Advanced Quantum Mechanics with Applications
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Lecture – 16
Qubits, EPR Paradox

For the next few classes we are going to discuss very interesting subject in quantum mechanics, which has gained a lot of importance in the recent times. And is expected to be in focus for at least a quite a few years from now, which is called as a quantum computation.

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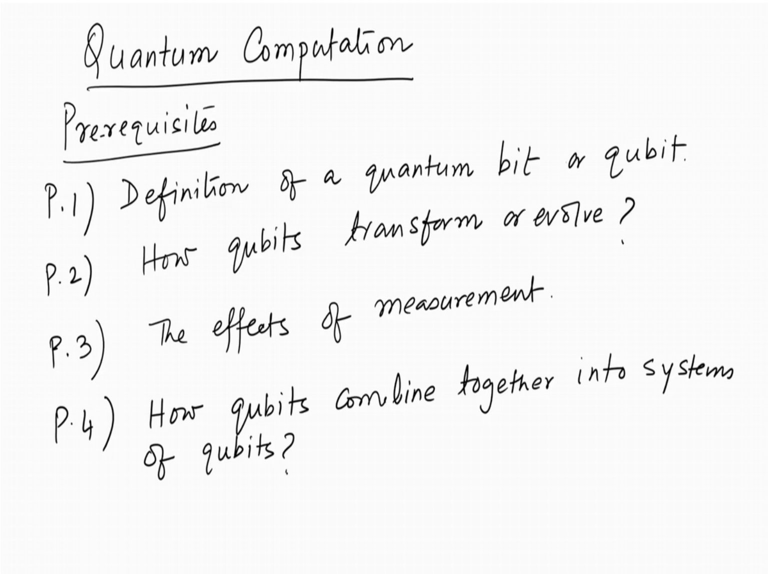

In order to discuss quantum computation, we will look at it is a various prerequisites, and its relevance to a number of things such as EPR paradox and so on. So, let us start with some prerequisites of quantum computing. Let us call them as a P 1 so, the definition of a quantum bit which is called as a qubit. Second one is that how qubits transform or evolve?

Third is, the effects of measurement. And the fourth one is how qubits combine together into systems of qubits? So, these are the 4 prerequisites that we should we would be talking about briefly, and their relevance to the subsequent topics we will see how that comes about. So, let us talk about the first one that is a definition of a qubit.

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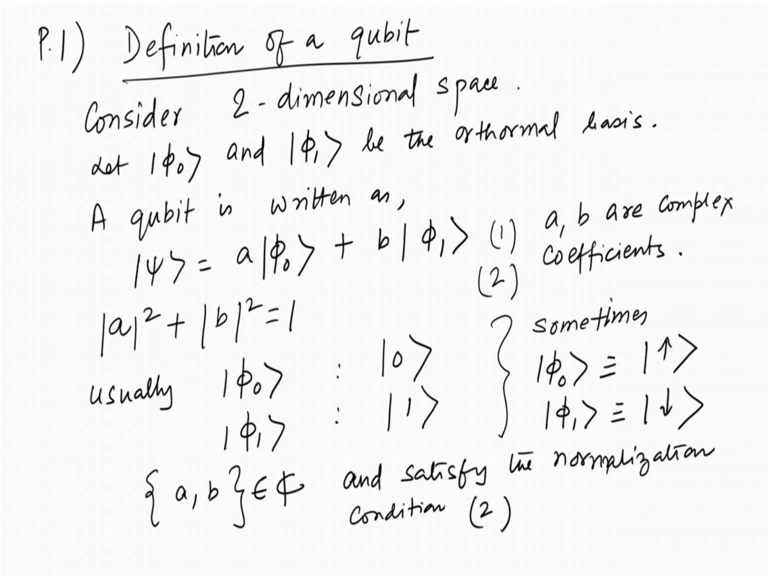

So, consider a 2 dimensional space. So, let phi 0 and phi 1 be the orthonormal basis for this 2 dimensional space. So, a qubit is written as I said that is just the short form of a quantum bit. It is sum a phi 0 plus b phi 1 with a constraint that a mod square plus b mod square equal to 1. And a's and b's are complex coefficients ok.

So, usually we write phi 0 as a taken as 0 and phi 1 is taken as 1. There are other notations such as the z component of a spin half particles so, sometimes these phi 0's and phi 1's are also written as a phi 0 is up and a phi 1 is down and so on, ok. We will go ahead with the 1 0 notation. Now you see that as opposed to a classical bit, which were using a 0 and 1, we would have written simply 1 number.

However, with this qubit there are infinite possibilities of writing numbers with differently choosing the a and b or the a mod square and the b mod square, and each will give rise to a new number. So, there are other things such as you know the voltage, the excited voltage and the ground state voltage. Those were also taken sometimes as the notations. So, a b they belong to the so, they belong to the complex space and satisfy the normalization condition. Let us call it as if we call this as 1, then this is called as 2 normalization condition 2, alright.

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P.2) Evolution of qubits

The evolution of a closed quantum system is described by a unitary transformation.

$$|\psi'\rangle = U|\psi\rangle \qquad U^{\dagger}U = \mathbb{1}$$

Example 1. $|\psi\rangle = a|0\rangle + b|1\rangle$

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$|\psi'\rangle = U|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$$

$$= b|0\rangle + a|1\rangle$$

So, let us go to the next one which is evolution of qubits. So, the evolution of a closed quantum system is described by a unitary transformation. So, if I have a state psi it upon operating it by a evolution operator becomes psi prime. And this operator is unitary in the sense that the U dagger U is equal to 1. So, let us take an example of this say psi is equal to a 0 plus b 1, just the way we have taken it.

So, U can be chosen as 0 1 1 0, you can check that U dagger U equal to 1. So, psi prime it is equal to U psi which is equal to 0 1 1 0 and a b. So, this becomes same as b a, which is nothing but equal to b operating on or b on b multiplied by 0 and a multiplied by 1. So, this unitary operator changes the original state psi from a 0 b 1. And the interchange their coefficients so, that it becomes b 0 and a 1.

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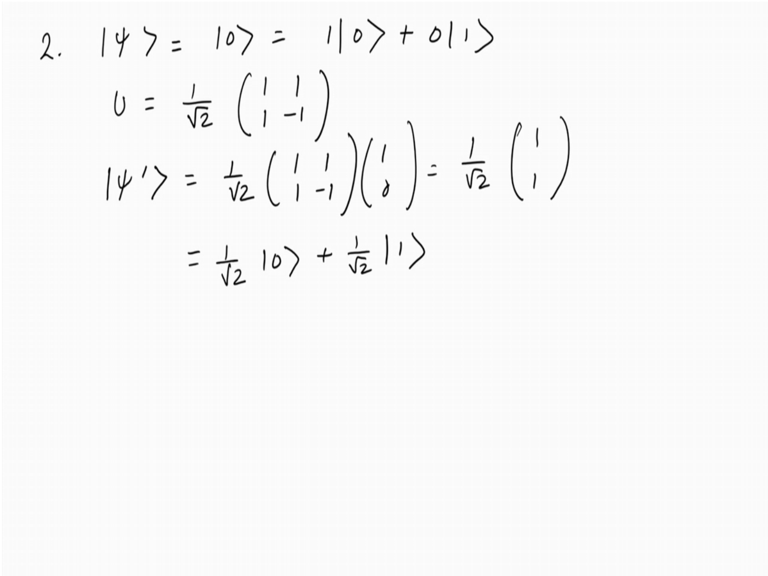

See your second example. So, psi can be purely in the 0 state which means that I can write it as 1 0 plus 0 1. And U can be taken as 1 by root 2 1 1 1 minus 1. In fact, one should again check that U dagger U equal to an identity matrix. So, the evolved psi under this operation of U is 1 1 1 minus 1 and 1 0. So, that becomes equal to 1 by root 2 1 1. So, this is 1 by root 2 0 plus 1 by root 2 1. And so, basically we have started from a state 0 and landed up with a half probability of having a 0 and 1 by evolving the system through this operator u.

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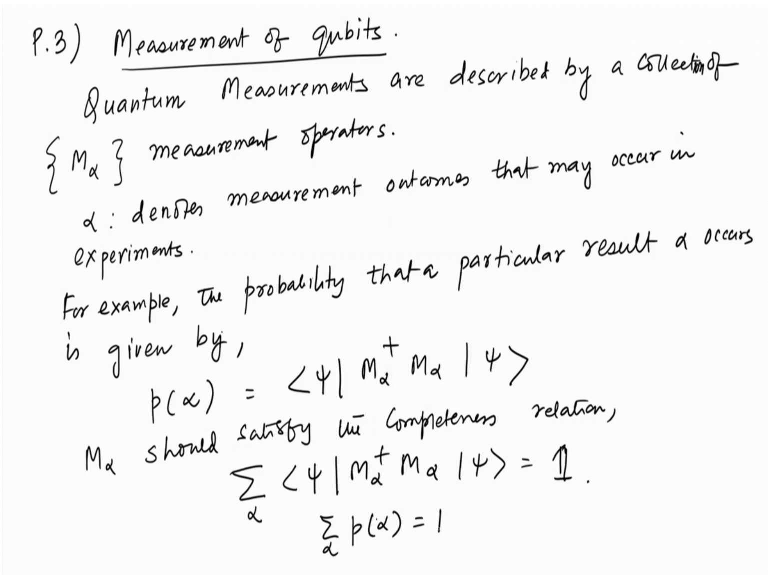

Third, so the quantum measurements are described by a collection of M alpha operators measurement operators. So, alpha denotes measurement outcomes that may occur in experiments. For example, the probability that a particular result alpha, alpha occurs is given by so, p alpha equal to the state, and M alpha dagger M alpha and the state. So, the measurement operators M alpha should satisfy the completeness relation sum over alpha psi M alpha dagger M alpha psi equal to 1, ok. And this happens because of the fact that alpha p alpha is equal to 1, ok.

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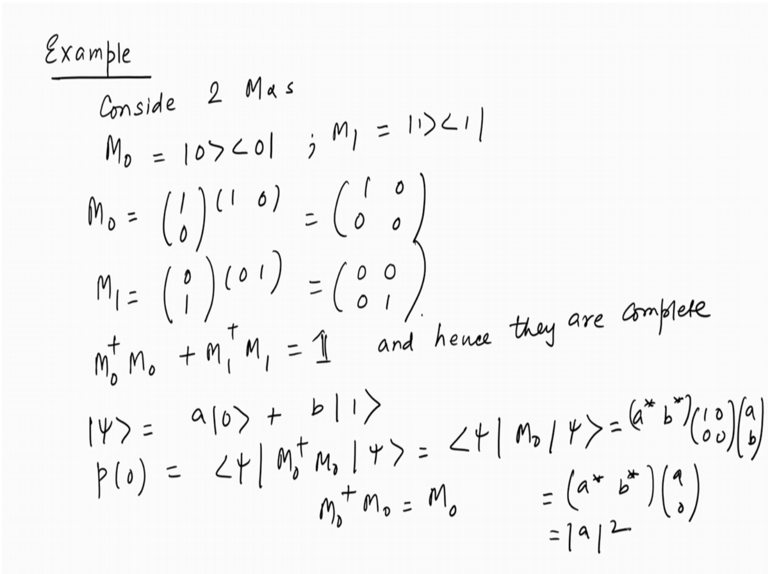

So, take an example. Consider 2 M alphas so, call them as M 0 which is equal to 0 0 and M 1 equal to 1 1. So, M 0 is equal to 1 0 1 0, it is equal to 1 0 0 0 and M 1 equal to 0 1 0 1 equal to 0 0 0 1, ok. So, one can check easily that M 0 dagger M 0 plus M 1 dagger M 1 equal to 1. And hence, they are complete. So, if I now choose a psi to be equal to a 0 and b 1; p 0 would require psi M 0 dagger M 0 psi, which is nothing but psi M 0 psi, because you can check that M 0 dagger M 0 equal to M 0. And this thing becomes a star and b star. And now I have a 1 0 0 0 and a b that typically gives equal to a star b star and a 0 which gives a mod square.

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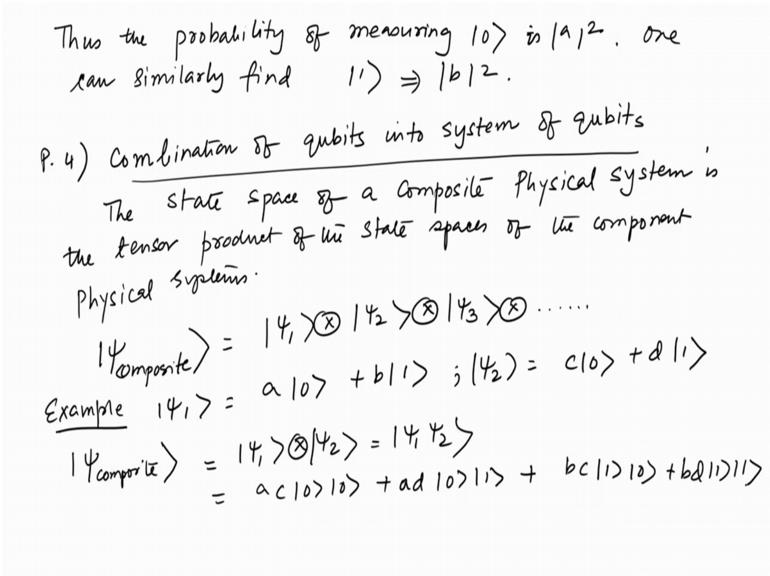

So, thus the probability of measuring 0 is a square. And one can similarly found find that the probability of measuring one would be b square and so on, ok. So, let us look at the 4th one how p 4 a combination of qubits into system of qubits. So, the state space of a composite physical system is the tensor product of the state spaces of the component physical systems. By this what I mean is that so, let us write a psi composite it is equal to the tensor product.

Take an example and psi 2 equal to C 0 plus d 1. So, the composite space is psi 1 multiplied by psi 2, it is equal to psi 1 psi 2, which is equal to a C 0 0 plus a d 0 1 plus b C 1 0 plus b d 1 1. So, that is the composite space. Let us now take on a concept called as quantum entanglement.

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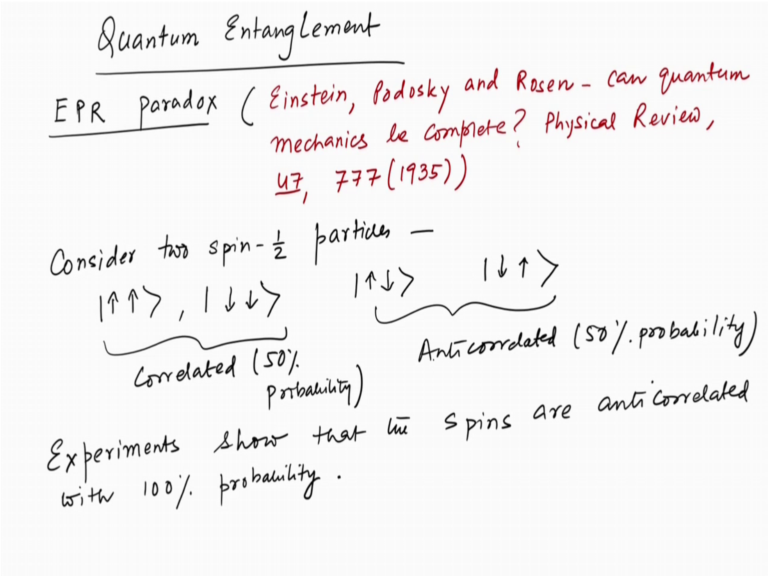

So, before we actually discuss quantum entanglement, let us look at the EPR paradox. So, this is by Einstein Podosky and Rosen, who wrote a paper in 1935. The paper is called can quantum mechanics be complete. And the reference of the paper is it is in physical review. It is volume is 47 pages 777 and it is in the year 1935. So, let us try to understand what is it.

So, consider 2 spin half particles, 2 spin half particles. So, they have a priori all these 4 states equally probable. Up up, down down, and up down, and down up. So, these are called as correlated, and with 50 percent probability. And these are called as anti-correlated. And again it should be 50 percent probability ok. So, what it means is that because, this is spin half particles the z component of spins are up and down. So, if you make a measurement of them, and then you would find them both to be in the up state, both to be in the down state, the first one to be in up, second one to be in down and the second one to be in down and the, the first one to be in down and the second one to be in the up state.

So, either they are correlated or they are anti correlated, each having equal weightage or possibility. And the correlated ones will enjoy a possibility of 50 percent, or and the anti-correlated ones will enjoy a possibility of 50 percent. And this is what nearly one should expect. But experiments do not say that the experiments say that that the spins are anti correlated; that is, if you get an up spin then you get a down spin with 100 percent probability. So, that is somewhat strange, but it is true let us try to you know understand this that what it means. I will take a 2 keys, 1 key being here and the other key is a smaller. So, one is a bunch of keys the other is just a set of 2 keys. And I put them in my both hands. And say a prime one does not know that which hand has which key.

Now, if you ask me to show my right hand, the right hand has this bunch of 2 keys that we had just shown; which automatically mean my other hand has the bunch of keys or rather the bunch which is a bigger bunch of keys. And the small, because now by making a measurement of one hand, we automatically know that the measurement of the other hand which has the other kind of key. And so, this is taken up as a faster than light measurement.

Because, I did not have to do a measurement of my left hand which had that big bunch of keys; it is automatically clear that since I had in my right hand the small bunch of keys the bigger the other hand must be having the bigger bunch of keys. And this had Einstein had difficulties Einstein and Podolsky had difficulties, because this are talked about faster than life, light faster than light measurements (Refer Time: 27:08). And if the 2 hands or 2 these 2 objects are far apart, to the tune of you know several galaxies apart or several light years apart, even then it is going to be true that the other hand or the other possibility, even without making a measurement that it has this bigger bunch of keys.

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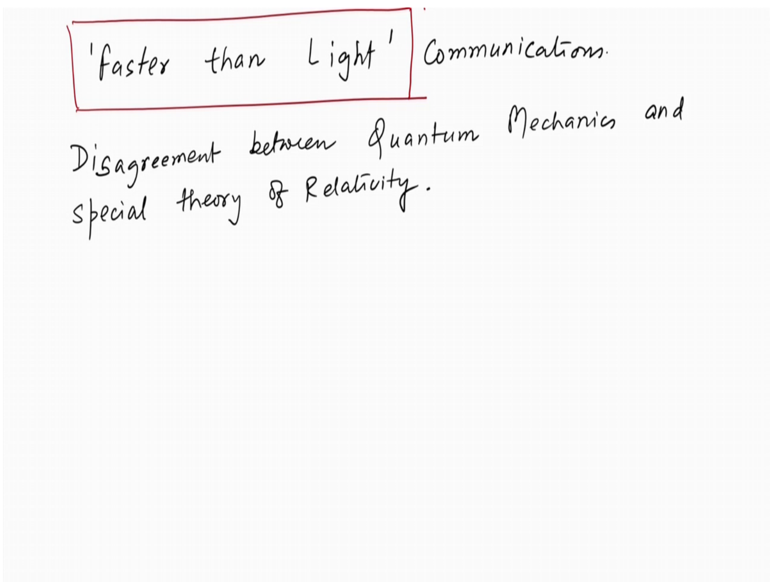

And this brings the notion of faster than light communications, ok. And let me box this, and faster than light is completely disagreed by the special theory of relativity. Because special theory of relativity by Einstein says, that light is the supreme speed and it remains invariant in all frames. So, faster than light communications not possible and so, there is a disagreement between quantum mechanics and special theory of relativity.

So, what Einstein proposed is that there must be some hidden variables, which are facilitating this faster than light communications. And they wrote down this paper which I have referred to here the and which essentially ask this question that what are the hidden variables and so on. So, there are hidden variables which are needed in order to explain this faster than light measurement or communications. However, Rosen did not agree to it, and he still said that there are no hidden variables and quantum mechanics is complete. How does this bring us this issue of quantum entanglement let us see.

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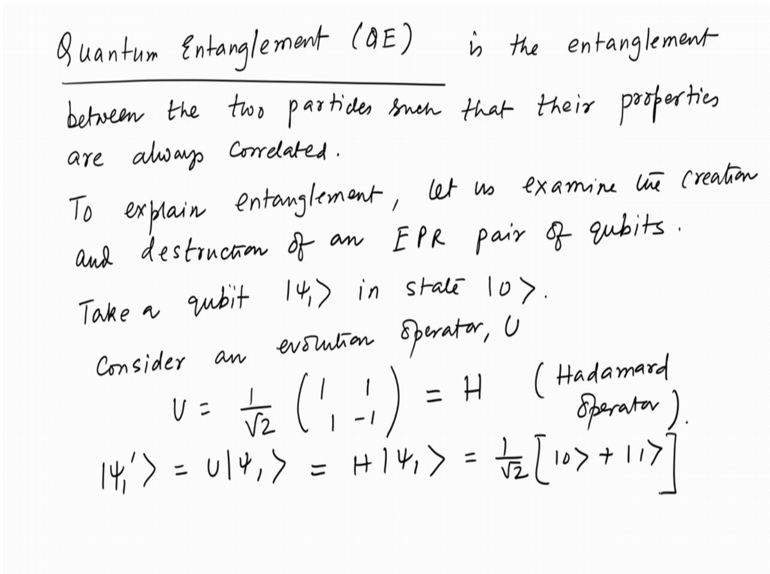

So, quantum entanglement is the entanglement between two particles or two objects just like the 2 keys that you have seen; such that their properties are always correlated. So, to explain entanglement, let us examine the creation and destruction of an EPR pair of qubits, just like the 2 keys that we had just said. So, let us take a qubit.

Qubit psi 1 in state 0 consider an evolution operator U such that U equal to 1 by root 2 1 1 1 minus 1. It is of course, an unitary operator, it is also written as by H which is called as a Hadamard operators operator. So, I operate the evolution operator of course, this unitary operator U dagger U has to be equal to 1. So, this one psi 1 which gives something or this is equal to H of psi 1 which tells it gives that it is equal to 0 plus 1. Just do a bit of digression and tell you the properties of this Hadamard operator.

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Properties of this Hadamard operator

$$H = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$$

$$H^2 = 1$$

$$H\sigma_x H = \sigma_z$$

$$H\sigma_z H = \sigma_x$$

H can be considered (upto an overall phase) as a rotation operator which rotates a state by $\theta = \pi$ about axis $\hat{n} = \frac{1}{\sqrt{2}}(\hat{n}_x + \hat{n}_z)$ that rotates $\hat{x} \to \hat{z}$ and vice versa,

$$\hat{R}(\hat{n},\theta) = \mathbb{1}\cos\frac{\theta}{2} + i\hat{n}\cdot\vec{\sigma}\sin\frac{\theta}{2} = \frac{i}{\sqrt{2}}(\sigma_x + \sigma_z) = iH$$

So, H the way it is written H or U, it is a 1 by root 2 sigma x plus sigma z. H square equal to 1, H sigma x H is equal to sigma z. H sigma z H is equal to sigma x. Also that it can be considered up to an overall phase as a rotation operator, which rotates a state by theta equal to pi about an axis, which is n cap equal to 1 by root 2 n x cap plus n z cap. That rotates x to z and vice versa.

So, that r n theta equal to cos theta by 2 plus i n cap dot sigma sine theta by 2 which for theta equal to pi or theta by 2 equal to pi by 2; it becomes equal to I by root root sigma x plus sigma z and this is nothing but i H. So, apart from that phase factor i, which can be taken as exponential i pi is the same as this rotation operator. Now let us come back to our discussion original discussion.

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Take another qubit $|\psi_2\rangle$ also in the $|0\rangle$ state.

The joint state space probability vector is the tensor product of the two.

$$|\psi_1'\rangle \otimes |\psi_2\rangle = |\psi_1'\psi_2\rangle = \frac{1}{\sqrt{2}}\left[1|00\rangle + 0|01\rangle + 1|10\rangle + 0|11\rangle\right]$$

Now define a new unitary transformation

$$C_{NOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad (C_{NOT})^2 = 1$$

Apply C_{NOT} to the combined system,

$$|(\psi_1'\psi_2)''\rangle = C_{NOT}\,|\psi_1'\psi_2\rangle$$

Take another qubit also in the 0 state. The joint state space probability vector is the tensor product of the 2, which is what we have learnt. It is like this, this is how the tensor product is taken, one can write in a slightly shorthand notation like this. And it is 1 by root 2 1 0 0 plus 0 0 1 plus 1 1 0 plus 0 1 1.

Now, define a new unitary transformation. Call it as C not which is equal to 1 0 0 0, 0 1 0 0, 0 0 0 1, 0 0 1 0. And operate on this the tensor space, but before that check that C not square is equal to 1. Now you apply C not to the combined system; that is a psi 1 prime psi 2. And so, this becomes psi 1 prime psi 2, and this is like a prime of that we will write it with the double prime. It is a C not and the psi 1 prime psi 2, ok. So, that psi 1 prime psi 2 state is now operated by this C not which is a unitary operator.

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$$|(\psi_1'\psi_2)''\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$$

Note that this state cannot be represented as,

$$[\alpha_0|0\rangle + \alpha_1|1\rangle] \otimes |\beta_0|0\rangle + \beta_1|1\rangle]$$

for any complex numbers $\alpha_0, \beta_0, \alpha_1, \beta_1$.

So, this is then psi 1 prime psi 2, this equal to 1 0 0 0, 0 1 0 0, 0 0 0 1, 0 0 1 0, then this 1 by root 2 0 0 1 by root 2. So, this is equal to 1 by root 2 0 0 plus 1 1, ok. And now if you actually look at this qubit this combined qubit, this thing, this state cannot be represented as alpha 0 0 plus alpha 1 1. And a tensor product of beta 0 0 and beta 1 1 for any complex number numbers alpha 0 beta 0 alpha 1 beta 1. So, what does it mean?

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Thus we cannot analyze the state of the individual qubit in the system, because the states of the combined qubit are entangled.

If we make a measurement of the first qubit, the state of the other qubit is determine by the outcome of the measurement of the first one.

Thus there is a perfect anti coorelation that exists.

That means, analyze the state of the individual qubit in the system. Because the states state of each individual qubit, we cannot state with the individual, because the excuse me here states of the combined qubit are entangled. So, if we make a measurement, if we make a measurement of the first qubit, the state of the other qubit is determined by the outcome by the outcome of the measurement of the first one. So, there is so, thus there is a perfect anti correlation hat that exists. We will elaborate more on this.