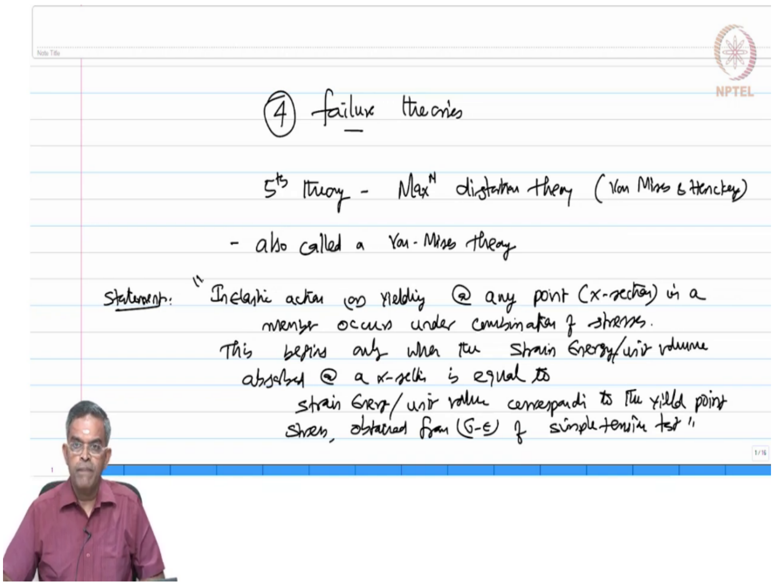
Advanced Design of Steel Structures
Dr. Srinivasan Chandrasekaran
Department of Ocean Engineering
Indian Institute of Technology, Madras

Lecture - 05
Failure theories - 3

(Refer Slide Time: 00:30)

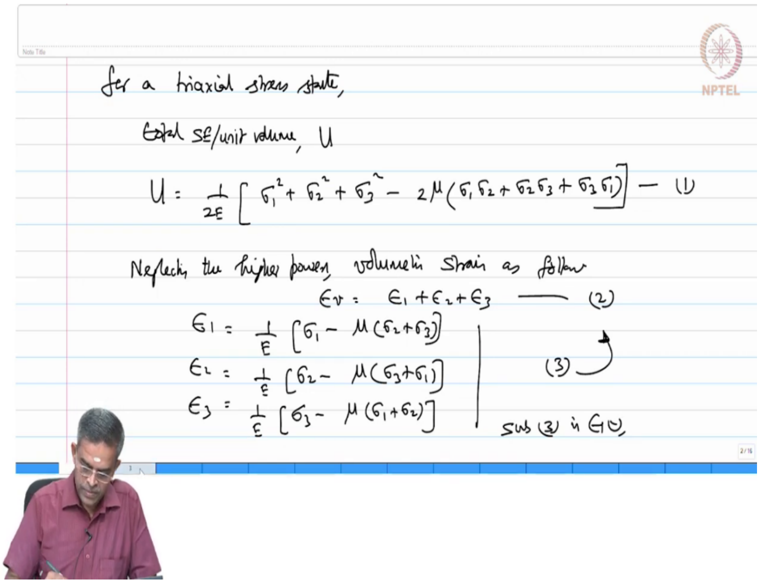


Friends, let us continue to discuss about the Failure Theories more in detail in this lecture. So, we have discussed about 4 failure theories so far. We will now discuss the 5th theory which is the maximum distortion theory, this theory was proposed by Von Mises and Henckey and famously also known as Von Mises theory.

The statement of the theory is as follows. Inelastic action or yielding at any point or cross section in a member occurs under combination of stresses. This begins only when the strain energy per unit volume observed absorbed at a section is equal to the strain energy per unit volume corresponding to the yield point stress obtained from the stress strain curve of simple tension test.

The statement looks similar to what we have in the strain energy theory, but there is a marginal variation in this theory, which we will explain as we proceed.

(Refer Slide Time: 03:34)

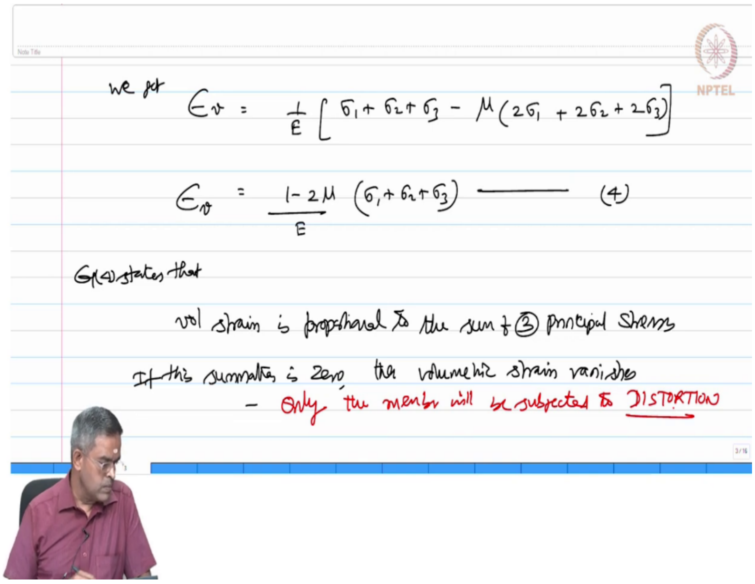


So, for a triaxial stress state we already wrote this equation total strain energy per unit volume which is given by U is expressed as follows, we call the equation number 1.

$$U = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$$

Neglecting the higher powers one can also write the volumetric strain as follows: the volumetric strain is given as a sum of the strains in all the 3 axis neglecting the higher powers where the strain in all the 3 axis are known to us; we call this equation number 3. I can substitute 3 in 2 and let us see what do we get.

(Refer Slide Time: 05:33)



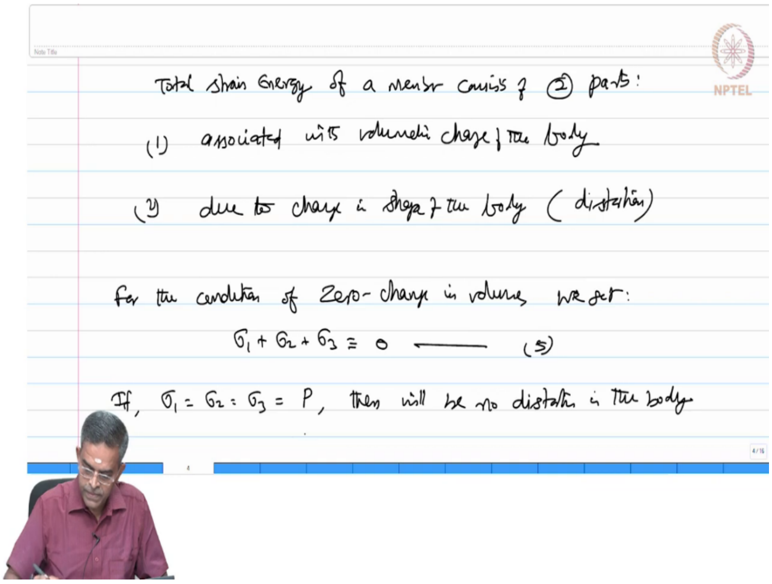
So, we get the volumetric strain as

$$\varepsilon_v = \frac{1}{E}[\sigma_1 + \sigma_2 + \sigma_3 - 2\mu(\sigma_1 + \sigma_2 + \sigma_3)]$$

which can be said as $= \dfrac{1-2\mu}{E}[\sigma_1 + \sigma_2 + \sigma_3]$. We call this equation number 4, look at this equation carefully this equation 4 states that volumetric strain is proportional to the sum of 3 principal stresses.

If the summation is 0, then volumetric strain vanishes. So, if the volumetric strain vanishes what happens to the member or the body? Only the body or the member will be subjected to distortion.

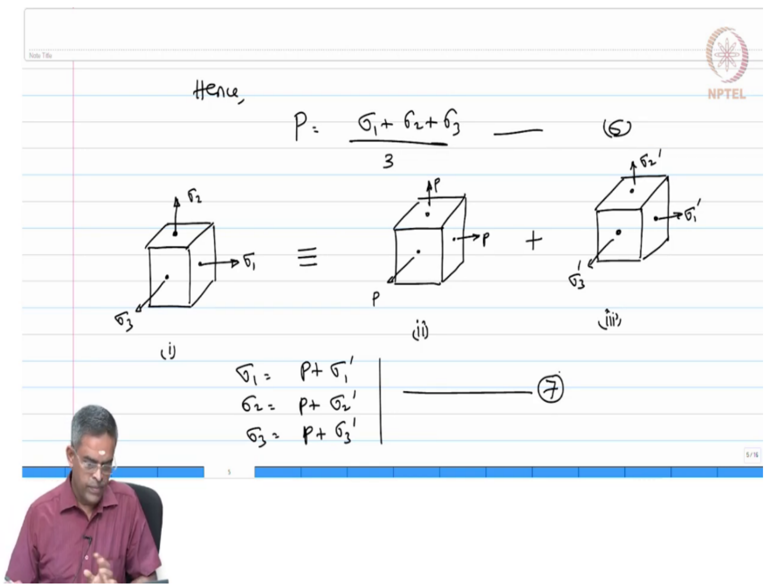
(Refer Slide Time: 07:51)



So, friends the total strain energy of a member or a body consists of 2 parts; one the one which is associated with volumetric change of the body, the one which is due to change in the shape of the body which we call as distortion.

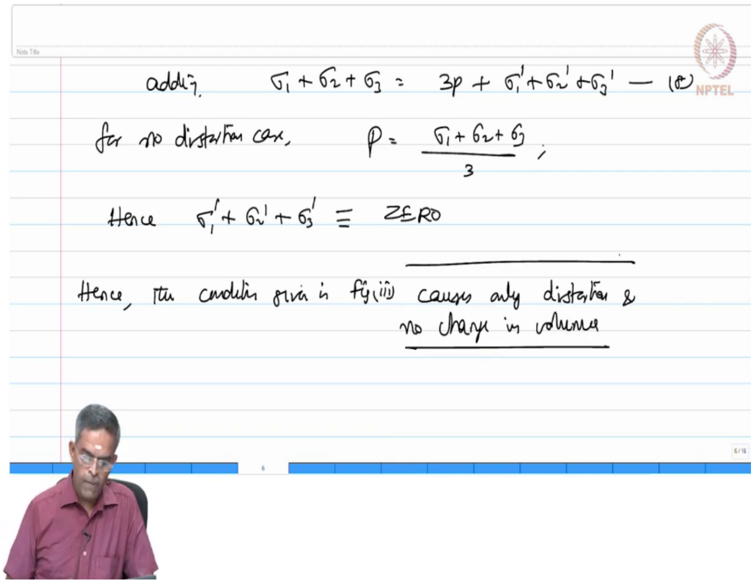
Therefore, for the condition of zero-change in volume we get **σ**1+ **σ**2+ **σ**3 as 0. Now, if these stresses are equal to magnitude P, then there will be no distortion in the body because the stresses are equal in magnitude and direction. So, no volumetric change and no distortion.

(Refer Slide Time: 09:53)



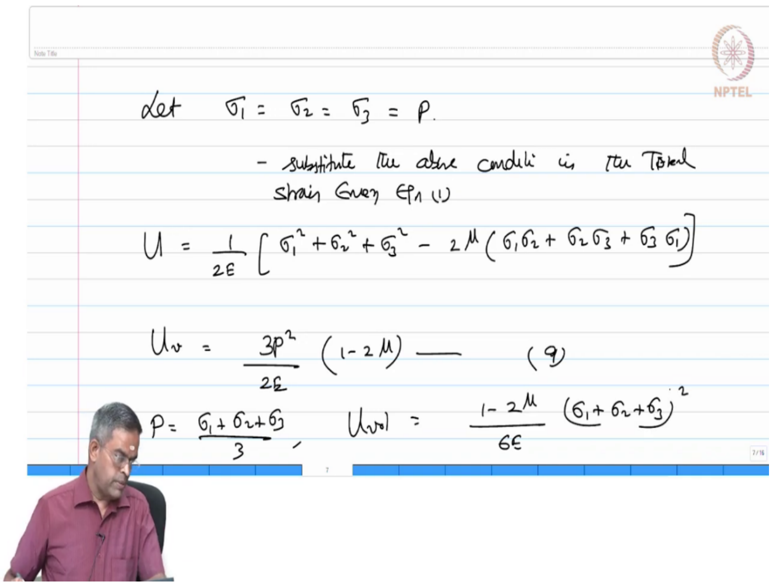
Hence, P should be **σ**1, 2 and 3 divided by 3. Let us explain this graphically, if I have a body subjected to triaxial stress state, let us say this figure 1, this can be expressed as plus. So, **σ**1 is equal to, the following relationship holds good, I call this equation number 7.

(Refer Slide Time: 11:49)



Now, adding **σ**1 plus **σ**2 plus **σ**3 is 3 P plus these variables. So, for no distortion case P should be equal to this relationship, hence **σ**1, 2 and 3 dash should be 0; For no distortion case, hence the condition given in figure 3 that is this condition given in figure 3 causes only distortion and no change in volume.

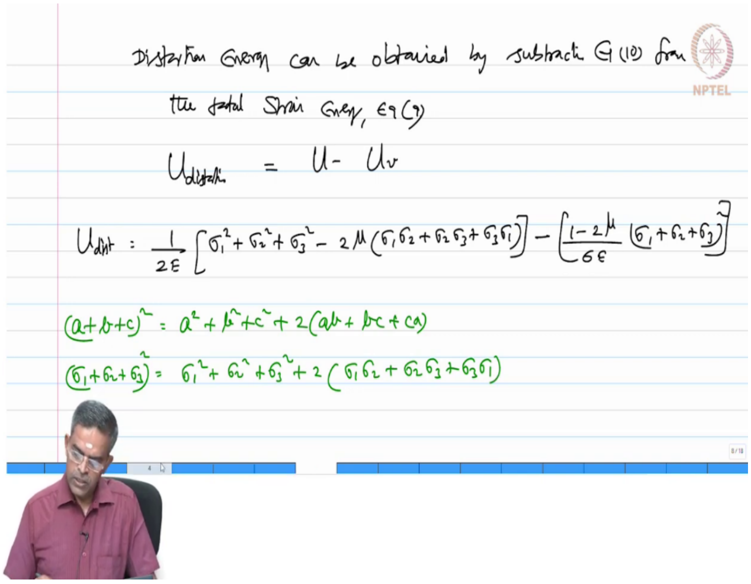
(Refer Slide Time: 13:17)



Then further let us say the stress intensities are equal. Substitute this condition in the total strain energy equation. Which says u will be now $U = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$, which now becomes $U_v = \frac{3P^2}{2E}(1 - 2\mu)$. We will call this as equation 9.

Since, P is sum of these expressed by this equation u volumetric, will be now $U_v = \left(\frac{1 - 2\mu}{6E}\right)(\sigma_1 + \sigma_2 + \sigma_3)^2$

(Refer Slide Time: 14:59)

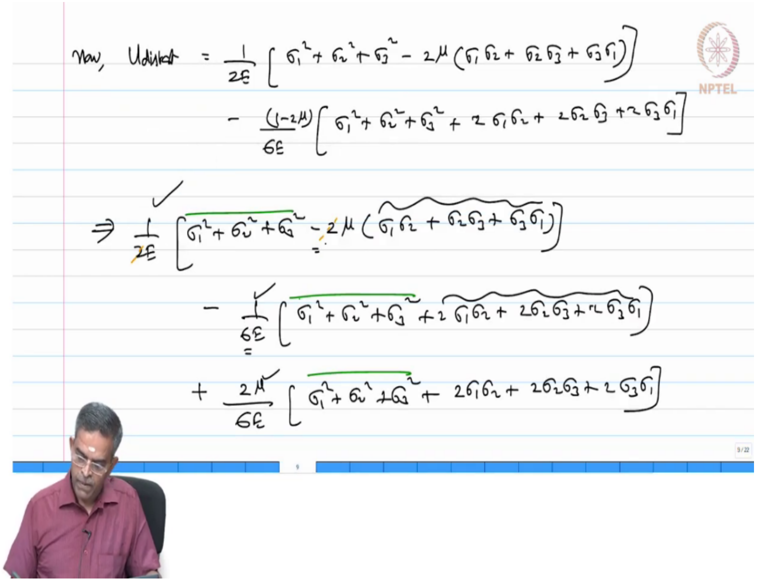


Distortion energy can be obtained by subtracting equation 10 from the total strain energy because total strain is summation of volume plus distortion; is it not, which now given the equation 9.

So, I should say $U_{distortion}$ should be total u minus u volume, let us do that. So, let us write down that u distortion will be 1 by 2 E of 1 square 2 square 3 square minus 2 mu of the products. Minus 1 minus 2 mu by 6 E, let see 2 mu by 6 E of **σ**1, 2 and 3 the whole square.

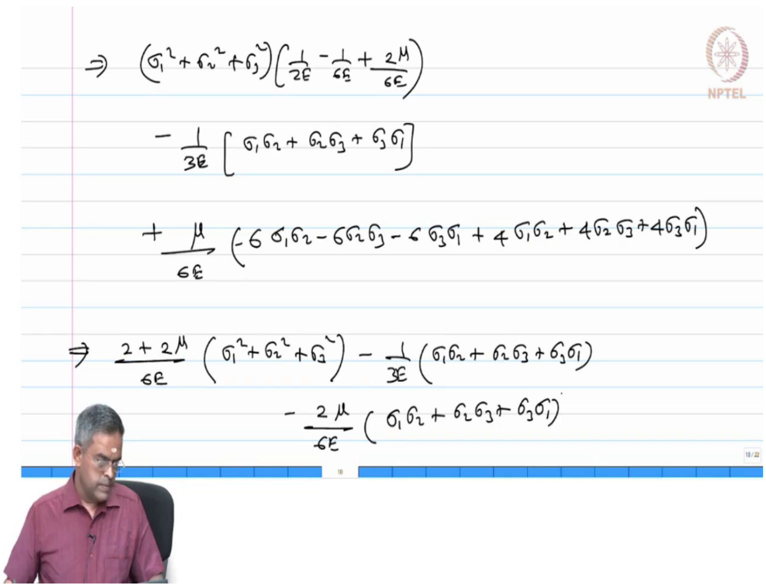
So, let us do a simple mathematics here, $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$. Hence, **σ**1, 2 and 3 the whole square will become 1 square 2 and 3 square plus twice of 1 2, 2 3 and 3 1. So, let us substitute this in the previous expression.

(Refer Slide Time: 17:26)



So, now $U_{distortion}$ will be 1 by 2 E minus 1 by 6 E. Let us simplify this which will be 1 by 2 E of minus 1 by 6 E of plus 2 mu by 6 E of.

(Refer Slide Time: 19:24)

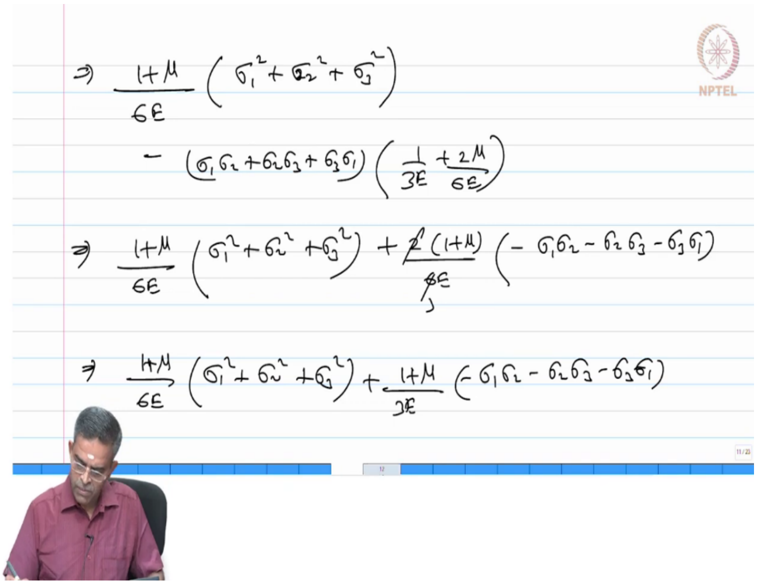


Which will become **σ**1 square 2 square and 3 square. Let us take it common and see what happens. If I take this terms common then I can say which will be 1 by 2 E, which is this term minus 1 by 6 E which is this term plus 2 mu by 6 E which is this term.

Then we can also the remaining terms minus 1 by 3 E of **σ**1 **σ**2 3 and 3 1; This term comes from this, because twice 6 E. So, 1 by 3 E correct plus mu by 6 E of I can say here I get only mu by 6 E, where I also have mu by 2 E here because mu by E because this 2, this 2 and this 2 goes away so mu by E. So, I can now say it is going to be 6 times of **σ**1 **σ**2 **σ**2 **σ**3 **σ**3 **σ**1, that is these terms all are negative ok.

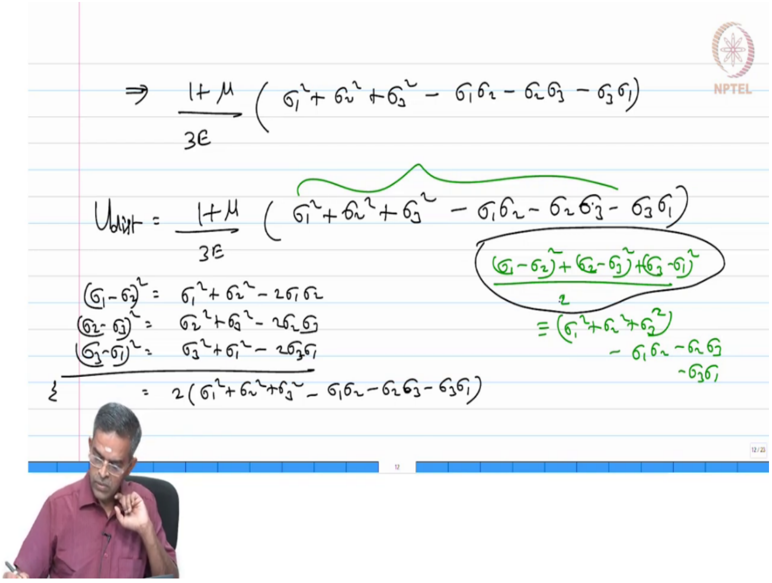
Because you see they are all negative plus I have this term also. So, there is 2 and 2 = 4 here. So, I should say plus 4, let us simplify this. So, which I do that I will get 2 plus 6 mu by 6 E of squares minus 1 by 3 E of products minus 2 mu by 6 E of products.

(Refer Slide Time: 22:40)

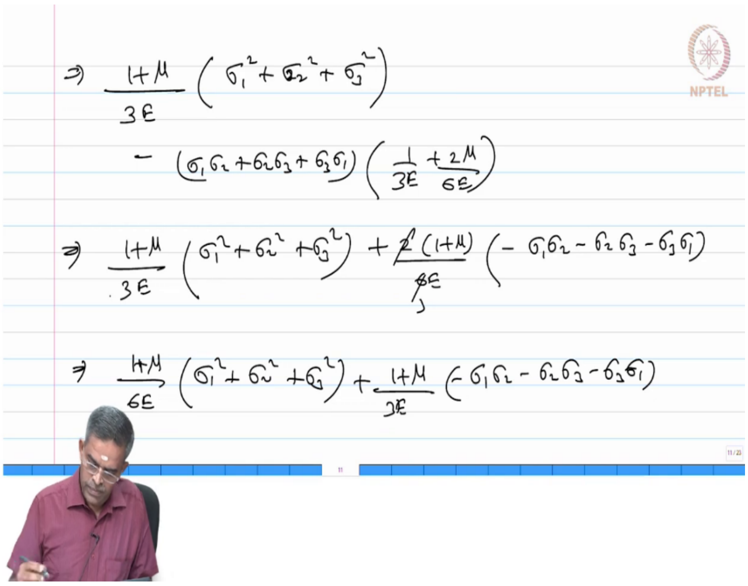


Let us simplify this further which will become minus **σ**1 **σ**2, 2 3 and 3 1 of 1 by 3 E plus 2 mu by 6 E; See here. 1 by 3 E 2 mu by 6 both are negative. So, I take a negative here I further simplify this, I get 1 plus mu by six6 E of the squares plus 2 into 1 plus mu by 6 E of minus products, which says 1 plus mu by 6 E of plus 1 plus mu by 3.

(Refer Slide Time: 24:24)



(Refer Slide Time: 24:45)



I can further simplify, this which will now become 1 plus mu by 3 E of the squares minus I think I will get here let me make this correction I will get here 3 Es right; please make that correction. So, I get here 3 E because 2 times of 1 plus mu by 6 E becomes 1 plus mu by 3 E ok; so 3 E.

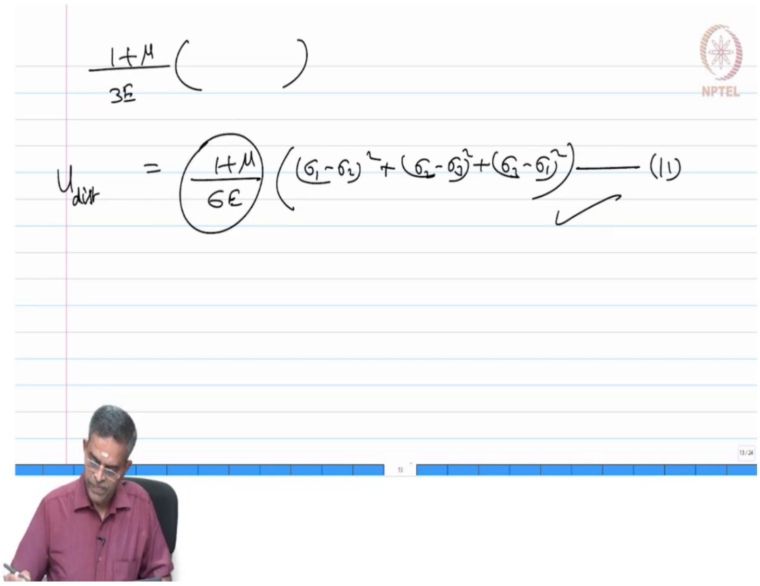
So, I take 1 plus mu by 3 E as common. So, **σ**1 square minus the products, my u distortion is actually equal to 1 plus mu by 3 E of 1 square 2 square and 3 square minus the products ok.

So, let us do little bit more mathematics in this, you know **σ**1 minus **σ**2 the whole square is this way, the whole square is this way ok.

Let us sum them up, which will be equal to twice of **σ**1 square 2 square and 3 square ok, minus the products. What I can say now is, **σ**1 minus **σ**2 the whole square plus **σ**2 minus **σ**3 the whole square plus 3 minus 1 the whole square by 2 should be equal to **σ**1 square 2 square 3 square minus **σ**1 **σ**2 2 3 and 3 1.

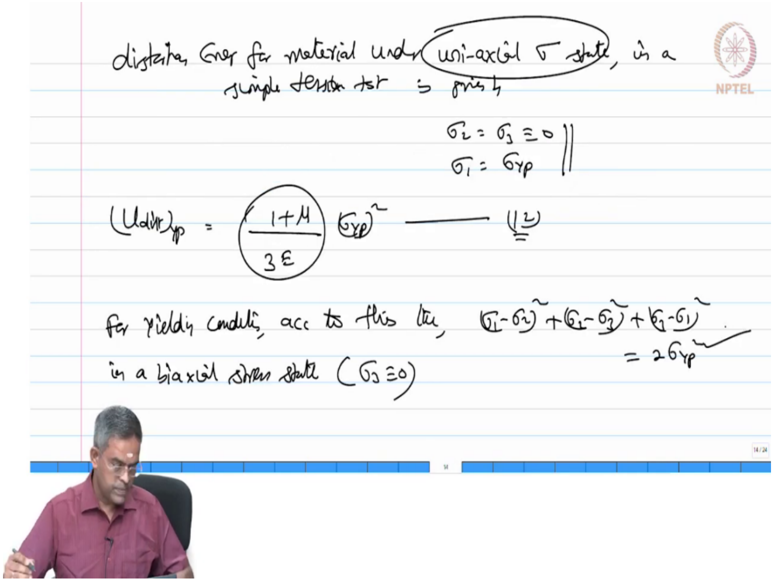
So, what I have it here? So, I am going to replace this bracket term with this. So, there is 1 plus mu by 3 E, there is 2 here.

(Refer Slide Time: 27:42)



So, I can now write 1 plus mu by 3 E of, this will now become 1 plus mu by 6 E of **σ**1 minus **σ**2 the whole square, am I right. So, therefore friends u distortion is given by this equation now. The equation number will be we can say 11 ok, let us say 11 will be given by 1 plus mu by 6 E of this equation.

(Refer Slide Time: 28:45)



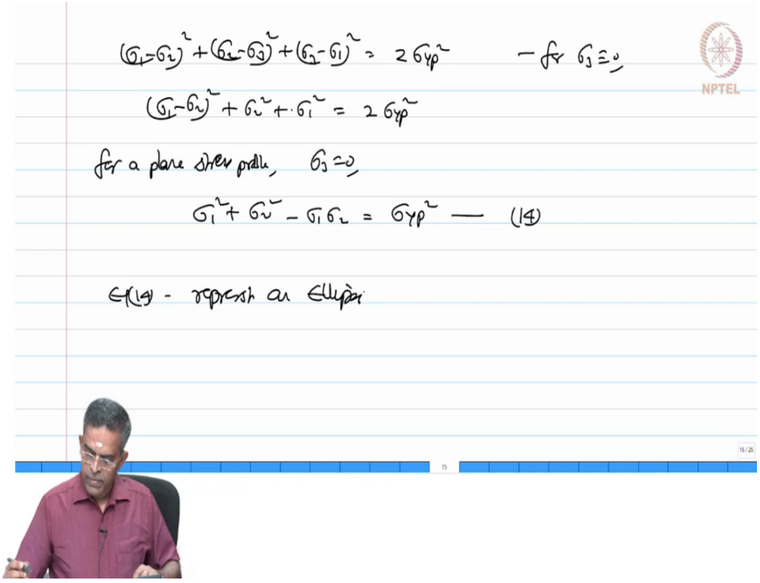
Now, having said this distortion energy for material under uniaxial stress state in a simple tension test is given by the following expression.

So, in uniaxial stress state **σ**2 and **σ**3 will be said to 0 and **σ**1 will reach **σ**yp, Hence, u distortion at yield point will be 1 plus mu by 6 E of **σ**yp square, Simply substituting this expression and make **σ**2 and **σ**3 is 0, you get this; We call this equation number 12.

For yielding condition according to this theory in a biaxial stress state, where **σ**3 will be said to 0 **σ**1 minus **σ**2 the whole square plus **σ**2 minus **σ**3 the whole square plus 3 minus **σ**1 the whole square should be equal to 2 **σ**yp square;. How do you get this? It is very simple, this is 1 plus mu by 6 E here and this is 1 plus mu by 6 E, this is going to be 3 E.

So, I can now straight away say equating this with the previous case. So, I can write this equation know, I can write this equation is it not; $\boldsymbol{\sigma}_{YP}^2$.

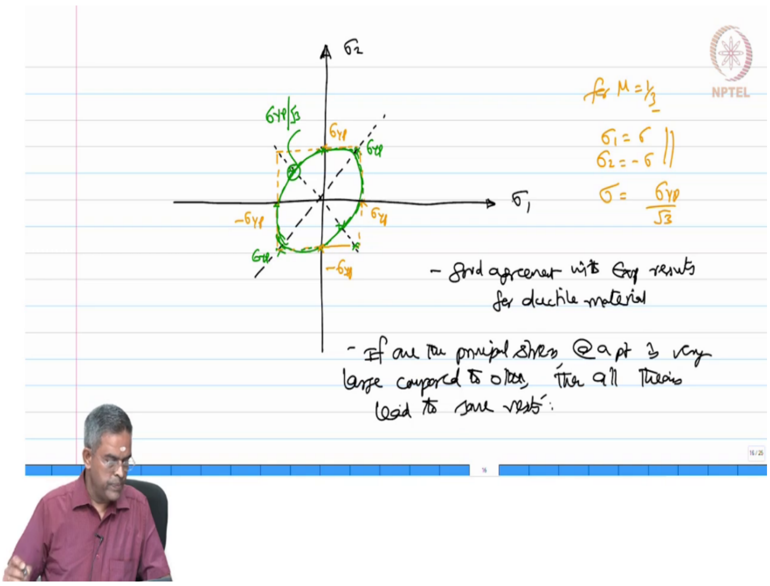
(Refer Slide Time: 31:22)



So, **σ**1 minus **σ**2 the whole square is the whole square **σ**3 is the whole square is 2 **σ**yp square, for **σ**3 equals 0 **σ**1 minus **σ**2 the whole square plus **σ**2 the square plus **σ**1 square should be equal to 2 **σ**yp square.

So, for a plane stress problem **σ**3 is 0. So, **σ**1 square plus **σ**2 square minus **σ**1 **σ**2, should be equal to **σ**yp square. Again, if you look at this equation, equation 14 also represents an ellipse.

(Refer Slide Time: 32:44)

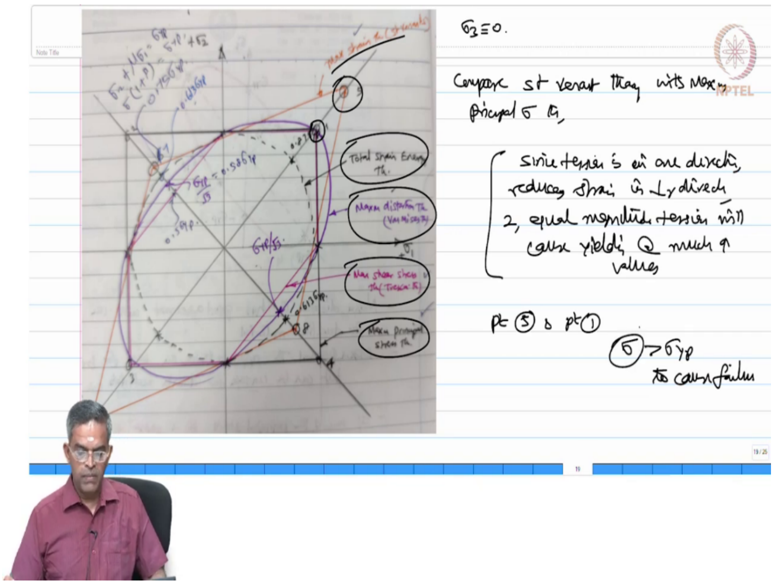


Let us try to plot this envelope, let us say we get **σ**yp in all the cases, I will mark that as an envelope for our control line.

So, these are all **σ**yp and minus **σ**yp ok. So, for mu equals 1 by 3 and **σ**1 equals sigma, **σ**2 equals minus **σ**you will find the **σ**value will become **σ**yp by root 3 when you substitute in this equation. Now, let us draw the envelope which is going to be these are the points, it must touch here ok, this is envelope.

So, this value **σ**yp by root 3. Whereas, this values are **σ**yP. So, this theory has a good agreement with the experimental results, for ductile material. Further, if one of the principal stresses at a point is very large compared to others, then all theories lead to same results that is very interesting.

(Refer Slide Time: 35:46)

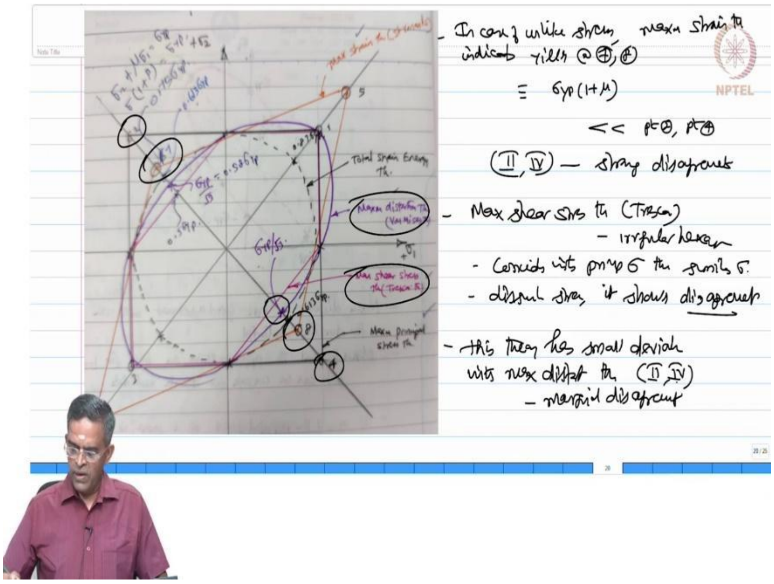


This is a graphical comparison of all the theories, as we just now derived and plotted in this previous 3 lectures. So, in this plot **σ**3 is taken as 0. So, we are plotting for a biaxial stress state. So, all theories are the maximum principal stress theory, maximum strain theory, total strain energy theory, Von Mises and maximum shear stress theory ok. This is the plot what you have obtained please turn back your notes and compare them is what we are getting. We can write down some important observations for this.

When we compare St. Venant's theory with maximum principal stress theory, it is seen that since tension is in one direction, reduce a strain in the other one in the perpendicular direction, 2 equal magnitude tension will cause yielding at much higher values. We can compare this point 5 and point 1, point 5 is here point 1 is here when you have got both stresses as tensile, you will see that yielding is caused at much higher stress exceeding **σ**yp, is it not.

So, stress exceed **σ**yp to cause failure, if I say failure is a yielding then this stress is exceeding **σ**yp to cause yielding.

(Refer Slide Time: 38:25)



In case of unlike stresses maximum strain theory indicates yielding at point 7 and 8, the values are equal to **σ**yp of 1 plus mu these values are much lesser than 0.2 and 0.4.

So, it means in quadrants II and IV, there is a very strong disagreement. What are these quadrants II and IV? When the stresses are dissimilar in nature of the same magnitude the disagreement between the theories are very large, maximum shear stress theory that is given by Tresca, which is indicated by this curve. Indicated by an irregular hexagon coincides with the principal stress theory in case of similar stresses.

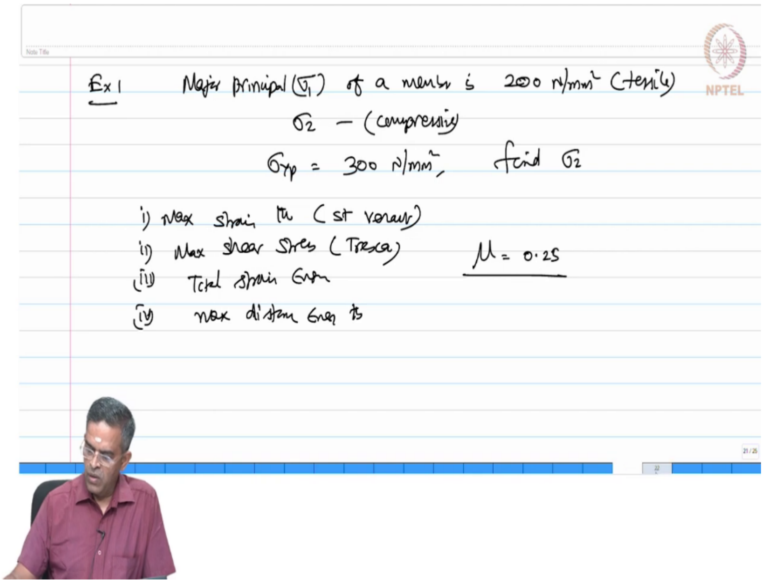
In case of dissimilar stresses, it shows disagreement. Maximum shear stress theory has only a very small deviation with maximum distortion theory. Maximum distortion theory is this, the purple line. Let us look into quadrants II and IV, the maximum shear stress theory is in cyan line, is got a very marginal disagreement.

So, friends, if I really wanted to find out what would be my stress at failure and if you believe yielding is one of the failure modes and I want to estimate the stress at yield under a different stress state conditions and different stress magnitudes different theories give me disagreemented and non-proportional results.

So, it is very difficult for an engineer to predict what would be my deciding stress value because certain theory say stress can be even more than **σ**yp, to cause failure. Certain theories and certain quadrant says stress is definitely lesser than **σ**yp to cause failure. It means even at lesser stresses than **σ**yp failure can be caused.

So, for causing a failure stress need not be equal to yield value. So, that is what we have learnt by comparing these theories and assessing failure.

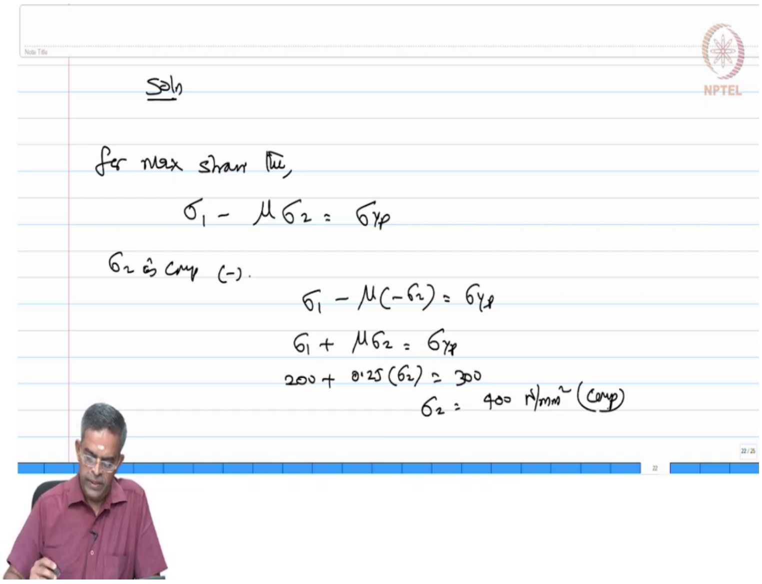
(Refer Slide Time: 42:48)



Let us quickly do a problem and understand this more in detail. The major principal stress let us say **σ**1 of a member is 200 N/mm^2 tensile.

The minor principal stress seems to be compressive. If the yield strength of the material is 300 N/mm^2, find the minor principal stress using the following theories. Maximum strain theory, which is St. Venant's theory, maximum shear stress theory which is called a Tresca theory, using total strain energy theory, also using maximum distortion energy theory. We can take μ as 0.25 for the problem.

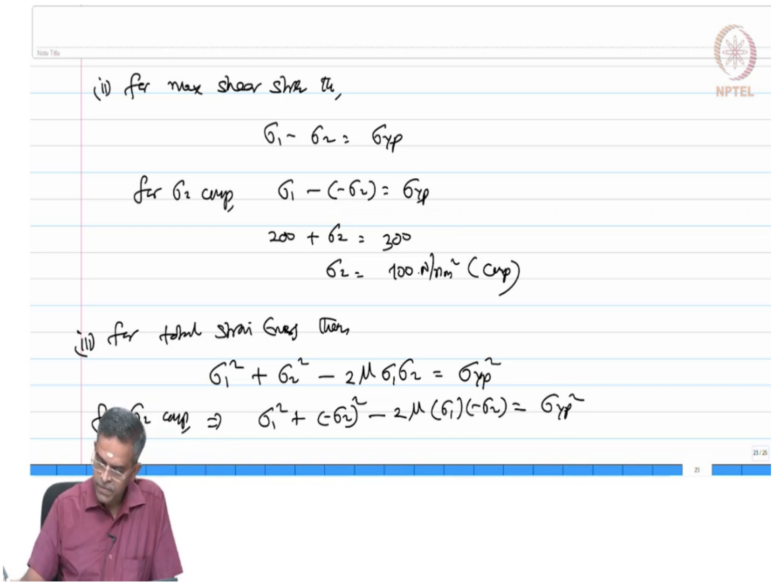
(Refer Slide Time: 44:21)



Let us solve this problem, for maximum strain theory, the governing equation is **σ**1 minus mu **σ**2 is **σ**yp, please check the derivation what we did. In this case **σ**2 is compressive, so we taken as negative ok. So, **σ**1 minus mu of minus **σ**2 is **σ**yp. So, **σ**1 plus mu **σ**2 is **σ**yp.

So, **σ**1 is known and mu is known, **σ**2 is not known and **σ**yp is known, I can solve this to get **σ**2 as 400 compressive, when I use this first theory.

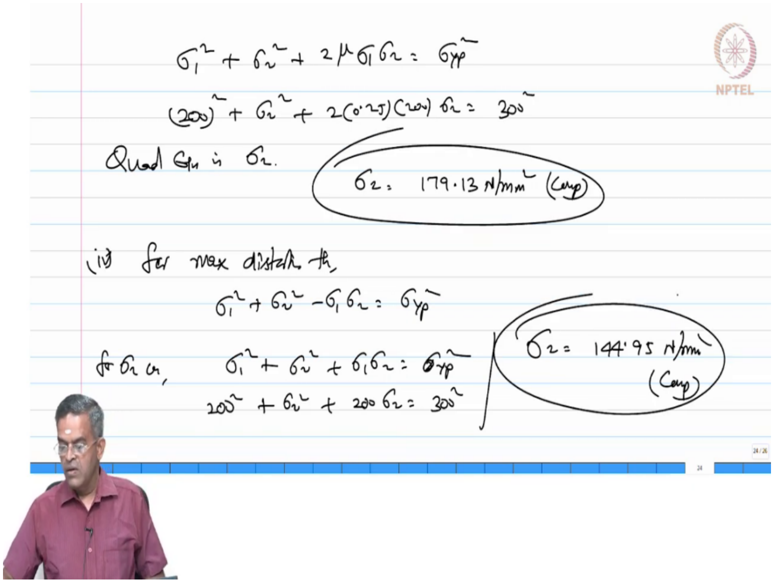
(Refer Slide Time: 45:26)



For maximum shear stress theory, the governing equation is **σ**1 minus **σ**2 is **σ**yp. So, for **σ**2 compressive, **σ**1 minus of minus **σ**2 is **σ**yp. So, 200 plus **σ**2 is 300. So, **σ**2 is 100 N/mm^2 compressive.

For total strain energy theory this is 2, the governing equation is **σ**1 square 2 square minus 2 mu **σ**1 **σ**2 is **σ**yp square. Please check the governing equation for **σ**2 compressive.

(Refer Slide Time: 46:57)

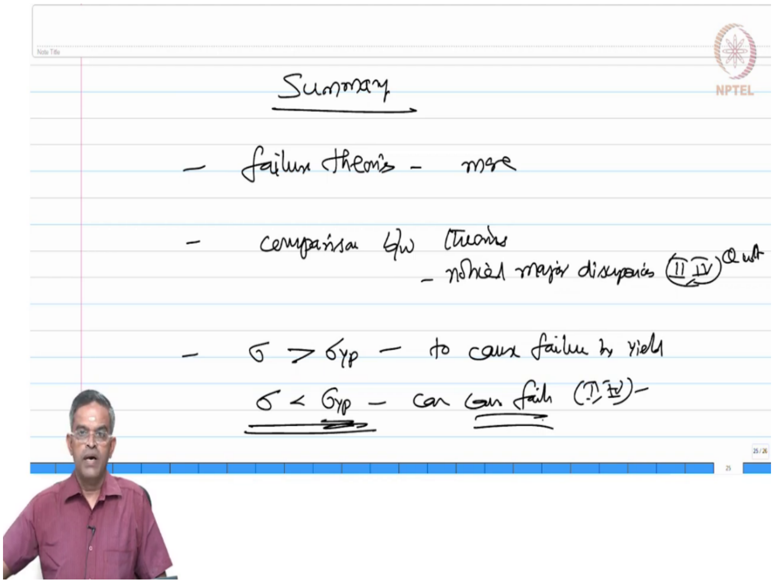


This equation now becomes **σ**1 square minus **σ**2 the whole square minus 2 mu of **σ**1 minus **σ**2 is **σ**yp square, which is **σ**1 square **σ**2 square ok plus 2 mu **σ**1 **σ**2 is **σ**yp^2.

So, 200^2 plus **σ**2^2 plus 2 of 0.25 of 200 of **σ**2 is 300^2. So, this becomes a quadratic in **σ**2, you can solve this quadratic and you find **σ**2 as 179.13 N/mm^2 compressive.

Now, for maximum distortion theory, the equation is given by for **σ**2 compressive. So, let us substitute, that again it is a quadratic in **σ**2 solve we get **σ**2 as 144.95 newton per mm square compressive. So, friends you see different theories estimate different principal stresses; is it not. So, that is the whole analogy what we learned from this theory.

(Refer Slide Time: 48:46)



So, in this lecture we learnt more about failure theories, we also understood a comparison between the failure theories and noticed a major discrepancy in II and IV quadrants, correct. We have also learnt the stress need to be more than **σ**yp to cause failure by yielding, that is what one of the theory says.

All the theory says stress even at less than **σ**yp can cause failure in quadrants II and IV, is it not, which is not acceptable because if I take yielding as a failure criteria no stress lesser than yielding can cause me failure. So, it is very interesting that we have to fix up the stress at which the failure is initiated, then only we can start proceeding that as a landmark for design.

So, that is a very interesting argument which we had in this couple of lectures and learnt how different failure theories give us different understanding of failure modes between fracture and yielding, that is for ductile and brittle materials.

Thank you very much, have a good day, bye.