Advanced Quantum Mechanics with Applications
Prof. Saurabh Basu
Department of Physics
Indian Institute of Technology, Guwahati

Lecture - 19
Entangled state for two spins

So we have looked at one quantum spin and how effectively measurement can destroy the state of the system. Now let us look at a little more complicated issue which is two quantum spins.

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Two quantum spins

$|\uparrow\rangle \quad |\downarrow\rangle$

Combined state of the system:

$|\psi\rangle = a|\uparrow\uparrow\rangle + b|\uparrow\downarrow\rangle + c|\downarrow\uparrow\rangle + d|\downarrow\downarrow\rangle$

$|a|^2$ is the probability that both spins are $|\uparrow\rangle$ state

$|b|^2$ " " " first spin in $|\uparrow\rangle$ and second in $|\downarrow\rangle$ state

$|c|^2$ " " " " " " $|\downarrow\rangle$ and " " $|\uparrow\rangle$ "

$|d|^2$ " " " " both spins are in the $|\downarrow\rangle$ state.

$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1.$

So, consider two spins up and down and they are in this state. So, the combined system or the combined state of the system is let us call it as say psi s equal to a up up plus a b up down plus a c down up and a d down down.

So, that is the most general state of the system where a b c d are complex unknown coefficients. So, we are actually talking about the z basis and in that basis a square is the probability that both spins are in upstate and b mod square is the probability that first spin in upstate and second in downstate c mod square is the probability that first spin in down and second in upstate. And finally, d mod square is the probability that both spins are in the down state. Clearly the total probability has to be equal to 1 so, the a mod square plus b mod square plus c mod square plus d mod square will be equal to 1.

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What happens if we just measure the first spin?

(i) for $|\uparrow\rangle$ outcome the probability is $|a|^2+|b|^2$

(ii) for $|\downarrow\rangle$ " " " " $|c|^2+|d|^2$.

After measurement, the state of the two spins corresponding to $|\uparrow\rangle$ outcome for the first spin:

$$|\phi_\uparrow\rangle = \frac{a}{\sqrt{a^2+b^2}}|\uparrow\uparrow\rangle + \frac{b}{\sqrt{a^2+b^2}}|\uparrow\downarrow\rangle$$

Similarly for the down outcome.

$$|\phi_\downarrow\rangle = \frac{c}{\sqrt{|c|^2+|d|^2}}|\downarrow\uparrow\rangle + \frac{d}{\sqrt{|c|^2+|d|^2}}|\downarrow\downarrow\rangle.$$

Now, let us ask a few questions relevant questions that too for a quantum teleportation. So, what happens if we just measure the first spin? For up outcome for the first spin, the probability is a mod square plus b mod square. Similarly for down outcome the corresponding probability is c mod square plus d mod square.

So, after the measurement the state of the two spins corresponding to up outcome for the first spin is a divided by root over a square plus b square plus. So, that is that corresponds to up up and a plus b a square plus b square up down. So, that is after the measurement, this is the state of the system. We can call it as a state phi.

So, this is the first spin is up and this corresponding to the second spin being up and down. And similarly for the so, this is let us call it as up and similarly for the down outcome, phi down equal to c divided by root over c square plus d square. And then you have a down up plus d divided by a c square plus d square down down.

So, what it basically means is that if we measure the first spin in the upstate, then we remove all terms in the superposition except the ones that preserve the first spin in the upstate. And similarly the same goes for the downstate.

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Basically if we measure the first spin being $|\uparrow\rangle$, then we remove all terms in the superposition where the first spin is in $|\downarrow\rangle$ state.

Introduce z-basis for one spin at a time.

First spin

$$|\uparrow\uparrow\rangle = \frac{1}{\sqrt{2}}|\rightarrow\downarrow\rangle + \frac{1}{\sqrt{2}}|\leftarrow\uparrow\rangle$$

$$|\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}|\rightarrow\downarrow\rangle + \frac{1}{\sqrt{2}}|\leftarrow\downarrow\rangle.$$

Second spin

$$|\uparrow\uparrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rightarrow\rangle + \frac{1}{\sqrt{2}}|\uparrow\leftarrow\rangle$$

$$|\downarrow\uparrow\rangle = \frac{1}{\sqrt{2}}|\downarrow\rightarrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\leftarrow\rangle.$$

Then we remove all terms in the superposition where the first spin is in downstate ok. And similarly for the first spin to be in the downstate, we remove all combinations which yield that the first pin is in the upstate.

So, again if we go to the term introduce the x basis for one spin at a time, so, the first spin. So, this is corresponding to the first spin we have a up up which is equal to a 1 by root 2 right down plus 1 by root 2 left up. So, this is for the first spin and then this is for the other one. So, these are the combinations that one can write down and this is again this minus sorry plus 1 by root 2 down and down here.

And for the second spin, so these are the linear combination of the z basis in terms of the x basis for the first spin. And for the second spin, so we have a up up its written as 1 by root 2 up plus a 1 by root 2 up left. And similarly for the down up, so this is for the second spin being up. So, it is a 1 by root 2 a down and a plus 1 by root 1 and so on ok. So, these are the linear combinations for the spins to be written in the z basis in terms of the x basis.

Let us now write down the most familiar state that we are aware of for a two spin system or a system consisting of two spins namely the singlet state.

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<u>Singlet state</u>

$$|\psi_s\rangle = \frac{1}{\sqrt{2}}\left[\,|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\,\right]$$

It is an entangled state of two spins. The properties are:

(a) If either spin along any axis, the outcome will always yield $|\uparrow\rangle$ with 50% probability and $|\downarrow\rangle$ with 50% probability.

(b) If we measure one spin along any axis, the result for the other spin is always anticorrelated. Knowing the state of one spin, we know the result that would yield for the other spin.

This you have seen it ok. So, let us write down a singlet state. So, this is the singlet state which you are aware of 1 by root 2 up down minus down up.

The important thing now which we have not said earlier we have of course, discussed it in the context of what kind of states or what are the different states that can take place in the context of this you know for a system of two spins. The singlet is just one and there are three triplet states that also come the singlet state is actually anti symmetric with respect to the change in the spins. But what is most important for us here is that it is an entangled state of two spins.

the properties are if either spin is measured along any axis; in this particular case any axis means the plus z axis or the minus z axis that is in the upstate or in the downstate any axis, the outcome will always yield up with 50 percent probability and down with 50 percent probability ok.

So, this is a property of this singlet state and it is important for us and it is also important to note that if we measure one spin along any axis. The result for the other spin is always anti correlated. So, what it means is that if a one of them is found in the upstate the other is always in the downstate and vice versa. So, this is what you see in this particular case that one is in the upstate and the other is in the downstate. And so, this is in the downstate the first one is in the downstate and the other is in the upstate. So, this is a property of that state singlet state ok.

So, they are always anti correlated. So, the basically this means that knowing one knowing the state of one spin, so we automatically know the result that would yield other spin. Let us just give you an example that if you ask your friend to draw an up arrow on a piece of paper a big up arrow and take a Xerox of it by reversing the page that is rotating the page by you know sort of 180 degree, then you know that up will become down. So, if you know that the friend has drawn an up arrow, the result of the outcome after a Xerox is taken with the page being rotated by 180 degree. You know that the result is going to be a down state. So, it is that kind of a situation that we are talking about.

But you see the important thing is that all these measurements just now also what we have said that the measurements actually destroy the states. But there are other kinds of measurement which do not destroy the states and these are called as unitary operations or unitary measurements.

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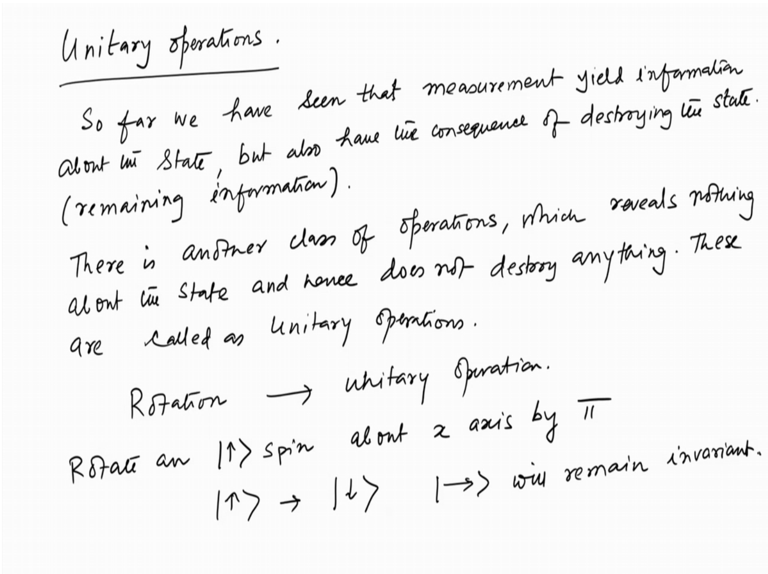

Let us see that. Because we did not want to destroy the state for the reason that as was discussed in the last discussion or deliberation that we have been conducting, once you do a measurement all these entangled states are gone and once you have this depletion of these resources of entangled states you cannot do a teleportation. So, it is very important for us to understand that what operations would not actually destroy the states.

And so, these are the unitary operations. So, we have; so, let us just write that so, far we have seen that measurement yield information about the state, but also have the consequence of destroying the state. I mean destroying the state what we want to say is that the remaining bit of information as just if you look at it here that we have, we have measured the up spin in the first spin in the upstate which means that there are no c and d coefficients because there the spins were in the downstate. And similarly when you have measured the first spin in the downstate, there are no information about a and b coefficients because which correspond to the amplitudes for the first spin to be in the upstate.

So, it is we are talking about that so, destroying the state means the remaining information. So, let us just make it more clear so, this is another class of operations as I said earlier which reveals nothing about the state and hence does not destroy anything. These are called as unitary operations. And of course, the most familiar unitary operation for you is a rotation.

So, what we mean by that is that if you rotate an up spin about z axis by about x axis by pi, we would get up we will to go down; however, for this will remain invariant so if we come back to the language of quantum teleportation then and call this rotation by pi about the x axis as X operation.

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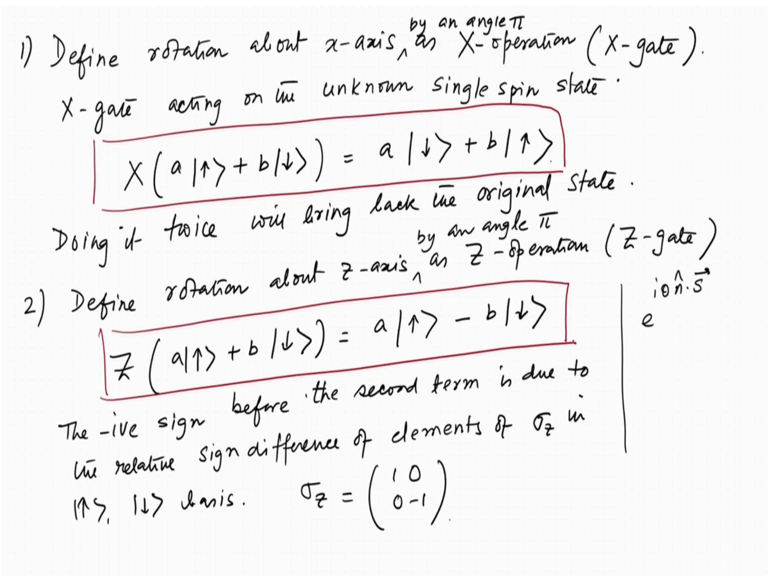

So, this you define as the X operation or it is also called as the X gate. Then X gate acting on the unknown single part single spin state. So, then X on a up plus b down and the reason we call it an unknown state is that these as and bs are unknown, these are unknown amplitudes or unknown coefficients. This will become a down and we will become b up. So, it does not destroy any information and you can also see that if I do it twice, then it will come back to its original state all right.

So, this is about the X gate operation. So, X gate operation simply reverses the up to down and the down to up and without changing anything and if you do it once, so that is X square acting on that would get back the state that we had seen. Now define so, this is this is 1, then define rotation about Z axis as Z operation basically by; so, this is by pi by an angle pi and this is also by an angle pi. So, this is called as the Z operation or it is also called as a Z gate.

So, Z gate acting on this unknown spin state is given by this it is equal to a up minus b down. You have to convince yourself that why there is a minus sign coming for that if you look at the rotation operator written in terms of an exponential, so which is basically it goes as exponential i theta and the n cap dot S that is the rotation operator for a spin and here theta is equal to pi. And because it is about the z axis and the so, it will the S Z or the sigma Z will come and since the sigma Z has in the up and down basis a written there are the two diagonal elements have a relative sign difference, that sign difference is appearing here.

So, let me write down that the negative sign before the second term is due to the relative sign difference of elements of sigma Z written in the up and down basis. Just to remind you that sigma Z is equal to 1 0 0 minus 1. So, that is the X and Z operations that you can see let me just block make a block of this is the X operation this is an important ingredient to the quantum teleportation. So, we will write this and this is a Z operation. So, these are important; we are going to only define another operation which is called as the constrained X operation so, we call it a C X 3.

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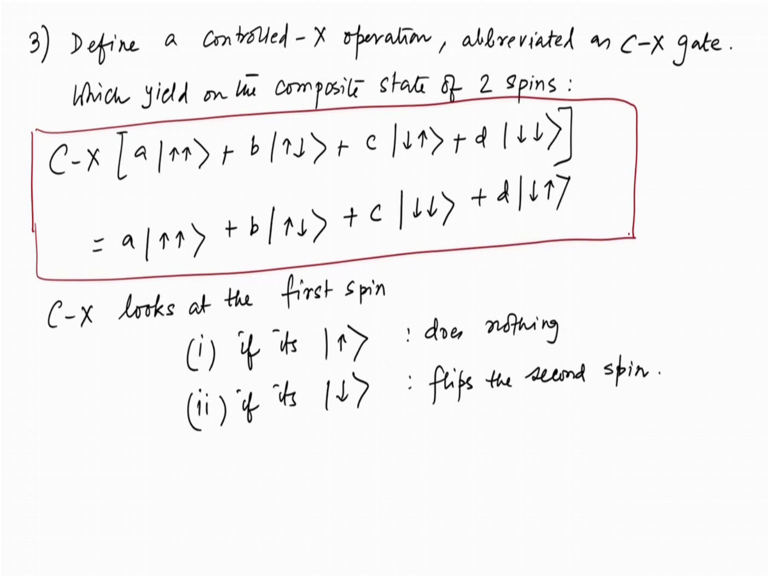

Define a controlled X operation the word controlled we will just make it clear which is abbreviated as again by a rotation pi C-X gate which yields the composite state of two spins. Let me first write down the relation and then we explain two spins. So, this is C-X acting on a up up plus b up down plus a c down up plus a d down down. This will yield a up up plus a b up down plus a c down down plus a d down up.

So, what it does is that it looks at the first spin that is the one of the left inside the ket and sees if it is up, then it does not do anything and if it is down then it flips the other spin. So, in these two terms here in these two terms here the up spin the first spin is in the up state, so it does not do anything. In the next two terms the first spin is in the down state. So, it flips the other spin makes it down here and makes it up here. So, this is called as the constraint X operation. C-X looks at the first spin; so, if it is up does nothing so if it is down, flips the next spin flips the second spin ok

So just to remind you that these are the three operations that we need for quantum teleportation of a spin let me also do a box of this operation. So, these three operations are needed in order to understand.

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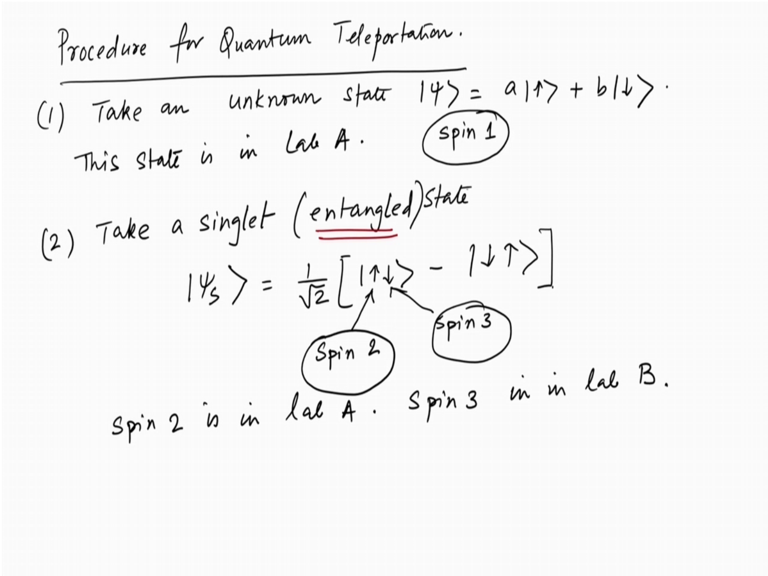

So, let us just so, we take an unknown state 1 take an unknown state psi equal to a up plus b down. This state is in lab A and the final goal of the teleportation is to transfer it to lab B.

Second is that take a singlet state which is an entangled state in our language now singlet or we call it entangled state. In fact, so, this is important for us. Now which let us write it as psi S; S for singlet is 1 by root 2 up down minus down up.

Now, remember that we call this as spin 1. So, this is called as a spin 1 and so, spin 1 is in lab A this one is called as spin 2 and this one is called as spin 3. So, spin 2 is in lab A again and importantly spin 3 is in lab B. So, these are the important requirements.

Now how an entangled state one is in lab A and one is in lab B is what we have seen in the last discussion in which they have physically met at some point in the past and they have created a lot of entangled state and they have carried it to the respective locations. There is only justification that we can give for this entangled state to have one spin in lab A and lab B and remember lab A and lab B are physically either very far apart or they are by some impenetrable wall they are separated by.

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Composite state of the system.

$$|\phi\rangle = \frac{a}{\sqrt{2}}|\uparrow\uparrow\downarrow\rangle - \frac{b}{\sqrt{2}}|\downarrow\uparrow\downarrow\rangle - \frac{a}{\sqrt{2}}|\uparrow\downarrow\uparrow\rangle + \frac{b}{\sqrt{2}}|\downarrow\downarrow\uparrow\rangle$$

$$a|\uparrow\rangle \times \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle = \frac{a}{\sqrt{2}}|\uparrow\uparrow\downarrow\rangle \quad : \text{first term.}$$

Flowchart for quantum teleportation.

Step 1 Apply C-X operation on $|\phi\rangle$.

$$C\text{-}X\left[\frac{a}{\sqrt{2}}|\uparrow\uparrow\downarrow\rangle - \frac{b}{\sqrt{2}}|\downarrow\uparrow\downarrow\rangle - \frac{a}{\sqrt{2}}|\uparrow\downarrow\uparrow\rangle + \frac{b}{\sqrt{2}}|\downarrow\downarrow\uparrow\rangle\right]$$

$$= \frac{a}{\sqrt{2}}|\uparrow\uparrow\downarrow\rangle - \frac{b}{\sqrt{2}}|\downarrow\downarrow\downarrow\rangle - \frac{a}{\sqrt{2}}|\uparrow\downarrow\uparrow\rangle + \frac{b}{\sqrt{2}}|\downarrow\uparrow\uparrow\rangle$$

So, with this we start our work. So, we will write down the composite state of the system. So, which is the composite state be phi which is equal to a by root 2 up up down minus b by root 2 down up down minus a by root 2 up down up plus b by root 2 down down up. These are basically obtained by multiplying the amplitudes ok. So, the first one if you consider so, it is a up and now it is getting multiplied with 1 by root 2 up down. So, that is how the first term came. So, this is equal to a by root 2 and we have simply taken it as up up down that is the first term that is the first term here ok. So, this term is written as this and again so, all these other terms can also be similarly justified.

So, once we have that let us just write down the flow chart for quantum teleportation. So, there are essentially five steps that will follow with the help of the gates which are X gates Z gate and C-X gate and we will achieve the teleportation of the unknown spin from lab A to lab B without really making it travel from lab A to lab B and anyway it is not a possible journey that we have talked about earlier. So, step 1 so, this is step 1 so, apply C-X operation on phi. So, C-X on phi on this a by root 2 up up down minus b by root 2 down up down down minus a by root 2 up down up plus b by root 2 down down up.

Remember the rule is that if it says the first spin, first spin is still the leftmost one inside the ket. If it sees the first spin in upstate, it will do nothing. If it sees it in downstate the downs state, then it will flip a spin 2. So, this is equal to a by root 2 up up down minus b by root 2 down down down minus a by root 2 up down up plus b by root 2 down down up up. So, that is the operation that C-X gives.

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Step 2 Now measure spin 2 in the z-basis.

Probability of getting $|\uparrow\rangle$ for spin 2 is

$$\left|\frac{a}{\sqrt{2}}\right|^2 + \left|\frac{b}{\sqrt{2}}\right|^2 = \frac{|a|^2+|b|^2}{2} = \frac{1}{2}$$

Same for $|\downarrow\rangle$ for spin 2.

We get 50% for each outcome

$$\frac{a}{\sqrt{2}}|\uparrow\uparrow\downarrow\rangle + \frac{b}{\sqrt{2}}|\downarrow\uparrow\uparrow\rangle. \rightarrow$$ state for $|\uparrow\rangle$ outcome for spin 2.

So, step 2 is now measured spin 2 in the Z-basis. So, probability of getting up for spin 2 is. So, it is that curtailed information that we get we have a by root 2 mod square plus a b by root 2 mod square which is equal to a mod square plus b mod square by 2 which is equal to half.

So, same for down spin for spin 2. So, basically we get 50 percent for each outcome. So, that is step number 2 let us. So, let us assume that we got a up outcome for spin 2, then at the state is equal to a by root 2 up up down plus a b by root 2 down up up; the rest of the information is gone as we have noticed earlier.

So, this is the state after this for up outcome for spin 2 and similarly it will be same for the spin 2 sorry. Similarly is this it will come out to be same for the down as well. So, that is your state and now we will go to step 3.

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Step 3

Now measure the first spin in the x- basis.

$$\frac{a}{\sqrt{2}}|\uparrow\uparrow\downarrow\rangle + \frac{b}{\sqrt{2}}|\downarrow\uparrow\uparrow\rangle \rightarrow a\left[\frac{1}{\sqrt{2}}|\rightarrow\uparrow\downarrow\rangle + \frac{1}{\sqrt{2}}|\leftarrow\uparrow\downarrow\rangle\right] + b\left[\frac{1}{\sqrt{2}}|\rightarrow\uparrow\uparrow\rangle - \frac{1}{\sqrt{2}}|\leftarrow\uparrow\uparrow\rangle\right].$$

Again we obtain $|\rightarrow\rangle$ outcome with 50% probability and $|\leftarrow|$ outcome with 50% probability.

For $|\rightarrow)$ outcome for spin 1

$$\frac{a}{\sqrt{2}}|\rightarrow\uparrow\downarrow\rangle + \frac{b}{\sqrt{2}}|\rightarrow\uparrow\uparrow\rangle$$

We will writing it not in roman so, we follow that same. So, now measure the first spin in the x basis. So, the resulting state from up up down plus a b by root 2 up down up it became a by 1 by root 2 right up down plus 1 by root 2 left up down plus b 1 by root 2 up up up minus 1 by root 2 this will be minus check this is up up ok. So, again after step 3 the right outcome with 50 percent probability and a left outcome also with 50 percent probability.

So, the state becomes if you want to do it measure, it in the right state or in the left state it becomes equal to a by root 2 up in the right state so, for right outcome for spin 1. Once again just to remind you spin 1 and spin 2 are in lab 1 and spin 3 is in lab 2. So, this would be this plus b by root 2 up right up up so, that is the outcome.

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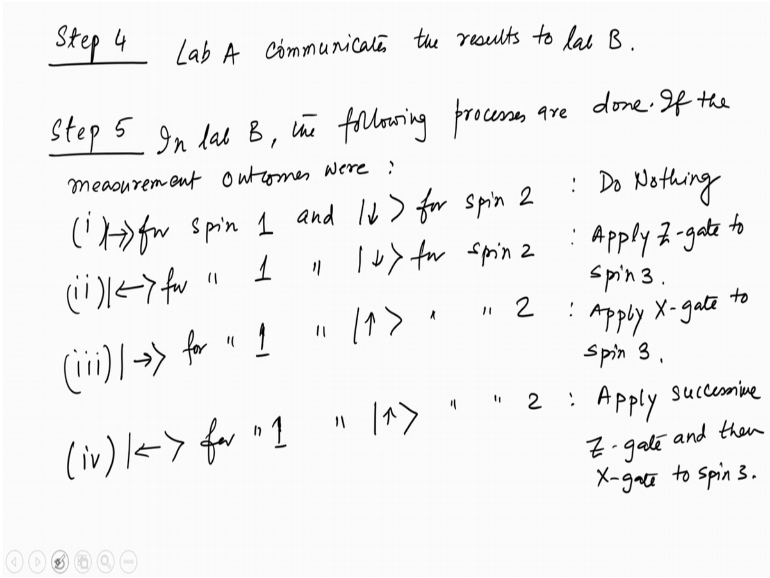

So, now let us go to step 4; lab A communicates the results to lab B. So all the results so far that is getting the right step right state for spin 1 etcetera. So, this state is essentially communicated to lab B; the state that we have obtained here the state that we have obtained here is communicated to lab B.

So, step 5 is a little involved so, the following so, in lab B, after they get all these outcomes if the measurement outcomes were number i for spin right for spin 1 that is labeled as 1 and down for spin 2 then do nothing. If it is left for spin 1 and down for spin 2 applies Z gate to spin 3 remember all these gates actually rotate by pi. So, that is it is not specified all the time, but that is what it means; iii right state for spin 1 and upstate for spin 2, then apply X gate to spin 3.

Finally if it is a left for spin 1 basically all combinations of right and left and up and down for 1 and 2 and this is down for sorry this is going to be up because we have taken the down already. This is going to be up up for 2, then it will should be written a little below for a left for 1 and up for 2 then apply successively Z gate and then X gate Z gate and then X gate to spin 3.

So, if you look at our case thus the lab whatever has been the final state so far which is here given by given by this state. So, we have the first spin to be in the right state and the second spin to be in the upstate. So, we basically should apply iii it is right and up. So, we will apply the X gate to spin 3. So, that is the conclusion from this and this is exactly what we will do.

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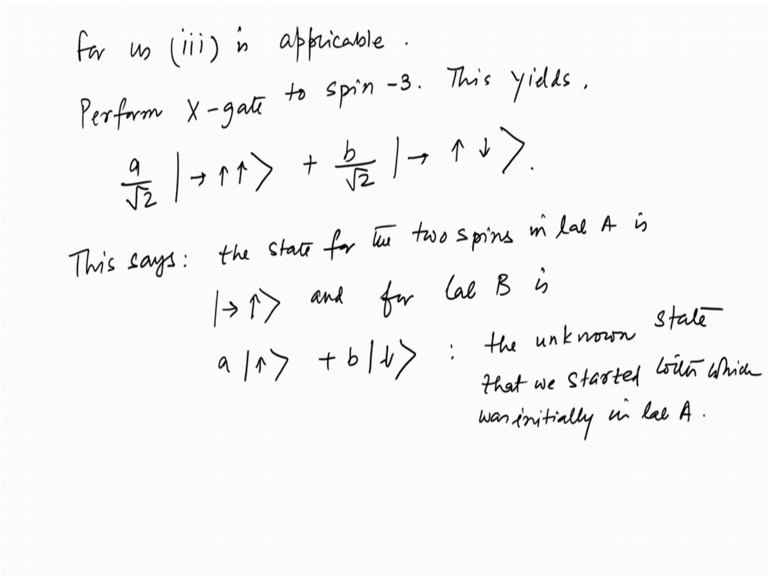

So, for us iii is applicable and that so, which means that perform X gate to spin 3 that is what is written apply X gate to spin 3.

This yields a by root 2 up up up. So, if it sees the first spin in the right state, then and then it will not do anything. So, basically if it sees the second spin in the, upstate then it will not do anything. But if it sees the second the third one, so this will be equal to so this is so, we have the first spin. So, we have a by root 2 and right up up. So, this is going to give that it is going to flip the spin for this third one and it is going to flip the spin if it is in the up state, then it will flip the spin for the third one to be in the down. So, this will be like this so, it is a up up plus b by root 2 and up up down.

So, this says that the state for the two spins in lab A is lab A is up right up that is this right up and for lab B is a up and a b down, this is essentially the unknown state that we started with; which was initially in lab A.

Now, it has come to lab B it has been teleported to lab b now whether such a simple operation or a series of operations that is applied to a single spin or entangled state would hold for humans that is a question. There is a lot of debate and disagreement that humans cannot be teleported for the simple reason that there are nearly 10 to the power maybe 30 or 10 to the power even more than that number of atoms which have their positions and their momenta etcetera all degrees of freedom. And transporting a single spin from lab A to lab B may be achievable. But so, many degrees of freedom each one will have to be transported or teleported whether that remains a possibility needs to be checked. In fact, it is very even seriously optimistic people will also oppose that such a thing can ever be achieved.

So, this is a simple demonstration of quantum teleportation that we can achieve for a single spin in the unknown state in a lab. We have talked about a spin half particle, but in principle it can be sort of extended to higher spins and so on.