging of terms with some simplification.

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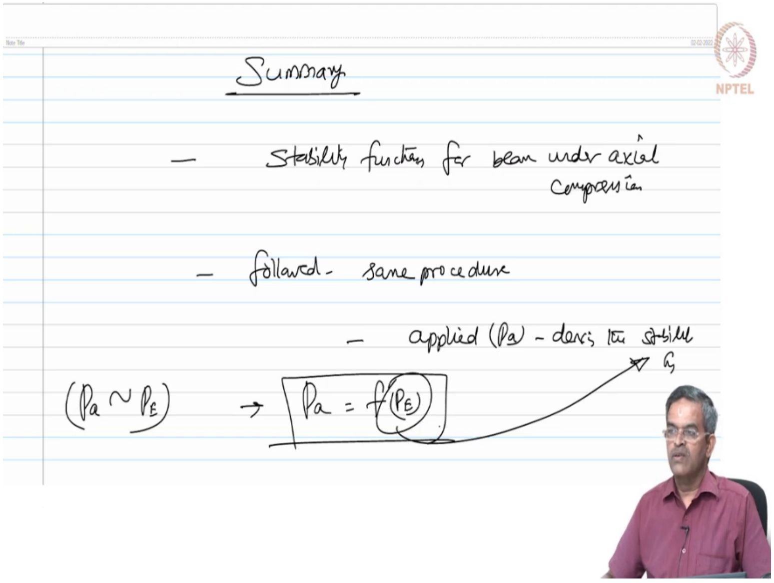

So, friends we look at the summary what we discussed in this lecture. In this lecture we are learning to derive the stability function for beam under axial compression. We have followed identically the same procedure with which we derive the stiffness matrix of a fixed beam.

But in this case, we applied axial compressive force P_a and deriving the stability function. So, while doing so, we made a simplification of expressing this particular as a function of Euler load. Why we did that, because we wanted to assess stability and Euler's load will help you to give stability because I can quickly compare P_a with P_E and say whether the member is stable or unstable. We already learnt it in the previous lecture how to define stability in Euler's criterion. Therefore, we are trying to use this relationship and derive the stability functions.

We will continue this derivation and do this stability functions in the coming lectures.

Thank you have a good day.