Advanced Design of Steel Structures
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Lecture - 04
Failure theories - 2

Friends, let us continue to discuss about the Failure theories. In the last lecture, we discussed about two failure theories. We gradually started understanding and learning the importance of assessing the failure criteria.

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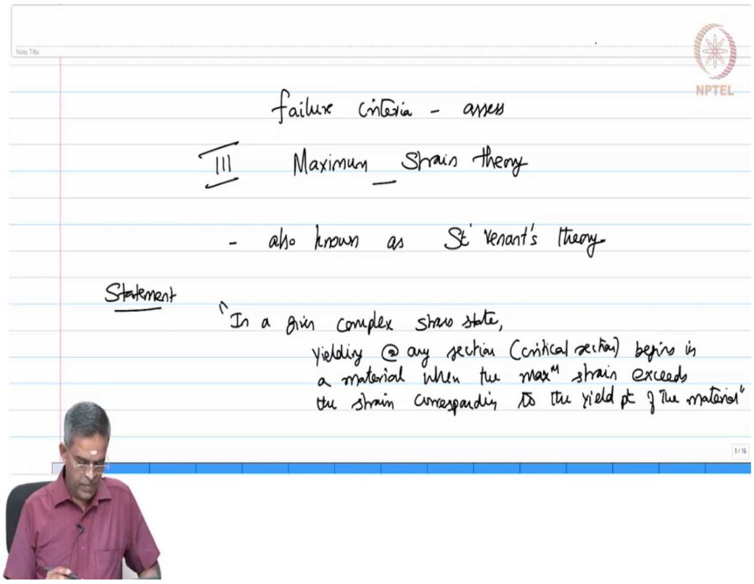

So, what we are trying to look here is to fix or to assess the failure criteria. Why is it necessary? I will explain this and interconnect this to the plastic design in the upcoming lectures. So, when the structural member is subjected to uniaxial, biaxial or triaxial stress states different theories predict the failure stresses at different perspectives for unlike and alike stresses in a biaxial or a triaxial stress states.

So, in this context we discussed about two theories in the last lecture. Now, we will talk about the third theory in this lecture which is the maxiμm strain theory. This theory is also known as St.Venant's theory. The statement of the theory is like this, in a given complex stress state yielding at any section, we will also say this as critical section begins in a material when the maxiμm strain exceeds the strain corresponding to the yield point of the material. So, this theory focuses essentially on the strain criteria that is why it is called maxiμm strain theory.

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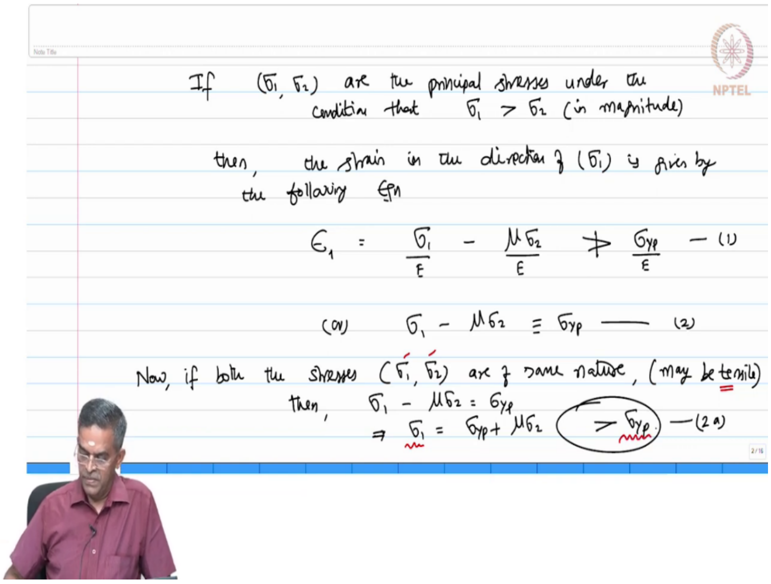

If σ_1 and σ_2 are the principal stresses under the condition that σ_1 is larger than σ_2 in magnitude. Strain in the direction of σ_1 is given by the following equation;

$$\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\mu\sigma_2}{E} \not> \frac{\sigma_{yp}}{E}$$

We will call this equation number 1. Or $\sigma_1 - \mu\sigma_2 = \sigma_{yp}$ that is what the theory states. Now, if both the stresses σ_1 and σ_2 are of same nature, let us say maybe tensile then substituting this condition in equation 2, we get $\sigma_1 - \mu\sigma_2 = \sigma_{yp}$ which means that σ_1 will be $\sigma_{yp} + \mu\sigma_2$ which exceeds even σ_{yp} , is it not. So, that is a very interesting condition; of course, this condition does not occur when σ_1 and σ_2 are dissimilar in nature.

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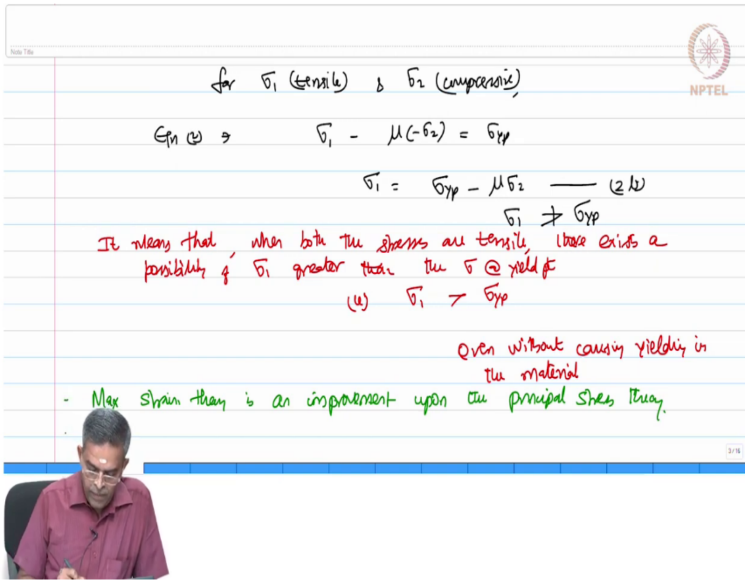

For $\sigma 1$ tensile and $\sigma 2$ compressive equation 2 will become σ_1 - $\mu(-\sigma_2) = \sigma_{yp}$ which says that σ_1 will be σ_{yp} minus $\mu\sigma_2$. So, in this case $\sigma 1$ does not exceed σ_{yp} is it not, but whereas, in the earlier case if both stresses are same in nature; let us say both are tensile, there is a probability that one of the principal stresses can even exceed the stress at the yield point.

So, this is an alarming condition which says that when both the stresses are tensile, there exist a possibility of $\sigma 1$ greater than the stress at yield point that is $\sigma 1$ even exceeds σ_{yp}, even without causing yielding in the material that is very interesting. So, one can then say that the maxiμm strain theory is an improvement upon the principal stress theory.

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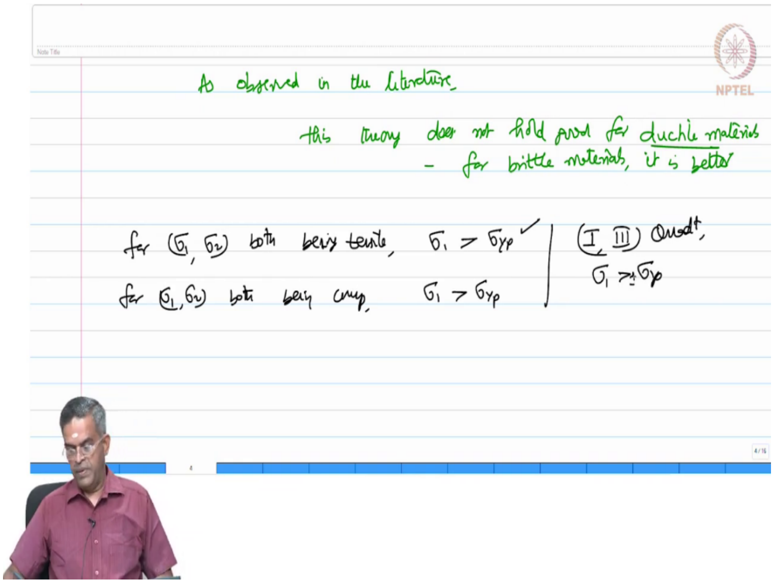

This theory, as observed in the literature, this theory does not hold good for ductile materials, it is better for brittle materials. So, let us try to plot this theory graphically. So, you µst understand one important statement for $\sigma 1$ and $\sigma 2$ both being tensile $\sigma 1$ can even go higher than σ_{yp}. Same story for $\sigma 1$, $\sigma 2$ both being compressive, still $\sigma 1$ can exceed σ_{yp}. It means in the first and third quadrants $\sigma 1$ exceeds σ_{yp}, I should say plus or minus σ_{yp}.

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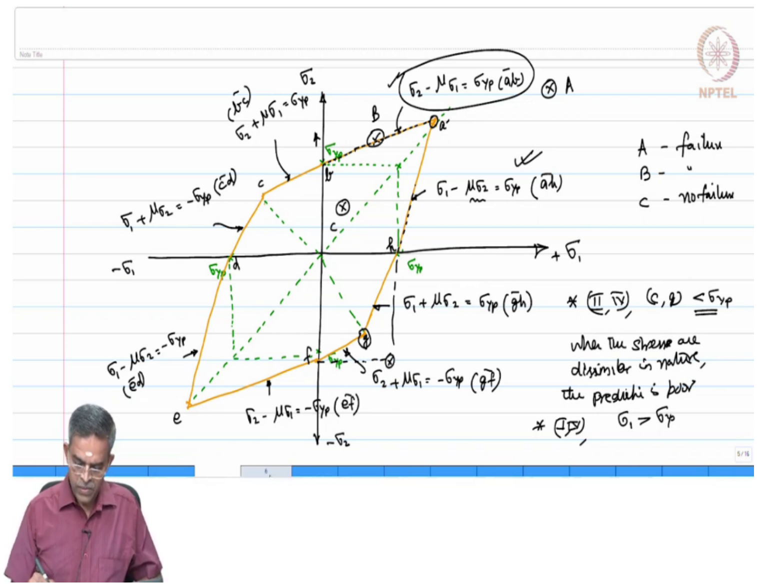

Let us say the x axis plots σ1 and y axis plots σ2, this is positive and this is negative σ1 and this is negative σ2. So, let us try to get a control line to plot this failure envelope which used to do for the remaining theories as well. Let us say my σ_{yp} is here, these are my σ_{yp}.

So, ideally speaking this value should be σ_{yp}, but for σ1 and σ2 both being tensile that is the first quadrant, I am getting σ1 more than σ_{yp}. So, let us take that point as somewhere here. Similarly, even in the third quadrant for σ1 and σ2 both remaining compressive even this is true it exceeds whereas, in the off-diagonal sides these values are lesser. I will come to that point how we work it out.

So, the envelope says it should cross σ_{yp} and on the dissimilar states; so, let us try to write down this equations for this particular line. So, in this case it will be σ2 minus μ σ1 which will be YP. So, σ2 exceeds σ_{yp}, this particular line the equation will be σ1 minus μ σ2 will be σ_{yp}. So, σ1 will exceed σ_{yp}.

Similarly, this particular line σ2 minus μσ1 will be minus σ_{yp}. This particular line it will be σ1 minus μ σ2 will be minus σ_{yp}. Let us talk about this particular line, let us say we will divide this line as a b c d e f g and h. So, this equation is for a h, this equation is for a b, this equation is for e f, this equation is for e d. Let us write down the equation for c d, you know here it is dissimilar state, σ1 is compressive, σ2 is tensile.

So, it should be σ1 plus μ σ2 will be minus σ_{yp}, this is for c d whereas, g f will have σ2 plus μ σ1 as minus σ_{yp}. Now, let us come to the line, this is for g f; let us come to the line equation b c. So, this equation will be σ2 plus μ σ1 will be σ_{yp} ok. This is for line b c and this line will have an equation σ1 plus μ σ2 will be σ_{yp}, this is for the line g h.

So, this is my failure envelope, as we understand any point lying on the periphery or inside or outside. So, if I say A, B and C, A indicates failure, B also indicates failure, C no failure. So, we can write a very interesting information here, in the second and fourth quadrants that is a value at c and g are lesser than σ_{yp}; it means when the stresses are dissimilar in nature the prediction of failure is poor.

Because, it says the system fails even before yielding is it not because, this value is not σ_{yp}, σ_{yp} is somewhere here; its somewhere here. But, it fails at g which is much lower than σ_{yp} whereas, in quadrants I and IV, when both stresses are either tensile or compressive that is similar in nature, it says one of the stress even exceeds σ_{yp}, then only the failure will occur.

Without yielding value starts whereas, when σ_{yp} is reached yielding should initiate in the material.

So, one can see very well here that this theory has lot of controversies in a biaxial stress state, when the principal stresses are same in nature or different in nature that is represented in all the four quadrants as you see in the figure on the screen. I think it is very easy and convenient for all of you to really know how do we generate these equations. For example, take this equation, this is same equation as equation 2 whereas, σ1 being positive σ2 will exceed σ_{yp}, is it not.

So, this line for example, I am marking in black dotted, this line is the rate of growth of σ2, the rate of growth of σ2 beyond σ_{yp} as an influence from σ1.

So, that is how this equation is written. Similarly, if you want to find the rate of growth of σ1 beyond σ_{yp} as an influence from σ2, this equation is written. So, by this algorithm you should be able to realize the equations of all other segments in various coordinates and various quadrants represented as shown on the screen. So, let us move on to the next theory which is total strain energy theory.

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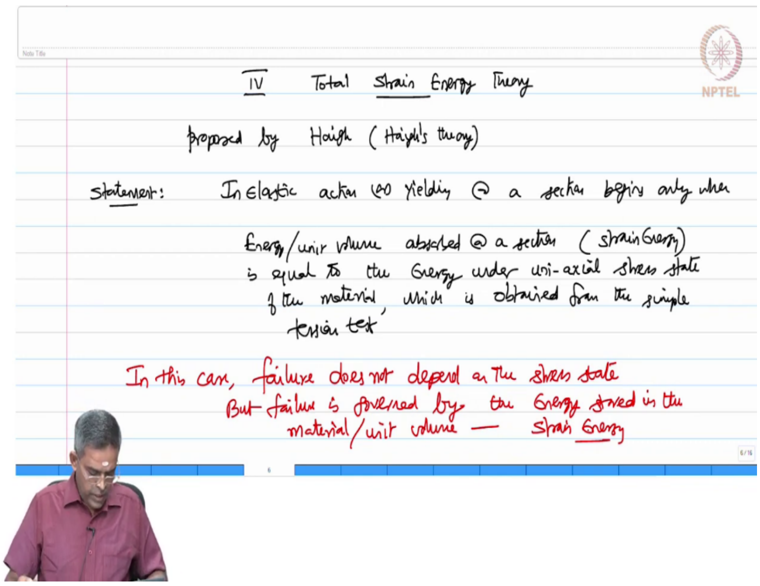

So, now we are looking at the IVth theory which is total strain energy theory. This theory was proposed by Haigh, it is also called as Haigh's theory in the literature. The statement of the theory is as follows: inelastic action or yielding at a section begins only when energy per unit volume absorbed at a point. So, energy absorption is actually the strain energy, is equal to the energy under uniaxial tensile test or uniaxial stress state of the material which is obtained from the simple tension test.

This theory has got a very interesting and major contradiction. This theory says in this case, failure does not depend on the stress state. That is a very interesting deviation this theory gives in comparison to the remaining three theories. But, failure is governed by the energy stored in the material per unit volume. So, the focus is on strain energy. Let us get more details of this theory and derive the controlling equation for the failure envelope.

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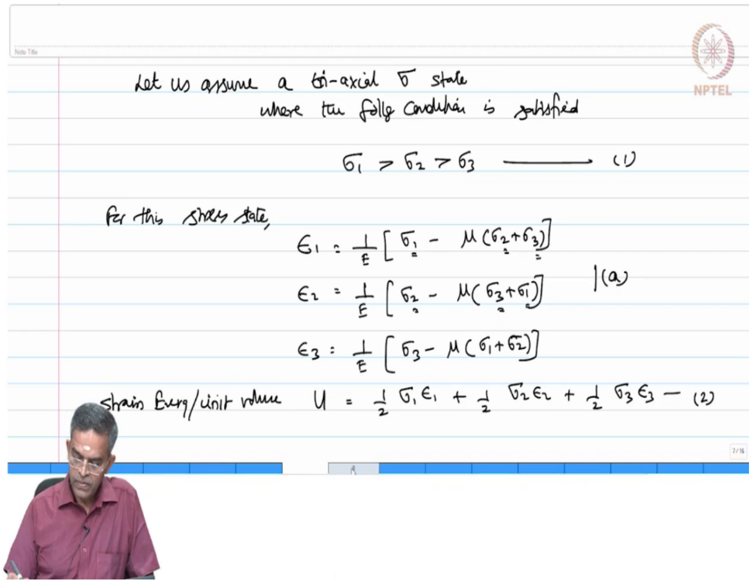

Let us assume a triaxial stress state, where the following condition is satisfied. The condition is σ1 exceeds σ2 exceeds σ 3. For this stress state ε1, ε2 and ε3 are given by the following equations, it is very easy. Please look at the screen carefully and then watch how I am writing this equation. When you talk about strain, it is stress by Young's modulus.

So, σ1 minus Poisson's effect on the remaining two axis. So, when you talk about ε 2, it is σ2 Poisson's effect on the remaining two axis. When you write please see the order, 1, 2 and 3, 2, 3 and 1. So, I think you have to follow this order. So, by this logic now you can easily write ε 3 which is σ 3 minus μ of 1 plus 2.

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)]$$

$$\varepsilon_2 = \frac{1}{E}[\sigma_2 - \mu(\sigma_3 + \sigma_1)]$$

$$\varepsilon_3 = \frac{1}{E}[\sigma_3 - \mu(\sigma_1 + \sigma_3)]$$

I call this equation as 1 a. Once I know the strain values, I can always find the strain energy per unit volume which is usually expressed as u, which is given as

$U = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3$. I call this equation as 2. Now, I have the equation for the strains along three principal axis 1, 2 and 3 which is given by equation 1 a, let us substitute 1 a in equation 2 and see what happens.

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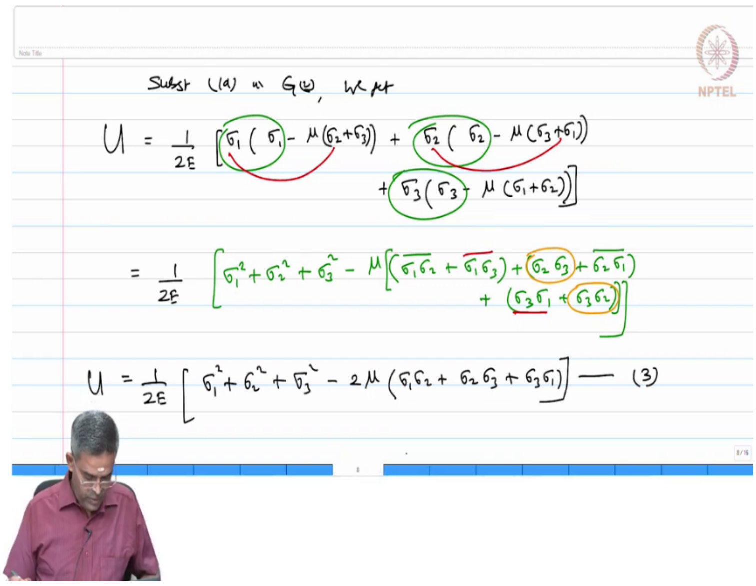

So, substituting 1 a in equation 2, we get the total strain energy per unit volume u, let's say

$$U = \frac{1}{2E}\left[\sigma_1\left(\sigma_1 - \mu\left(\sigma_2 + \sigma_3\right)\right) + \sigma_2\left(\sigma_2 - \mu\left(\sigma_3 + \sigma_1\right)\right) + \sigma_3\left(\sigma_3 - \mu\left(\sigma_1 + \sigma_2\right)\right)\right].$$

I think it is very easy to substitute, you can do it yourself. Let us simplify this. So, 1 by 2 E of one can see here this product will give me squares. So, let us say σ1 square σ2 square σ 3 square, then you will also see σ1 σ2 once and σ1 σ2 twice. So, you get two products of both repeatedly. Let us do that, minus μ times of σ1 σ2 plus σ1 σ 3 plus let us say σ2 σ 3 plus σ2 σ1 plus σ 3 σ1 plus σ 3 σ2.

$$U = \frac{1}{2E}\left[\sigma_1^{\ 2} + \sigma_2^{\ 2} + \sigma_3^{\ 2} - \mu\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 + \sigma_2\sigma_1 + \sigma_3\sigma_1 + \sigma_3\sigma_2\right)\right].$$

Let us put it in green. Now, you see one σ1 σ2 here, one is here. Similarly, σ1 σ 3 σ1 σ 3, similarly σ2 σ 3 and σ2 σ 3. So, there are two values here. So, therefore, let us simplify this further and say it is going to be 1 to 1 by 2 E of σ1 square 2 square 3 square minus 2 μ of σ1 σ2 2 3 and 3 1 agreed.

$$U = \frac{1}{2E}\left[\sigma_1^{\ 2} + \sigma_2^{\ 2} + \sigma_3^{\ 2} - 2\mu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3\right)\right]. \qquad \text{----------------(3)}$$

We will call this as equation number 3, I hope there is no confusion in this derivation. Please follow is very interesting and very important and I want you to derive as you do it on the screen. So, that you have no confusion later and you do not have to memorize anything. Learning is a very important skill friends. We are emphasizing on learning, not on memorizing. Once you learn, keep on doing it you will understand, once you understand there is no question of memorizing it, there is no requirement for that.

So, let us focus on that very clearly. So, equation 3 gives me the total strain energy per unit volume of the material.

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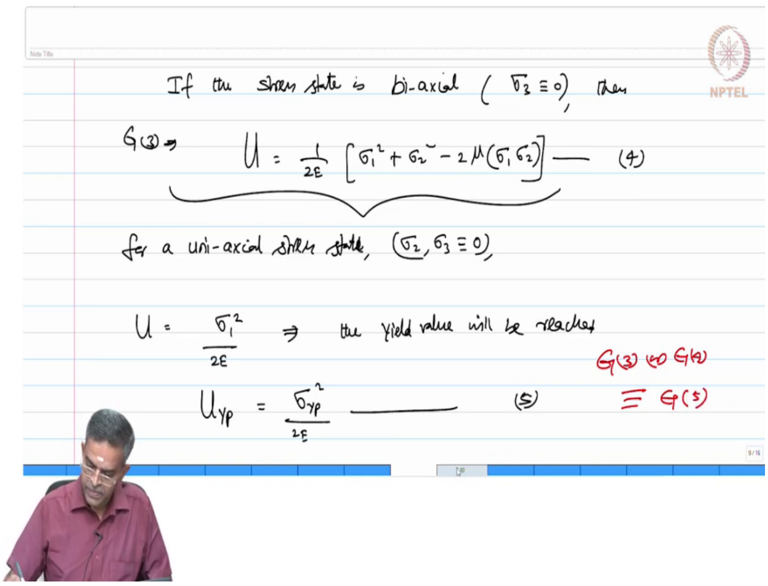

Now, if the stress state is biaxial that is σ 3 is 0, then let us see what happens. Then equation 3 will become

$$U = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 - 2\mu(\sigma_1\sigma_2)]$$

is it not, that is very obvious. Let us call this equation number 4. And, more simple for an uniaxial stress state put σ2 and σ 3 both as 0, then we get u is going to be σ_1^2 /2E.

Now, the theory says this is where the yield value will be reached. Hence, I can say strain energy at yield point will be similarly the stress at that point divided by 2 E, can I say this equation number 5. I think we all agree to land up in equation 5 without any confusion.

See equation 3 is for triaxial stress state, equation 4 is for biaxial stress state. The theory says equation 3 or equation 4 should be identical to equation 5. So, let us equate that.

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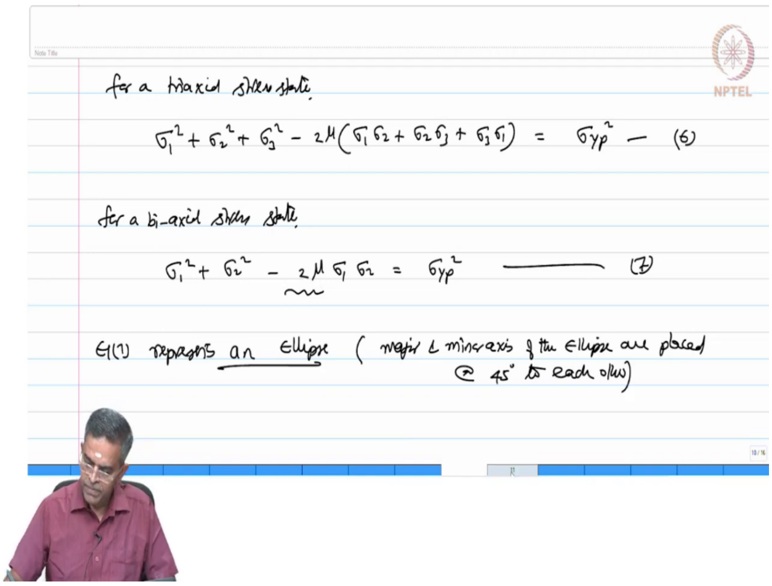

So, for a triaxial stress state should be equal to σ_{yp} square right, for a biaxial stress state we call this equation number 6 and this equation number 7. Take a simple case, please look at equation 7. Equation 7 actually represents an ellipse with the condition that the major and minor axis of the ellipse are placed at 45 degrees to each other, look at this factor.

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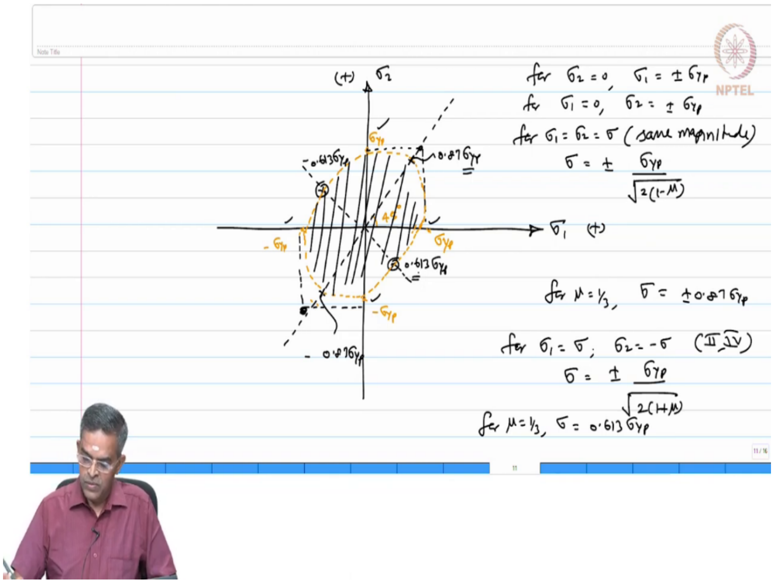

So, let us plot this envelope, let us call this as σ1 and this as σ2. Let us say this is positive axis, let us draw the major and minor axis of the ellipse at 45 degrees. Now, in this equation if any one stress value is 0 in equation 7, you will automatically see the corresponding axis will be representing the σ_{yp}. So, σ_{yp} should be marked and the curve should pass through σ_{yp}, let us mark.

So, let us draw the ellipse, that it cannot cross σ_{yp}. So, when you draw this ellipse, let us be very careful about this. So, I have drawn the ellipse. This angle is 45 degrees. Now, let us do some calculations for σ2 equals 0, we know σ1 is plus or minus σ_{yp} which is indicated here, for σ1 equals 0, σ2 will be plus or minus σ_{yp} is indicated here.

So, it has to pass through these four points which is passing through, the curve has to pass through this point. Let us say we will draw the curve passing through this point, For σ1 equals σ2 equals σ that is the stresses are same magnitude, then in that case σ will become plus or minus σ_{yp} /√2(1-μ), please do this calculation and check; same intensity, same magnitude this equation is true.

For a typical value of μ as 1/3, you will find this stress will become plus or minus 0.87 σ_{yp}. So, this value when the stress of the same intensity and nature may be tensile, may be compressive. This is 0.87 σ_{yp}, it is not σ_{yp} please understand that. So, the curve should not pass through this point. Further, for σ1 equals σ and σ2 equals minus σ, that is we are talking about now, the second and the fourth quadrant.

In that case, you will see σ will be plus or minus $\sigma_{yp}/\sqrt{2(1+\mu)}$. Please see this equation, which gives me for μ equals 1 by 3, this stress value will become 0.613 σ_{yp}. So, these values are 0.613 σ_{yp}. So, this is my failure region, I mean safe region and anything exceeding is a failure region. So, this theory predicts that at failure, yield stress is not reached. It is only about 87 percent the material fails or the failure happens in the member.

Even though the material has not touched the yield stress, when the stresses are similar in nature and same magnitude. When they are dissimilar in nature, failure occurs even much lower than σ_{yp}. So, that is a very critical and dangerous composition and interpretation this theory gives me.

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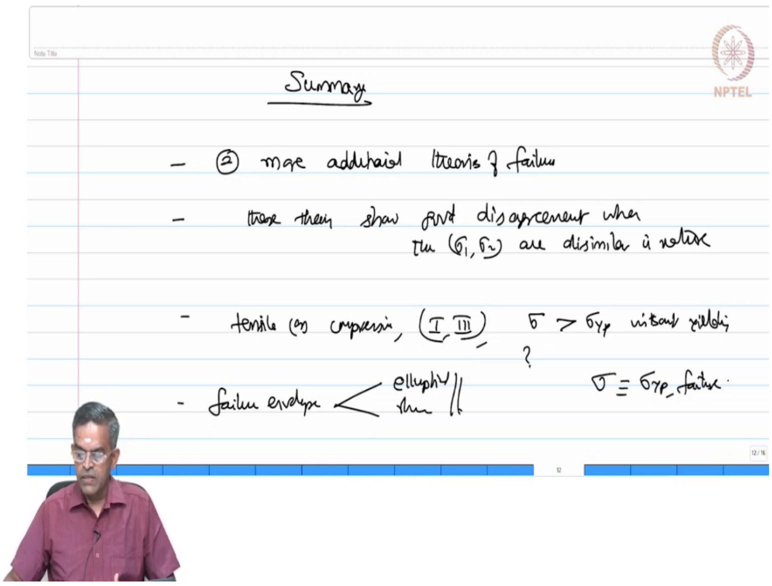

So, friends let us quickly see the summary what we learnt in this lecture. The summary says we have learnt two more additional theories. We have also saw that these theories show good disagreement when the stresses are dissimilar in nature. One of the theory says when the stresses are tensile or compressive, I mean both of them that is in first or third quadrant, the stress even exceeds σ_{yp} without yielding, which is not correct. Because, when the stress exceeds σ reaches σ_{yp} yield has to start in the material.

So, this theory one of the theories amongst these two I leave it to you, which theory is that which gives you this and the failure envelope is elliptical or a rhombus depending upon the pattern what you assume. So, it is very interesting and critical to really guess what theory will be used for assessing the failure condition. because, at failure condition the engineer desires to know what is the stress.

It is not always that when the stress reaches σ_{yp} failure starts, you know the theory says even at 0.87 the failure can happen, even at 0.613 the failure can happen and so on. So, there are serious discrepancies suggested by these theories when we talk about failure.

Thank you very much, have a good day.