Advanced Quantum Mechanics with Applications
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Lecture - 20
Quantum Gates, Walsh Hadamard Transportation, No cloning theorem

So, as a concluding class or rather the discussion on the quantum computation that we have been looking at for the past week and a half or so, let us look at a summary of the quantum gates that are important for the discussion of quantum computation.

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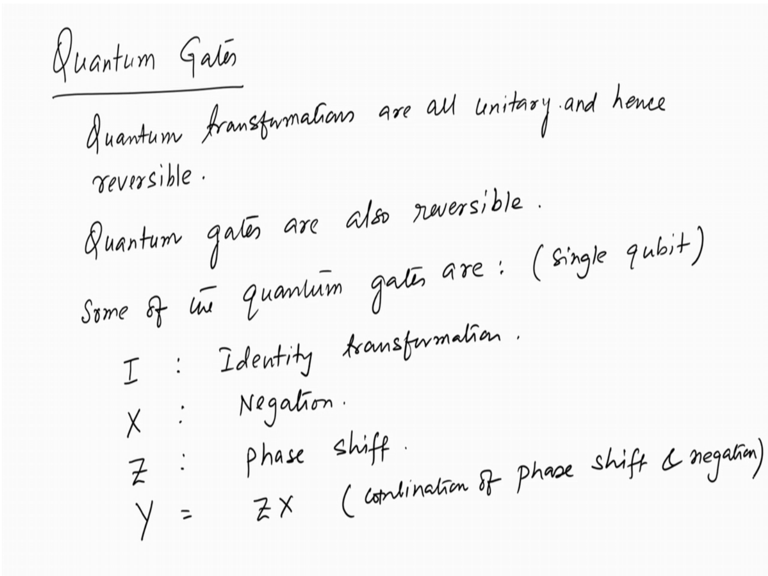

So, we will talk about quantum gates. Now and to a fair degree, you already know the quantum gates or have been introduced to quantum gates in some form. Just we want to you know sort of enumerate different gates that are important for our discussion.

So, these quantum transformations are all unitary. And as you know that the unitary transformations not only preserves the state, they are also reversible. Like we have seen that a rotation transformation by an angle pie can be easily reversed if you make a second rotation by an angle minus pi or in the other direction; if you make an, make a transformation or make a rotation by an angle pie.

So, these are reversible. So, let us and very importantly, they do not destroy the quantum states that we are interested, you know they do not disturb. So, this quantum gates are also reversible in that sense.

So, let us discuss some of the simple quantum gates that are used or rather useful for the single qubit quantum state transformation. So, we will look at this for the single qubit and then, we will also discuss 2 qubits and briefly on multi qubits as well.

So, this is I is an identity transformation. We will tell you what it means and then X is called as negation and then it is Z which is called as a phase shift. We will just give the forms of that and a Y gate which is a combination of Z X, that is a combination of phase shift and negation. So, what do we mean by that? Let us see.

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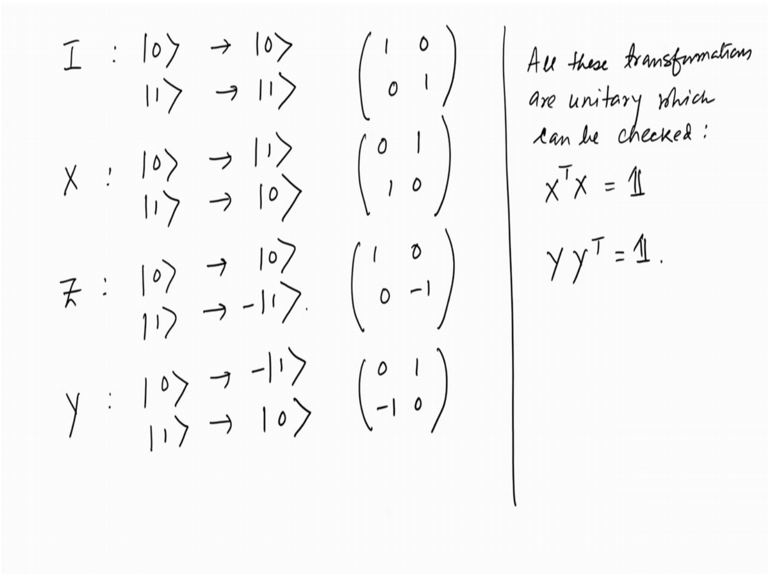

So, the identity I that on 0 gives a 0 on a 1 it gives a 1. So, this has a form which is 1 0 0 1. So, if it this in the standard basis of 1 and 0 in which the qubits are represented and then we have X which acts on a 0 gives it a 1 and acts on a 1 gives it a 0. So, this is a negation and this is 0 1 1 0. Then, let us write Z which acts on a 0 it gives a 0 acts on a 1 gives a minus 1. So, this is written as 1 0 0 minus 1.

Remember that, X gate looks like polymatrix for corresponding to the X component for a spin half particle. And similarly, Z looks like a sigma Z for a spin half particle again.

And, now the combination which is why is it acts on a 0, it gives minus 1 and it acts on a 1 and it gives a 0. So, this is nothing but equal to 1 0 0 1 minus 1 0. So, this is almost like the sigma Y; the Y component of the poly matrix excepting for a lack of a imagining I.

So, all these things are unitary which can be checked individually. For example, so, either you do X transpose X that will give you unitary matrix; similarly, a Y Y transpose also gives an unitary matrix. And so, is true for others. So, these are some of the things that are the sing the single qubit gates that are important. Now, there is another gate that we have talked about rather in details. That is called as a C not gate.

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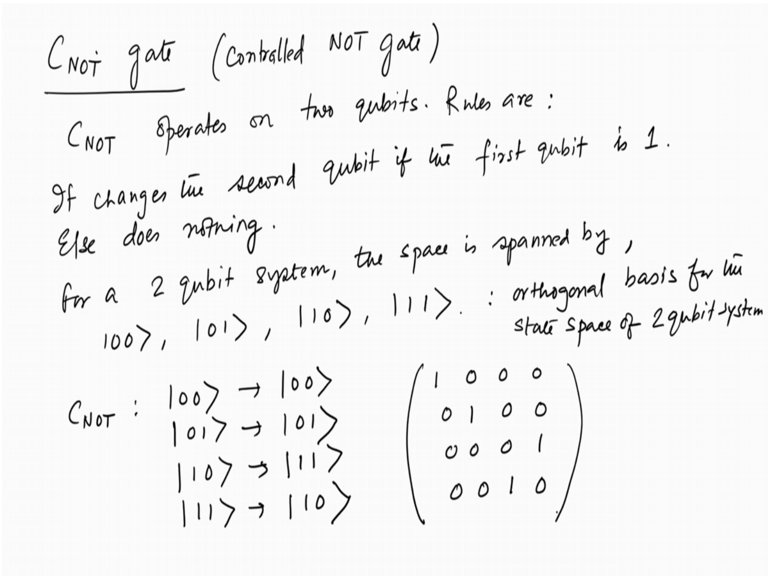

And, the full name is called a controlled not gate. This was used while we discussed the quantum entanglement and then a simple variant of that we have used in quantum teleportation which we called it as C X K. Now, this called as a C NOT gate and in this operates on basically a 2 qubit system. And the rules are so, it changes the second qubit. If the first qubit is 1, else does nothing.

So, what it means is that, so, as I said that these like this C X operation or the C NOT operation that we have seen earlier. So, for a 2 bit, the spaces is 0 0, 0 1, 1 0 and a 1 1. So, this forms the basis of that.

So, the C NOT on a 0 0, it transforms a 0 0. On a 0 1, it is just keeps it unchanged, that is 0 1 1 0. Since, the first qubit is 1; it will change the second qubit to 1 as well. And similarly, 1 1 it does it is 1 0 because the first qubit is 1, then the second qubit is changed to 0. So, the 4 by 4 matrix form is 1 0 0 0, 0 1 0 0, 0, 0 0 1, 0 0 1 0.

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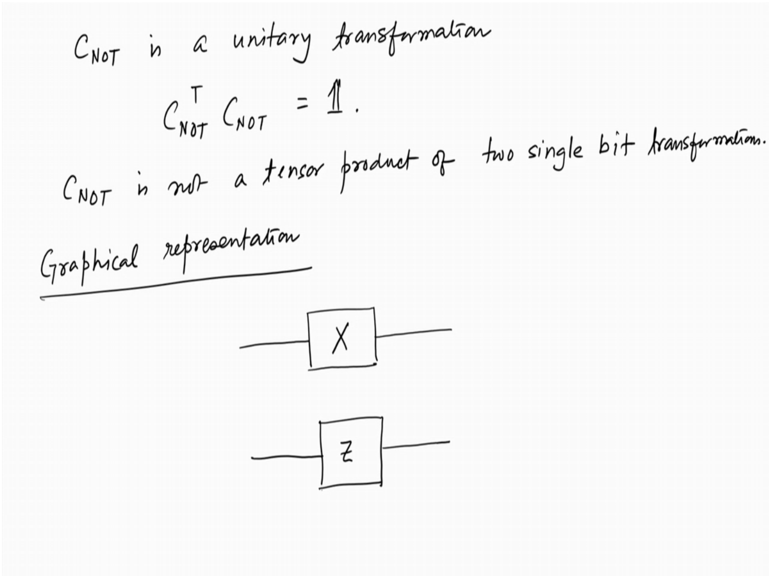

So, that is the matrix form of this C NOT gate and a one can check that C NOT is a unitary transformation. By checking that a C NOT transpose a C NOT which is equal to an identity matrix. So, in principle, the C NOT is or rather it cannot be is not a tensor product of 2 single bit transformations ok. So,. So, it cannot be represented as a tensor product of 2 single qubit transformation.

Let us look at the graphical representation of these gates. So, usually it is a box, the single qubit once they are by 2 lines. So, this could be X or it could be you know Z. For example, or a Y; so, it is just a box with 2 one line going in the other line going out; however, there is a the C NOT gate has a slightly different form than this.

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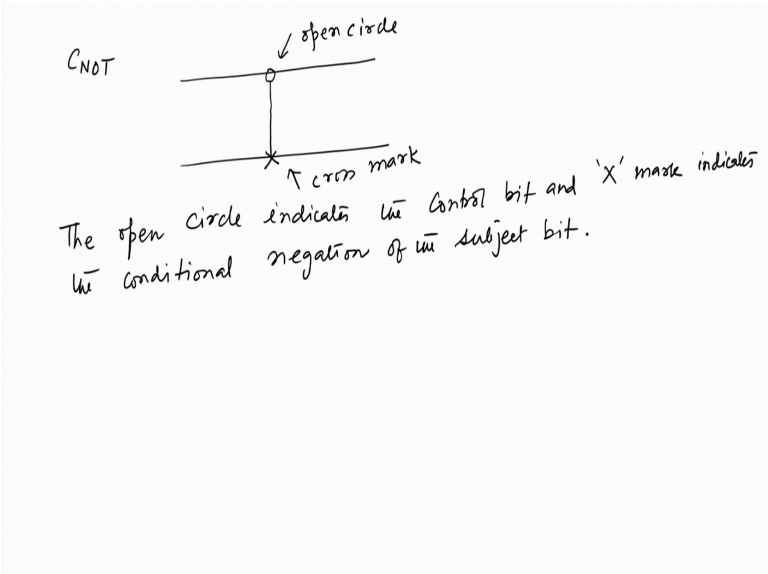

So, it is a 2 lines an open circle here a vertical line with a crossed mark. So, this is the; so, the open circle. So, this is the open circle and this is that cross mark. The open circle indicates the control bit and the cross mark indicates the conditional negation of the subject bit ok.

So, the single qubit transformations are replaced or represented by just a block box and 2 lines on either side and C NOT gate is represented by 2 horizontal lines with an open circle which denotes the control bit and the X mark which is at the bottom of a vertical arrow starting from the open circle that represents there is a conditional negation. That is, if the first bit is 1, then you change the second bit and if the first bit is 0, then you do nothing. So, we have looked at so far the single qubit and the 2 qubit transformations. Let us look at a little more complicated that is, an n qubit transformation.

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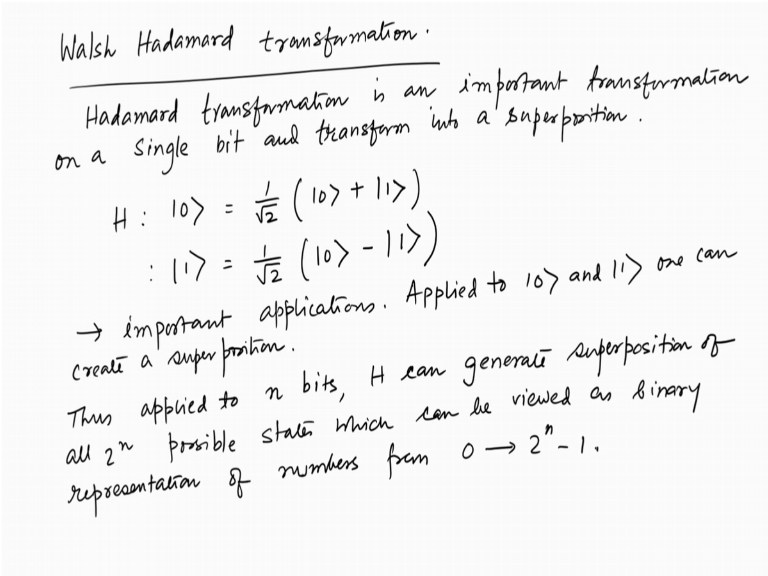

And, I will this called as the Walsh Hadamard transformation. So, we have looked at the earlier the Hadamard transformation. So, this is an important transformation on the single bit because a single qubit is now represented as a combination of superposition of 2 bits; so, on a single bit and transforms into a superposition.

So, H is represented by H which 11 acting on a 0. It is a normalized superposition of a 0 and 1 and on a 1 it is just a change in sign for the second term which is also super position like this. So, this has important applications applied to 0 and 1, one can create a superposition, all right.

So, this can in principle be applied to n bits. So, H can actually generate a superposition of all 2 to the power n possible states which can be viewed as a binary representation of the numbers between 0 to 2 to the power n minus 1. Let us see what we mean by that. In fact, this is an important statement.

So, let me write it now. This applied to n bits H can generate superposition of all 2 to the power n possible states which can be viewed as binary representation of numbers from 0 to 2 to the power n minus 1.

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$$(H \otimes H \otimes \cdots \otimes H)|00000\ldots0\rangle$$
$$= \frac{1}{\sqrt{2^n}}\left(|0\rangle + |1\rangle\right) \otimes \left(|0\rangle + |1\rangle\right) \cdots \otimes \left(|0\rangle + |1\rangle\right)$$
$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$$

This is called as Walsh Hadamard transformation, W.

W is a recursive decomposition,

$$W_1 = H, \quad W_{N+1} = H \otimes W_n$$

So, what this really means is the following that this Hadamard transformation being applied successively to this bit n bit system. This will create a 1 by 2 to the power n and each one will create as we know that it is 0 plus 1, then a tensor product of 0 plus 1 and so on and then it is 0 plus 1.

So, this is the successively applying this and this is called as sum over x equal to 0 to 2 to the power n minus 1 and we will write it as x. So, this is called as Walsh Hadamard transformation and it is represented by a W where it is a it is a W is a recursive. W is a recursive decomposition of the form as so, W 1 equal to H and W N plus 1 that is H and the tensor product of W n.

So, this is how a multiple quantum bit or rather multiple bits are being superposed or rather they are you know, this Walsh Hadamard transformation that creates a state which is a tensor product of multiple bits. Let us now look at another important thing which is called as a no cloning.

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No cloning

Unitary property is equivalent to saying that the states can not be copied or cloned.

Simple proof of no-cloning theorem.

U : unitary transformation.

$U|a0\rangle = |aa\rangle$ for all quantum states $|a\rangle$

Let have another $|b\rangle$, $\langle b|a\rangle = 0$.

$U|a0\rangle = |aa\rangle$ $\quad U|b0\rangle = |bb\rangle$.

Lets have $|c\rangle = \frac{1}{\sqrt{2}}\left(|a\rangle + |b\rangle\right)$ By linearity.

So, the unitary property of the quantum states say that, they cannot be copied or rather cloned. Basically, the quantum states cannot be copied or in this language, they are also called as cloned. So, we give a very simple proof of no cloning, no cloning theorem. Suppose, U is a unitary transformation which acts on a state a 0 and creates a state a a for all quantum states a and let us have another state orthogonal state b quantum state b where it is an orthogonal to a.

So, this is equal to 0 and. So, U on a 0 creates a a and U b 0 also creates a b b. So, let us say let us have a C which is equal to 1 by root 2 a and b the superposition of a plus b. So, that is a new state this is by linearity.

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$$U|c0\rangle = \frac{1}{\sqrt{2}}\left[U|a0\rangle + U|b0\rangle\right] = \frac{1}{\sqrt{2}}\left(|aa\rangle + |bb\rangle\right).$$

But if U is a cloning transformation.

$$U|c0\rangle = |cc\rangle = \frac{1}{2}\left(|aa\rangle + |ab\rangle + |ba\rangle + |bb\rangle\right)$$

These two transformations do not agree.

There are no unitary transformations that can clone an unknown quantum state.

It is not possible to create a n-particle state:

$(a|0\rangle + b|1\rangle) \otimes (a|0\rangle + b|1\rangle) \otimes \cdots\cdots \otimes (a|0\rangle + b|1\rangle)$ starting from an unknown state.

Now, what happens is that, U on C 0. So, this would create a e 1 by root 2 and then, U acting on a 0 plus U acting on b 0 that is going to give a 1 by root 2 a a plus a b b ok. But, if U is a cloning transformation, then U acting on C 0 should be giving me a C C which is equal to 1 half of a a plus a a b plus a b a plus a b b which is certainly not what we have written above. So, these 2 transformations do not agree ok.

So, basically, what it says is that there are no unitary transformations which that can clone an unknown quantum state. So, let us box this because, this is the statement of the proof that we had just given.

So, it is clear that the cloning cannot be done by measurement because measurements actually are probabilistic and not only that, they are destructive as well. So, they destroy the states which are not in the subspace of the measuring device; that is, if you want to have a up spin in a certain measurement, it cuts down the down spin for that particular particle.

So, it is not possible to create a n particle state such as a 0 plus a b 1 and a 0 plus a b 1 and so on a 0 plus a b 1 starting from an unknown state. So, this is about the quantum computation or the quantum information that we wanted to say, just let us have a quick summary of things that have been done.

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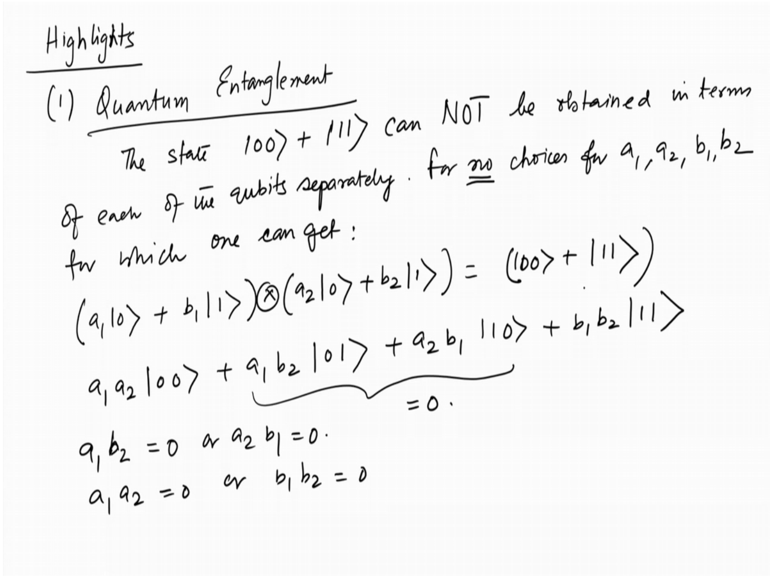

The highlights or the highlights of this thing: because we cannot spend a whole lot of time on this quantum computation. Because, there are other things which are equally important which are application oriented or rather their direct applications of the quantum mechanics that you learn. So, we will move on to another topic such as perturbation theory. But, before that, let us give the highlights.

So, the 2 things that we have done with a lot of emphasis or illustration; one is called as the quantum entanglement and it just simply says that the state 0 0 plus 1 1 you can normalize it with a 1 by root 2. It cannot be formed, obtained in terms of each of the qubits separately.

So, what we mean to say that for no choice or no choices for a 1 a 2 b 1 b 2 for which one can get a 1 0 plus a b 1 1 and then a 2 0 plus a b 2 1. This will give us 1 0 0 plus 1 1 and the reason is very simple. Because, this if you multiply, it will become a 1 a 2 0 0 plus a 1 b 2 0 1 plus a 2 b 1 1 0 plus a b 1 b 2 1 1.

Now, for these 2 become 0 then so, for the coefficients need to be 0. So, a 1 b 2 has to become equal to 0 or a 2 b 1 has to become equal to 0 in which case either a 1 a 2 will become 0 or b 1 b 2 will become 0. And hence, we cannot have a state which is a 0 0 and a 1 1 and this is called as a entangled state.

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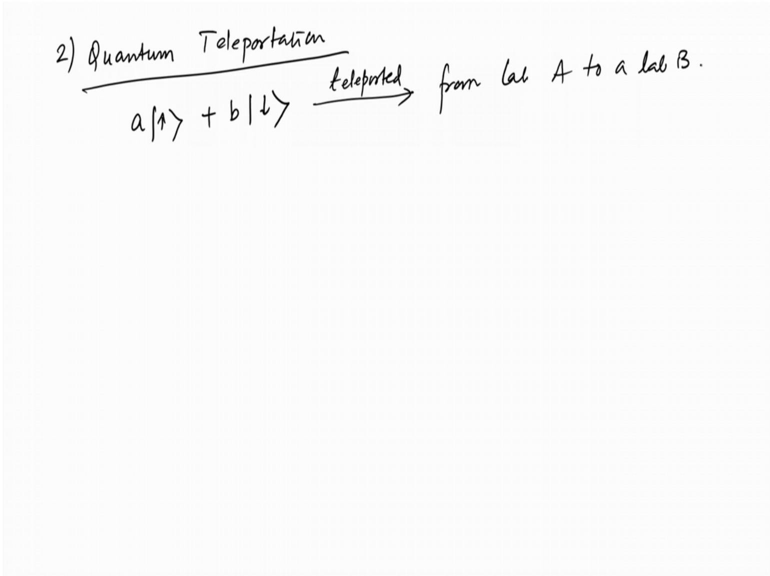

And, similarly for the quantum teleportation, we had a spin a single spin a up plus a b down that were teleported from lab A to a lab B which could be separated by universes, which means that in this there is no direct linkage from a to b. So, that this can be really taken and put it there, but it was teleported by a series of transformations which yield finally, yields that this unknown state quantum state which is a up and b down goes from lab A to lab B.