Advanced Quantum Mechanics with Applications
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Lecture - 26
WKB Approximation, Bohr Sommerfeld quantization condition

So we are going to look at the last topic in the Approximate Methods in Quantum Mechanics and namely the WKB Approximation which is by the name Wenzel Kramers and Brillouin approximation.

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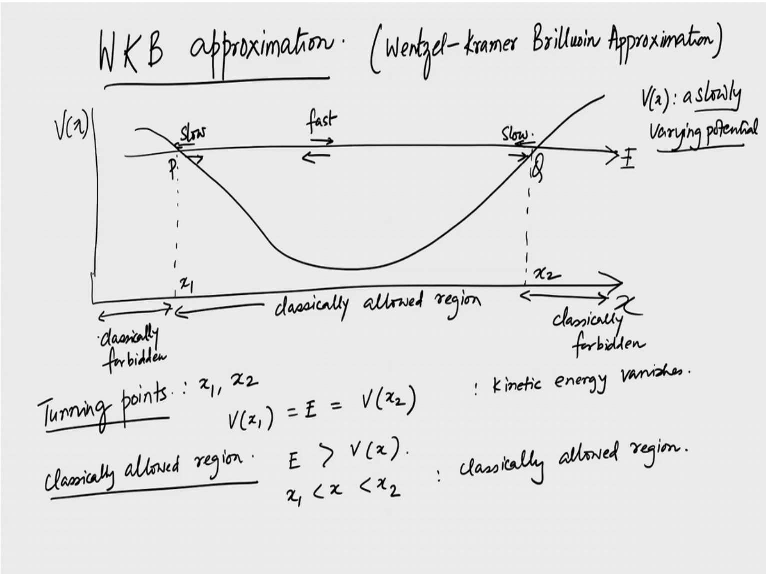

And this approximation is valid for slowly varying potential slowly and linearly varying potential I mean the main requirement is that it has to be slowly varying. So, suppose potential varies as something like this over a large distance. So, this is your V of x versus x. So, this is x and this is V of x. So, it is that kind of a potential.

Now, let us assume that the particle has energies which has a value which is like this represented by this E, now these are called as the turning points. So, we will call these points as P and Q P and Q are turning points. So, V x is of course, a slowly varying potential all right, and the particle has energy E. So, in the vicinity of this is a P the particle actually slows down. And it so, the velocity of the particle is slow and the particle has large velocities in this middle region and again it slows down and of course, so here it is slow and then of course, it you know it comes and then it goes back.

So, these regions are called as so, if we draw two vertical lines and let them meet at this x axis then this is called as the classically forbidden sorry this is called as a classically allowed region. And the regions beyond this so, these are Classically Forbidden region and this is Classically Forbidden as well all right.

So, this is the situation here, so we are talking about a potential of this kind and we certainly do not know the solution for this particular problem for the entire problem, but even if we do not know the solution for this entire problem at least we can get the asymptotic solutions. That is solutions for regions which are far away from this P and Q which are called as the turning points we will just define turning points.

So, these Turning Points are defined as x 1 and x 2 and these turning points are defined by where the total energy becomes same as the potential energy which means the kinetic energy goes to 0. So, that is the definition of turning points. So, the particle will come and turn from these two points of course, the quantum mechanical particle has a tunneling probability around these points as well. And so, we will get a dyeing solution or rather evanescent wave in the classically forbidden region. And so, the other region where the classically allowed region is where E is greater than E is greater than V of x and that tells that. So, any x between so this is the classically allowed region.

And so, near the turning points the kinetic energy is small and, so, the particle spends a significant amount of time close to the turning point and the motion gets slow and away from the turning point the kinetic energy is large and the particle is said to be moving with large velocities. And which can be understood quantum mechanically by calculating the probability density. So, the probability density of the particle being close to the turning points is large which means that the velocity is low.

And at a faraway regions that is somewhere the middle of this classically allowed region, you will get less probability for the particle to spend time there. And as the particle moves close to the turning point; then it gets reflected from these points P and Q which are as I told you that they are called as a turning points. And as it is also mentioned that this method is best suited for a slowly varying potential and if the potential is not slowly varying that is if it is rapidly varying then of course, this method does not work.

So, let us get an analogy with optics that this is the behavior of light in a varying in a medium where the refractive index is varying. So, if the refractive index varies too rapidly then of course, the light gets reflected and so, this is so and if it is a very gradual variation then of course, the light really does not get reflected.

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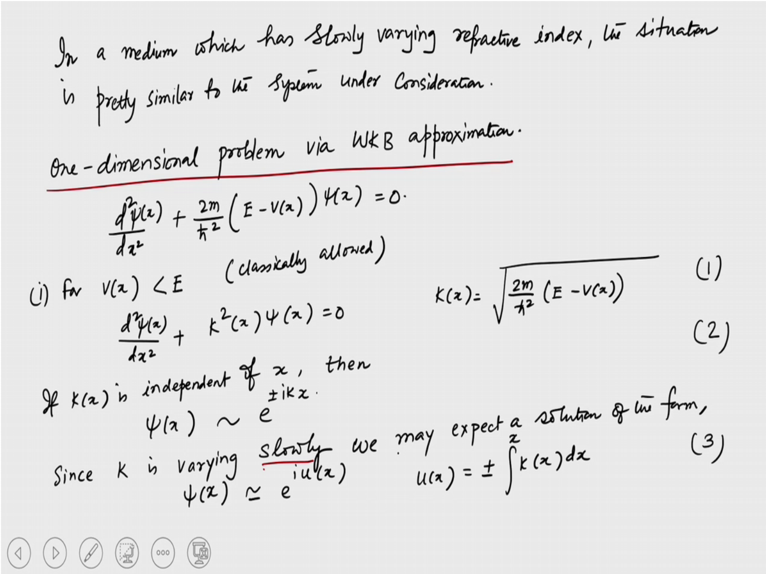

So, in a slowly varying or rather we should write it as in a medium, which has slowly varying refractive index; the situation is pretty similar under consideration for us in this WKB approximation all right.

So, this is the main idea of this problem that we are going to solve slowly varying potential and in terms of obtaining what are called as connection formula which is applying the boundary conditions, and these are very special formula because usually we have seen that the connection formula they actually connect the solutions at the boundaries. So, just epsilon left to the boundary to epsilon right to the boundary within the limit epsilon going to 0 the solution should match and, but here of course, there is no such things. In fact, very close to the turning points the solutions fail miserably it is only asymptotically that is far away from the turning point the solutions give good results.

So, let us take a one dimensional problem all right. So, how do we write Schrödinger equation I am simplifying it getting once getting rid off one step you should write all steps. So, this is d square side dx square plus a 2 m by h cross square E minus V of x psi x equal to zero. So, that is the equation that we are all familiar with it is a one dimensional time independent Schrödinger equation. This had to be solved so for V x less than E which means that we are in the classically allowed region the total energy is larger than the potential energy. So, this would be d 2 psi dx 2 plus a K square x psi x equal to 0, where the K of x which now depends of course, on x as opposed to earlier which it was a constant E minus V of x the x dependence of K comes from the x dependence of V and of course, we know that if K x is a constant K of x is independent of x then the solutions are simple and, but we still try to write down that solution. So, if K of x is independent of x then the solution is that that psi of x goes as exponential plus minus i K x, this is all known to you and these are well known solutions that you have looked at these are the travelling waves free particle solutions and so on.

Now, of course, since K is varying slowly and I would highlight this word slowly I just told you that the variation of K x is coming from V x so, if V x is a slow function of x, K x is also a slow function of x. And now we may expect a solution of the form which are psi of x psi of x is equal to. So, this is like exponential i u of x where u of x is nothing, but a plus minus K of x, dx.

Now, since K is not a constant it is variable or rather it varies on the space variable x. So, it has to be integrated over and we have not specified the limits of integration, but we could do that the lower limit is not important or rather even the limits are not important we simply can just put it there, but let us just for now put a limit there just is just any x arbitrary x.

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At the lowest level, one can expect the solution to have the form,

$$\psi_0(x) = \exp\left[\pm i \int^{x} k(x)\,dx\right] \qquad (4).$$

Putting (4) in (1),

$$\frac{d^2\psi_0(x)}{dx^2} + \left[k^2(x) \mp \frac{i}{2}\frac{dk}{dx}\right]\psi_0(x) = 0 \qquad (5)$$

Let write for a better (than the zeroth order) solution,

$$\psi(x) = F(x)\psi_0(x) \qquad (6).$$

$F(x)$ is a monotonic function and a simple polynomial in x

Putting (6) in (1),

$$\frac{1}{F(x)}\frac{dF(x)}{dx} = \frac{1}{2k(x)}\frac{dk(x)}{dx} \qquad (7).$$

We have neglected $\frac{d^2F}{dx^2}$

So, at the lowest level, one can expect the solution to have the form psi 0 just to say that it is a lowest level approximation, it is exponential plus minus i K x dx and with an upper limit put there. So, that is the lowest level approximation.

So, if we name these equations as so, this is equation 1, this is equation 2; this is equation 3 and call this as equation 4. So, putting a 4 in say 2 or rather its put in actually 1 that is the Schrödinger equation, one gets the Schrödinger equation now cast in the form of psi 0 and plus this is a K square x minus plus i by 2 d K dx psi 0 of x equal to 0.

Just let me tell you a priori that this is mathematically very intensive and I will have to practice at least a few times in order to get a hang of things, but these are not too difficult algebra these are simple algebra, but you would still have to practice it there are a few change in variables and there are knowledge about the special functions that are required for this.

So, this is the equation that is the Schrödinger equation for psi 0 if of course, d K dx is neglected then we get the equation to the lowest order which is d d 2 psi naught dx 2 plus K square psi naught equal to 0 this is what we have been looking at.

So, let us write for a better solution better means; better than the or rather than the zeroth order solution. So, psi of x equal to some F of x and a psi naught of x where F of x either you call it a slow slowly varying function or you can say that it is a monotonic function, and a simple polynomial in x. So, call this as equation 5 and let us call this as equation 6.

So, if you substitute these one putting 6 in 1, I get the same equation that is the Schrödinger equation, now written in terms of the variable F or the function F which is 1 over 2 K x d K x dx call this as equation number 7 now here of course, we have neglected d 2 f dx 2 and the rationale being that we want a simple polynomial in the vicinity of in the vicinity of the psi naught and moreover it is a monotonic function, so it does not have a curvature. So, these are the simple arguments that we can give in order to neglect d 2 f dx 2.

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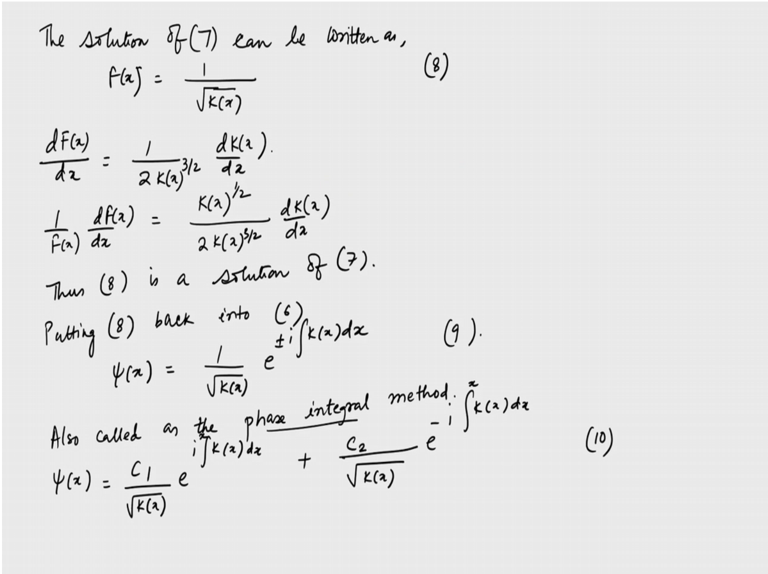

So, the solution of 7 can be written as F of x equal to 1 by root over K x. So, now with this is only a trial solution one can check that with this dF dx becomes equal to 1 divided by 2 K of x whole to the power 3 by 2 and the dK dx. So, you can write a 1 over F of x d F dx equal to a K x to the power half 2 K x 3 by 2 and a d K dx and so, this half and this 3 by 2 will cancel and will give me a 1 over K x which is what is wanted. So, this is a solution of so, thus 7 or rather 8 is a solution of 7.

So, we have gotten a solution for the Schrödinger equation in some form, and so this if we write 8 and put it back into 6. So, just to your 6 is the equation that you have made an unsearch for psi of x which is a polynomial f of x into the zeroth order solution.

So, then this becomes equal to psi of x which is just a slightly better approximation than the zeroth order approximation it is equal to 1 by root over of kx exponential plus minus i K x dx. So, what is the difference the difference is that that if you had a zeroth order approximation then the amplitude would have been a constant here the amplitude depends on x and its of course, has also the K in the denominator which of course, creates a problem that if that denominator blows up as K goes to 0 which is what happens in the vicinity of the turning points ok.

So, this is because of this it is also called as the Phase integral method all right. So, what is what are the solutions now we can write down this as equation 9 and hence the solution we can write down a full solution as C 1 divided by K x exponential i K x d x with an upper limit here plus a C 2 divided by root over K x exponential minus K x dx because the phase is integrated over that is why it is called as a Phase integral method. So, that is the most general solution so far. So, that is the one order better than or one level better than the solution that we had earlier proposed.

Now, this is for the classically allowed region this is what we had said that we lets write down this for V x less than E now we can go to the V x greater than E.

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(ii) for $V(x) > E$ (classically forbidden)

$$\psi(x) = \frac{D_1}{\sqrt{k'(x)}}\, e^{\int^x k'(x)dx} + \frac{D_2}{\sqrt{k'(x)}}\, e^{-\int^x k'(x)dx} \qquad (11)$$

is a solution of $\qquad (12)$

$$\frac{d^2\psi}{dx^2} - k'^2(x)\psi(x) = 0$$

$$\text{Where } k'(x) = \sqrt{\frac{2m}{\hbar^2}(V(x)-E)} \qquad (13)$$

Eqns. (10) and (11) are solutions to the "first" approximation for a slowly varying potential.

So, this is number 2 this is the classically forbidden. If you wish let us write also classically allowed, this is of course, classically forbidden for the simple reason that we are talking about the kinetic energy being negative.

So, this one will give me just proceeding as earlier. So, this is D 1 root over K prime x exponential x K prime x dx plus D 2 divided by a K prime x exponential x K prime minus K prime x dx. So, that is the solution for this call it 10 and this is our 11 and a K prime is of course, given by all right. So, we could have written down the.

So, there is a solution of is a solution of d 2 psi dx 2 minus K prime square x psi of x with equal to 0. So, let us call this as number 12 and where K prime x equal to 2 m by h cross square V of x minus E root over of that. So, that is my equation number 13.

So, my equations are 10 and 11 are, so Equations 10 and 11 are solutions to the first approximation I will put it in quote unquote which means; one level better than the zeroth order approximation for a slowly varying potential all right.

So, but these are fine this writing them down formally it is fine, but; however, you would see that there are some problems in trying to connect the solutions on both sides of the barrier and this is what could bring us to what are called as the connection formula, let us try to analyze these things a little more details.

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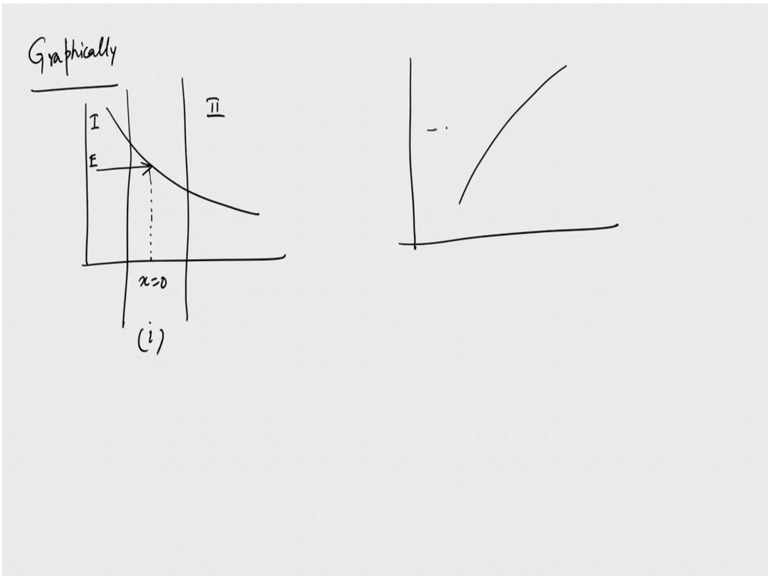

So, let us write graphically so, this is the solution. So, this is what the left curvature is, so this is the energy and this is of course, x equal to this is x equal to a and.

So, this is I am taking a thin layer which is in the immediate vicinity of the turning point. So, this is my E and this is will be called as region 1 and this will be called as region 2. And let us call this figure as figure 1 and that same figure I would draw it on the other side where it looks like this. So, then again the energy is like this.

So, let me look at the other end of the potential which is given by this the curve of another slope the other slope.

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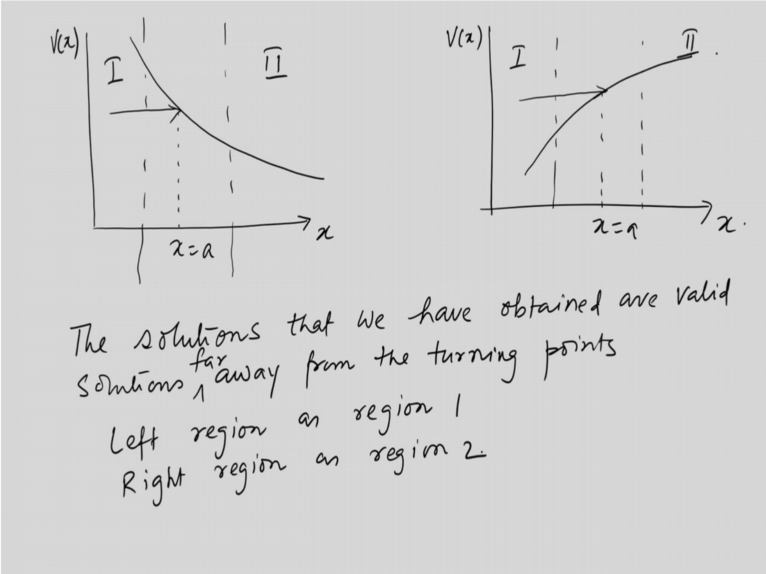

So, again we have this turning point at x equal to a and we also take two vertical lines in the vicinity of the turning point and.

So, we have already told that this point x equal to a is called as the turning point, look at the solutions that we have written down they had a 1 by root over k x in the denominator root over k prime x in the denominator both of them vanish as v becomes equal to e. So, in which case, these solutions are not valid in the vicinity of these turning points. So, they are the solutions that we have obtained are valid solutions away from the turning points away means far away from the turning point. So, let us just write far away from the turning points.

So, let us also demarcate these regions 1 and 2 here and regions 1 and 2 here so, we call the left region as region 1 and the right region as region 2 all right.

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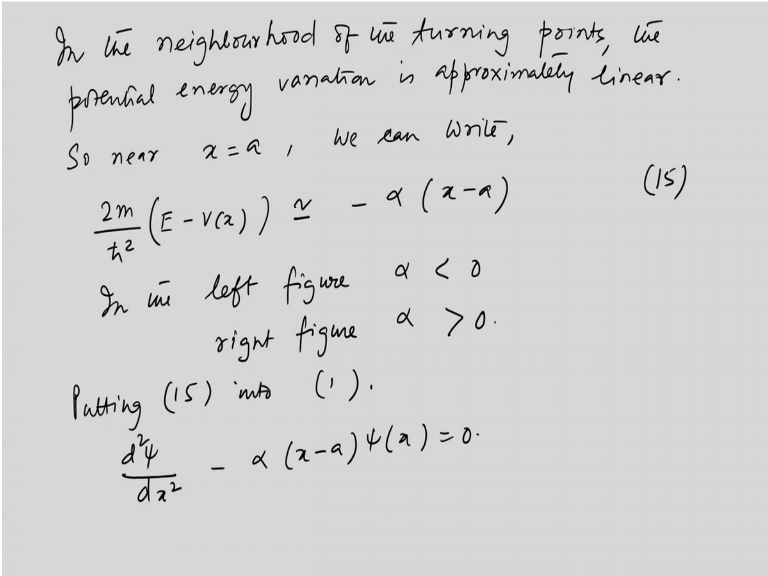

So, in the neighborhood of the turning points, the potential energy variation is approximately linear. So, near x equal to a which are the turning points, we can write 2 m by h cross square E minus V of x which is nothing, but the k square is minus alpha x minus a.

So, if you look at the left figure then we have alpha is less than 0 and in the right figure alpha is greater than 0. So, these are the respective slopes of the potential energy profile as we have drawn.

And so, if we substitute these form into the Schrödinger equation, that is this form let us call this as equation 15 in continuation with our earlier notations. So, we have putting 15 into 1 we have a d 2 psi d x 2 minus alpha x minus a psi of x equal to 0. So, this is just like a linear potential and the solutions are called as the AIRY functions which are written as AI and BI. And that is so, these are exact solutions are available, but let us just get them a little more in a familiar form and.

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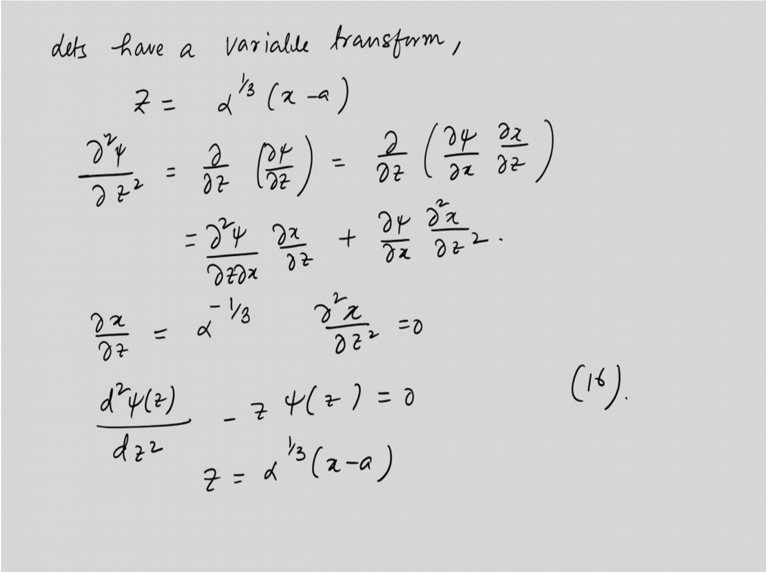

So, let us have a variable transform in which we write z equal to alpha to the power 1 by 3 x minus a and so, d 2 psi dz 2 its equal to a del del z of del psi del z or we can simply write it as del 2 psi del z 2, this and this is equal to del del z of del psi del x and del x del z.

And this is nothing, but equal to del 2 psi del z del x and del x del z plus a del psi del x del 2 x del z 2 from the given condition del x del z is simply equal to alpha to the power minus 1 third and del 2. So, this is minus one third not half pardon me for this. So, del 2 x del z 2 equal to 0. So, the second term is equal to 0 and the first term only contributes, and if you do this simplification or this variable transform.

Then we get this equation as d 2 psi z i I am skipping one step which you can fill it up and, this is equal to minus z psi z equal to 0 and this is the linear potential. So, this is a Schrödinger equation for a linear potential call this equation number 16 where of course, you should remember this z equal to alpha to the power 3 x minus a as written at the top of this slide ok.

So, we have to now write down the solutions in terms of the airy functions; however, the airy functions are less lesser known functions than the Bessel function. So, we will establish also a relationship between the Bessel functions and the airy functions.

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Relationship between the Airy function and Bessel function

$$\frac{d^2\psi(z)}{dz^2} - z\psi(z) = 0$$

(i) $\underline{z>0}$ Do a transformation.

$$\psi(z) = \sqrt{z}\,\phi(z). \qquad (17)$$

$$z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} - \left[z^3 + \frac{1}{4}\right]\phi(z) = 0. \qquad (18)$$

Again we make another transformation ((19)

$$\xi = \frac{2}{3}z^{3/2} \qquad (20)$$

$$\xi^2\frac{d^2\phi}{d\xi^2} + \xi\frac{d\phi}{d\xi} - \left(\xi^2 + \frac{1}{9}\right)\phi(\xi) = 0. \qquad (21)$$

– Modified Bessel function

So, at this moment we are a priori introducing on an ad hoc basis this Airy functions, but later on they will be shown to be the solutions of this equation 16 and Bessel functions you must have already seen in the context of either electrodynamics or quantum mechanics say particle in a spherical box.

So, this is your equation that you want to solve dz square minus z psi z equal to 0. So, let us first look at z greater than 0 which also means that x is greater than a. So, that is the lets do another transformation, psi of z equal to root over z phi z. So, then this if you substitute this equation 17 into 16 then, we get a differential equation which is equal to in terms of phi its nothing, but just rewriting the Schrödinger equation this and minus z cube plus 1 by 4 phi of z equal to 0.

So, again we make another transformation introduce a variable called as xi which is equal to 2 third z to the power 3 by 2 and that, lets us or arrive at this differential equation plus as xi d phi d xi minus xi square plus 1 over 9 phi of xi equal to 0.

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The solutions are :

$$I_{1/3}(\xi) \quad \text{and} \quad I_{-1/3}(\xi)$$

$$\psi(z) = \sqrt{z}\, I_{\pm 1/3}(\xi) = z^{1/2}\, I_{\pm\frac{1}{3}}\left(\frac{2}{3} z^{3/2}\right)$$

(ii) $\underline{z < 0}$ $\quad \eta = -z.$ (22)

$$\frac{d^2\psi}{d\eta^2} + \eta\psi(\eta) = 0$$ (23).

$$\psi(\eta) = \sqrt{\eta}\, \phi(\eta)$$

$$\zeta = \frac{2}{3}\eta^{3/2}$$

$$\zeta^2 \frac{d^2\phi}{d\zeta^2} + \zeta \frac{d\phi}{d\zeta} + \left[\zeta^2 - \frac{1}{9}\right]\phi(\zeta) = 0.$$ (25)

Bessel function

So, this is called as the modified Bessel function and the solutions are I to the power I for 1 third and this is I of minus 1 third xi. And so, the solution is obtained as psi z equal to root over z I plus minus 1 third xi which is also equal to z to the power half I plus minus 1 third 2 third xi to the power 3 by two. So, that is the solution for z greater than 0.

Now, what happens for z less than 0 once again we shall do this transformation that we will call a eta equal to minus 1 over z. So, the d to psi d eta 2 plus eta psi of eta equal to 0 is the solution which we can write it as we can write down these equation numbers of course, so, these are. So, this is 18 and this is 19 this is 20 and this is 21. So, this is equal to 22, 22 and now making another transformation that psi of eta. So, all these transformations are making sure that we are trying to simplify the situation as much as possible.

And introduce another variable called as zeta which is equal to 2 third eta to the power 3 by 2 now we are doing it for z less than 0, and that gives a differential equation with this for phi in terms of these zeta, which is this is not this is that zeta and plus zeta d phi d zeta plus zeta square minus 1 by 9 phi of this is not xi, but it is zeta this is equal to 0. So, this is called as the Bessel function and the solutions are J plus minus 1 third.

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$$\psi(z) = z^{1/2} J_{\pm 1/3}\left(\frac{2}{3}|z|^{3/2}\right) \qquad (26)$$

Summary

$$\psi(z) = \sqrt{z}\left[C_1 I_{-1/3}(\xi) + C_2 I_{1/3}(\xi)\right] \quad z>0 \qquad (27)$$

$$\psi(z) = \sqrt{z}\left[C_3 J_{-1/3}(\xi) + C_4 J_{1/3}(\xi)\right] \quad z<0 \qquad (28)$$

Airy functions.

$$Ai(z) = \frac{1}{3} z^{1/2}\left[I_{-1/3}(\xi) - I_{1/3}(\xi)\right] \quad z>0$$

$$Ai(z) = \frac{1}{3} z^{1/2}\left[J_{-1/3}(\xi) - J_{1/3}(\xi)\right] \quad z<0.$$

$$Bi(z) = \frac{1}{\sqrt{3}} z^{1/2}\left[I_{-1/3}(\xi) + I_{1/3}(\xi)\right] \quad z>0$$

$$Bi(z) = \frac{1}{\sqrt{3}} z^{1/2}\left[J_{-1/3}(\xi) - J_{1/3}(\xi)\right] \quad z<0$$

So, the solutions are written as psi of z which is equal to z to the power half J plus minus 1 third and 2 third, now it is a mod z to the power 3 by 2 and this is equation number 26. So, we can write down the solutions so, the summary of these solutions are psi of z equal to root over of z C 1 I minus 1 third zeta plus C 2 I 1 third zeta, that is equation number 27. So, this is for z greater than 0 and there is the other solution is psi of z is root over z plus C 3 J minus 1 third zeta plus a C 4 J 1 third zeta. So, this is zeta and this is equation 28 and this is for z less than 0.

So, this is the situation so far, that we have been trying to solve this equation and this in presence of a linear potential or a slowly varying potential now it has become a linear potential because as you come closer to the turning point whatever may be the variation if you come very close it looks like linear potential, and this is what we have obtained. So, far in terms of the modified Bessel function and the Bessel function.

Now, there are airy functions which are simply called as so, these are Airy functions, and these airy functions are written in terms of the Bessel functions and the modified Bessel functions, which are in this form a minus I minus 1 third minus 1 third xi minus I 1 third xi z greater than 0 and there is a B i z or this is equal to so, B i will just. So, that is for z greater than 0 and it is again the A i z that is equal to 1 third z to the power so this is not 1 third this is really half.

So, this is half and J minus 1 third xi minus J 1 third for z less than 0 and similarly a Bi z which is equal to 1 by root 3 z to the power half and there is a I minus 1 third xi minus I 1 third xi with a plus sign here which is for z greater than 0 and this is equal to B i z this is 1 by root 3 z to the power half and there is a J J minus 1 third xi minus J 1 third xi this is for z less than 0.

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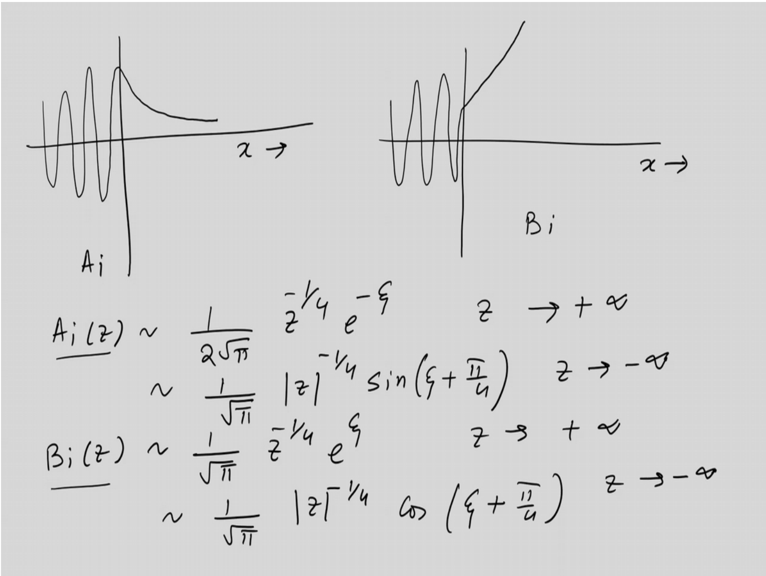

So, just a bit graphically we are so, this is equal to so, this is; so, this is my A i. So, this is x greater i mean x this way and so these are. So, this is your Ai function and the Bi function looks like so this is the Bi function and so on.

So, we can start from the to get the asymptotic forms of these Ai and Bi we could start from the Bessel functions and look at their asymptotic behavior rather we would write down straight away the asymptotic behaviors of this. And so, that tells us that the A i z that goes as 1 by 2 root pi z to the power minus 4 exponential minus xi for z going to plus infinity.

This is 1 by root over pi mod z to the power minus 4 a sin xi plus pi by 4 for z going to minus infinity. So, that is the behavior of the airy functions for the arguments to be in that region which is shown, there and a B i z has this 1 by root pi z to the power minus 4 z to the power minus 1 by 4 e to the power xi for z going to plus infinity and it goes to 1 by root pi mod z going to minus 1 by 4 cosine of xi plus pi by 4 for z going to minus infinity.

So, this means that the airy functions the Ai the first kind these are called airy functions of first kind these are called airy functions of second kind and they have these asymptotic behaviors which the for z going to infinity the airy functions of the first kind goes as exponential minus xi the other has a 4 x xi z going to minus infinity it has a sin or it sin vary sinusoidally whereas, for the Bi function for z going to infinity it goes as e to the power xi and for the other one it goes as cosine.

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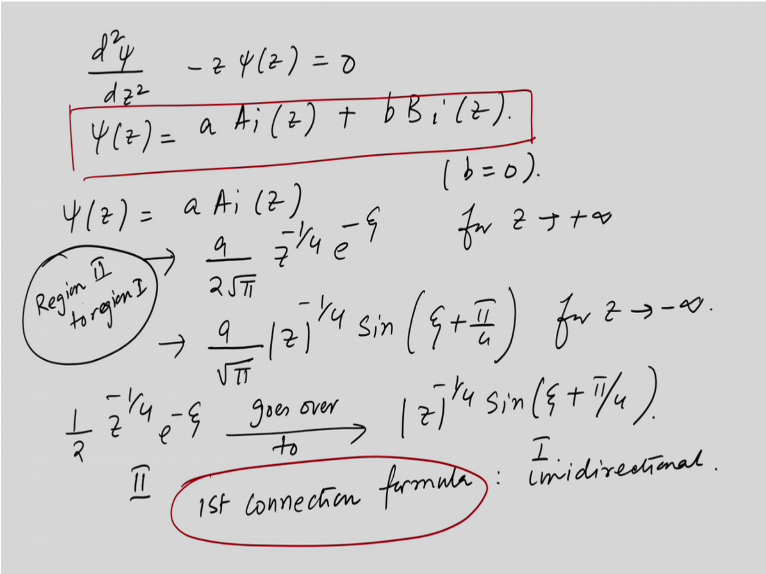

So, if we write down finally, the equation which we had obtained in terms of the z. So, this is our equation whose solutions we have we could have written it down simply, but we wanted to still introduce this Bessel functions which are. So, finally, these are these airy functions, so with a and b as the coefficients it is z. So, a and b are the so, this is the final solutions of this slowly varying potential a and b are amplitudes which are needed to be obtained from the boundary condition.

Now, this boundary condition is a little tricky for the simple reason that we will just explain. So, now, this psi of z that goes as a Ai z which goes as a by 2 root pi z to the power minus just look at this the last slide for z going to plus infinity and it goes as a by root pi mod z whole to the power minus 1 by 4 sin of xi plus pi by 4 for z going to minus infinity.

So, the solution is that that 1 by 2 z to the power minus half exponential minus xi it goes over to z to the power minus 1 by 4 sin of xi plus pi by 4. So, this is coming from the airy function of the first kind where we have put b equal to 0 b equal to 0 is that because of the square integrability of the wave function because bi diverges for. So, this term is increasing. So, we have just put the coefficient of that equal to 0 because otherwise you will land up with this problem of the square integrability of the wave function mod psi square dx integrated over all space will not be equal to 1 and the.

So, this basically what I we are trying to say is that this is the solution in region 2 and this is the solution in region 1 and this solution in region 2 should go to solution in region 1 and the other way around that is solution of region 1 is connected by to the solution of region 2 by the same formula is not allowed for the simple reason is that, as you change as you trying to go from region 1 to region 2 this could cause a change in the phase of the sin and you know the sin will if it changes its phase by a pi by 2 psi becomes cosine, and cosine is not connected to an exponentially dying solution which are we are going to see cosine is actually related to the exponentially growing solution. So, this is the reason that it is a connection formula it is called as a connection formula and it is purely unidirectional all right.

So, let us look at the other case.

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Now try to connect solutions from region I $\rightarrow$ region II.

$a = 0$

$$\psi(z) = b\, B_i(z)$$

$$B_i(z) \sim \frac{1}{\sqrt{\pi}} z^{-1/4} e^{\xi} \qquad z \to +\infty$$

$$\sim \frac{1}{\sqrt{\pi}} |z|^{-1/4} \cos\left(\xi + \frac{\pi}{4}\right) \qquad z \to -\infty.$$

$$\underset{\mathrm{I}}{\frac{1}{\sqrt{\pi}} |z|^{-1/4} \cos\left(\xi + \frac{\pi}{4}\right)} \longrightarrow \underset{\mathrm{II}}{z^{-1/4} e^{\xi}}$$

2nd. Connection formula.

Now, try to connect solutions from region 1 to region 2. So, this was actually from region 2 to region 1 that is what you see. So, we can we can write it since we know so, region 2 to region 1 and now we are trying to connect it from region 1 to region 2 we will put a equal to 0 and we will right psi z equal to b Bi z that is the airy function of the second kind and just to remind you that, this B i z it goes as 1 by root over pi z to the power minus 1 over 4 exponential of xi for z going to plus infinity and this is equal to 1 by root over of pi z going to minus this cosine of xi plus pi by 4 and z going to minus infinity.

So, now I will connect this as the other connection formula and cosine of xi plus pi by 4, this is in region 1 should connect to z to the power minus 1 by 4 exponential xi. So, this is in region 2 this is called as the Connection formula.

So, this is the second connection formula so, let us call it as a first connection formula. And this is the second connection formula. So, just to go over it, again that the recipe for connecting solutions from region 2. So, if you look at the region 2 so, region 2 is actually the classically forbidden region. So, region 2 is actually the classically forbidden region where e is less than V of x and region, 1 is the classically allowed region where e is greater than V of x of course, that is the situation changes when you are talking about the two sides of the barrier.

So, in any case these are the connection formulas that are or these are the boundary conditions that are used; now these boundary conditions are distinctly different than the boundary conditions that we have seen for all these other problems. The last thing that we would do in this regard is that we will derive.

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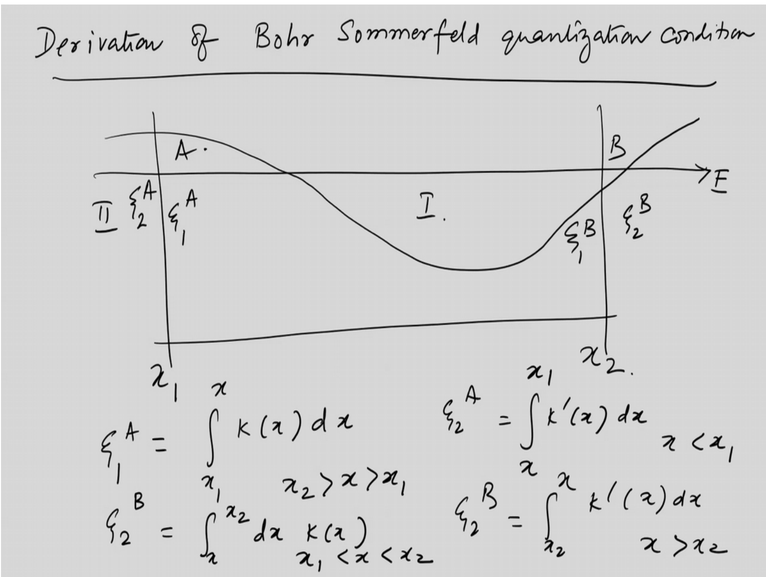

The Bohr Sommerfeld quantization condition using the connection formula; so, we draw this picture once again and we call it. So, we have let us say we have this varying like this where we have the energy is like this. So, this is the energy and this is my x 1 this is my x 2 this is my region 1 and this is my region 2, that is the energy so, this. So, this is a xi 2 a this is a xi 1 a. So, this is region 1 and this is again xi 2 B and this is xi 1 B and so on. So, these are points A and B and these are x 1 and x 2. So, let us define some new parameters xi 1 a equal to x 1 to x k of x dx.

So, this is between x 2 greater than x greater than x 1 xi 1 xi 2 a rather, xi 2 a equal to a k prime x these are the phases that we have seen x less than x 1 x 2 some x 1. So, this is x 1 x 2 x 1 and this is x 1 to x which is x is some arbitrary variable. So, xi 2 B this is equal to x 2 x 2 and dx k of x and a xi 2 B equal to x 2 to x k prime of x dx for x greater than x 2 and this is of course, x 1 less than x less than x 2.

So, these are my new variables and what we are trying to do is the following.

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At A

$$\frac{1}{2\sqrt{k'(x)}}\, e^{-\xi_2^A} \longrightarrow \frac{1}{\sqrt{k(x)}} \cos\left(\xi_1^A - \frac{\pi}{4}\right)$$

At any point r (inside the region)

$$\psi(r) = \frac{C_1}{\sqrt{k(r)}} \cos\Big(\underbrace{\int_{x_1}^{x_r} k(x)\,dx - \frac{\pi}{4}}_{\alpha}\Big) = \frac{C_1}{\sqrt{k(r)}} \cos\alpha$$

At B

$$\frac{1}{2\sqrt{k'(x)}}\, e^{-\xi_2^B} \longrightarrow \frac{1}{\sqrt{k(x)}} \cos\left(\xi_1^B - \frac{\pi}{4}\right)$$

At any arbitrary pt r

$$\psi(r) = \frac{C_2}{\sqrt{k(r)}} \cos\left(\int_{x_r}^{x_2} k(x)\,dx - \frac{\pi}{4}\right)$$

So, at A we have these going to the connection formula will have a 2 k prime x exponential minus xi 2 a it is connected with 1 by k of x cosine of xi 1 a minus pi by 2 or pi by 4 sorry this is pi by 4 and at any point r. So, that is inside the region. So, the wave function is like C 1divided by root over K of at r cosine of x 1 to xr and Kx dx that is the phase multip[lied]- minus pi by 4 called this one as alpha. So, this is equal to C 1 root over K of r cosine alpha.

Similarly, at B the connection says that it is a k prime x exponential minus xi 2 B xi 2 B it goes as 1 by root over k of x cosine of xi 1 B minus pi by 4. So, at any arbitrary point r psi of r is equal to some C 2 divided by K of r cosine of x r to x 2 K of x dx minus pi by 4 and, this one we are going to cast it in the form of alpha. So, let us call this as beta.

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$$\cos\beta = \cos\left(\int_{x_r}^{x_2} k(x)\,dx - \frac{\pi}{4}\right)$$

$$= \cos\left(\frac{\pi}{4} + \int_{x_2}^{x_r} k(x)\,dx\right) \qquad \cos(-\theta) = \cos(\theta)$$

$$= \cos\left(\frac{\pi}{4} + \int_{x_1}^{x_r} k(x)\,dx + \int_{x_2}^{x_1} k(x)\,dx\right)$$

$$= \cos\left[\alpha + \frac{\pi}{2} - \int_{x_1}^{x_2} k(x)\,dx\right.$$

$$= \cos(\alpha - \eta) \qquad \eta = \int_{x_1}^{x_2} k\,dx - \frac{\pi}{2}.$$

$$= \cos\alpha\cos\eta + \sin\alpha\sin\eta$$

At a general point, $\psi(r) = \frac{C_2}{\sqrt{k(r)}}\left(\cos\alpha\cos\eta + \sin\alpha\sin\eta\right)$

And Cos of beta equal to cosine of xr; so, just rewriting the Cos beta term as a pi by 4 plus x 2 divided by x r and a K x dx. So, see that we have just used the fact that the Cos of minus theta equal to Cos theta because Cos is an even function.

So, this can be a slightly modified in the form of a pi by 4 plus x 1 to xr k of x dx plus x 2 to x 1 k of x dx this is just simply writing down the x 2 to x r by splitting it into 2 terms, and this is nothing, but equal to cosine of alpha plus pi by 2 minus x 1 to x 2 Kx dx and this is nothing, but equal to cosine of alpha minus eta. So, cosine beta is cosine alpha minus eta where eta equal to x 1 to x 2 k dx minus a pi by 2.

So, this is nothing, but equal to cosine alpha cosine eta plus a sin alpha sin eta. So, at a general point the wave function is C 1 equal to so, it is so, psi of r. So, psi of r equal to C 2 divided by k of r cosine alpha, cosine eta plus a sin alpha sin eta and. So, this wave function has to be equal to this wave function that we have written down.

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$$C_1 = C_2 \cos\eta$$

$$0 = C_2 \sin\eta \quad \because C_2 \neq 0 \quad \sin\eta = 0 = \sin(n\pi).$$

$$\eta = n\pi = \int_{x_1}^{x_2} K\,dx - \frac{\pi}{2}$$

$$\int_{x_1}^{x_2} \hbar K\,dx = \left(n+\frac{1}{2}\right)\hbar\pi$$

$$\int_{x_1}^{x_2} p\,dx = \left(n+\frac{1}{2}\right)\hbar\pi$$

$$2\int_{x_1}^{x_2} p\,dx = \left(n+\frac{1}{2}\right)h \Rightarrow \int_{x_1}^{x_2} p(x)\,dx + \int_{x_2}^{x_1} p(x)\,dx$$

$$= 2\int_{x_1}^{x_2} p(x)\,dx$$

$$= \oint p(x)\,dx.$$

So, C 1 so, equating the coefficient C 1 equal to C 2 cosine eta and 0 equal to C 2 sin eta since C 2 is not equal to 0. Then a sin eta has to be equal to 0 which is equal to sin n pi for n equal to 0, 1, 2 and so on, sin n pi.

So, that tells that eta equal to n pi equal to x 1 to x 2 K dx minus pi by 2 and I can multiply by h cross and then it becomes x 1 to x 2 h cross K dx which is equal to n plus half h cross pi, where h cross is equal to h over 2 pi. Now I can write this down equal to p dx x one to x 2 equal to n plus half h cross pi which a little bit of algebra it shows that it is 2 p dx x 1 to x 2 this is equal to n plus half h simply h. So, this can be written as x 1 to x 2 p of x dx plus x 2 to x 1 p of x dx which is equal to 2 x 1 to x 2 p of x dx which is nothing, but equal to a closed integral of p p of x dx.

So, that tells that the left hand side of this equation the condition that we have gotten from here is equal to closed integral of px dx.

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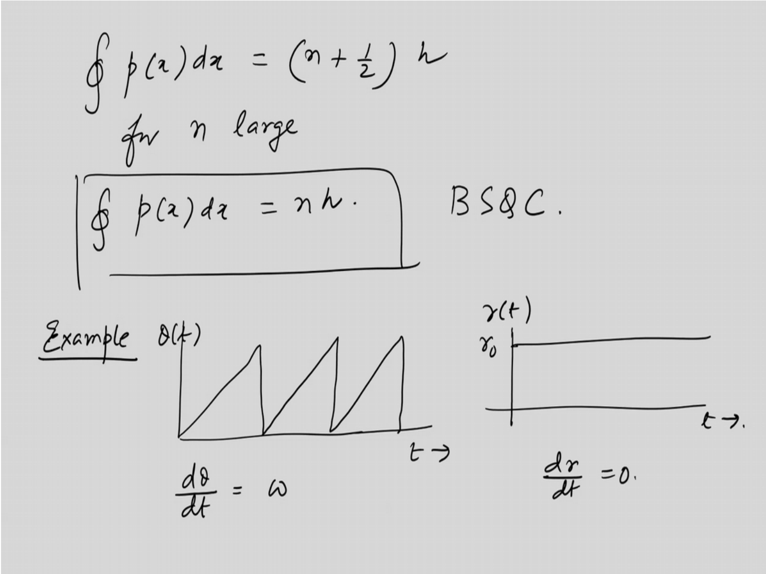

So, that is the Bohr Sommerfeld quantization condition that a p x dx is equal to n plus half h and for n to be large this is simply equal to P x dx equal to n h and this is the we will call it BS QC which is called as the Bohr Sommerfeld quantization condition.

So, just give one simple example the example; is that the in a certain system the theta is like a sawtooth wave as a function of t. So, this is theta of t that is the angular variable and the r simply is just a constant at a value. So, d theta dt equal to omega and dr dt equal to 0.

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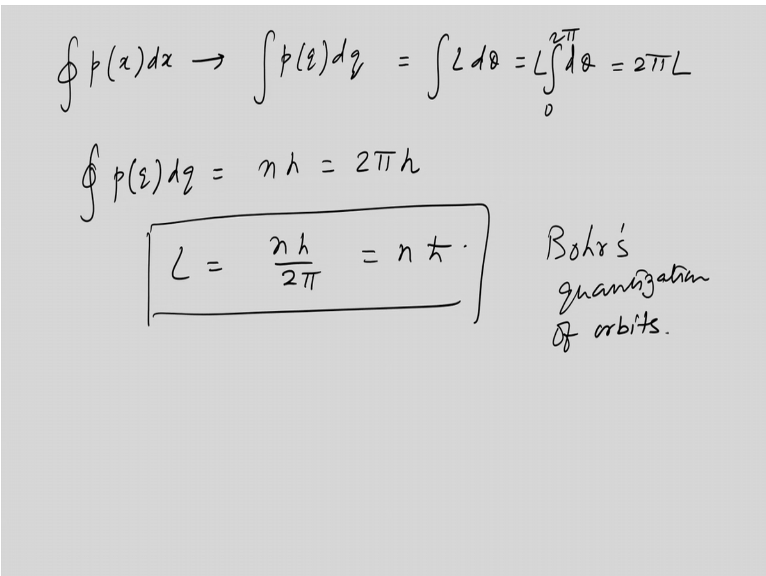

So, that tells that my P now its x dx we just simply write it as pq dq, where Q is the canonical coordinate and P is the canonical momentum which are nothing, but equal to ld theta which is equal to l l is a constant here d theta from the full range which is 0 to 2 pi which is equal to 2 pi l. So, P q dq is nothing, but equal to nh which is equal to 2 pi l. So, l becomes equal to nh by 2 pi, which is equal to nh cross and this is called as the Bohr's quantization.

So, this is one of the Bohr's postulate where he said that the angular momentum is quantized in terms of h cross and those are the allowed orbits in which the electrons are allowed to move around the nucleus where they do not emit electromagnetic radiation and they are called as the stationary orbits.

So, just to go back rerun the whole thing again there is a lot of mathematics that we have done, but what we have finally, said is that for a slowly varying potential the solutions that you write down fails miserably close to the turning points because the amplitudes are proportional to 1 over k x or 1 over root over Kx where k is the wave vector, which is obtained from this energy and the potential energy relation the total energy and the potential profile relation and.

But these are good solutions away from the turning points, if they are good solutions away from the turning point there has to be a way to connect the solutions into the from the classically forbidden region to the classically allowed region, that gives us one very important factor that you cannot do this arbitrarily that is connect the two solutions and write it as equality, it has to be done with care that sometimes you can go from region 2 to region 1 via 1 relation.

But to come back from region 1 to region 2 you will write another relation, and these are called as the connection formula this connection formula have important applications where one can actually compute what is called as a Bohr Sommerfeld Quantization condition we have showed a simple case where we recover the Bohrs postulate starting from the quantization condition, which are which are of course, the byproduct or artifact of the connection formula.