Advanced Quantum Mechanics with Applications

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Lecture - 28

Time dependent Perturbation Theory, Fermi's Golden rule, Einstein's A and B Coefficients

So, short recap to begin today's discussion. So, we were talking about time dependent Perturbation Theory. So, we have a time dependent perturbation which is likely to cause transitions of from one state to another maybe from a ground state to an excite state or vice versa.

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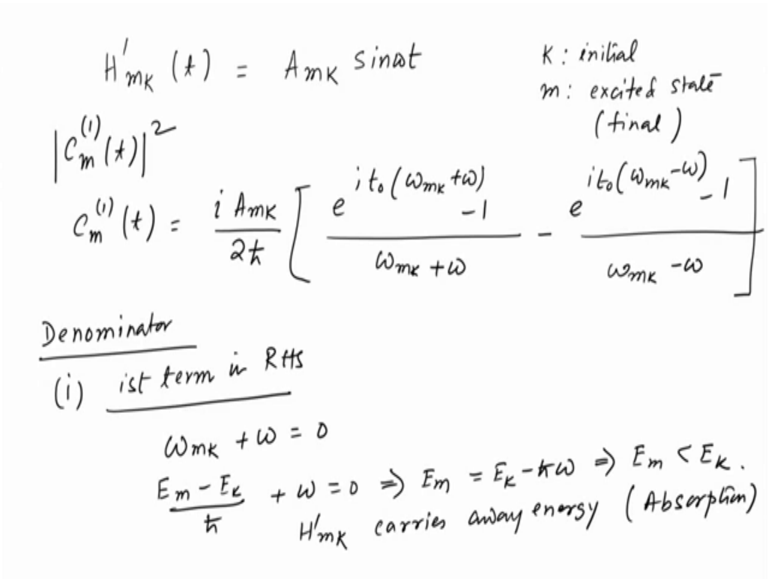

$H'_{mK}(t) = A_{mK} \sin\omega t$

K : initial
m : excited state
(final)

$|C_m^{(1)}(t)|^2$

$$C_m^{(1)}(t) = \frac{i A_{mK}}{2\hbar}\left[\frac{e^{it_0(\omega_{mK}+\omega)}-1}{\omega_{mK}+\omega} - \frac{e^{it_0(\omega_{mK}-\omega)}-1}{\omega_{mK}-\omega}\right]$$

Denominator

(i) ist term in RHS

$\omega_{mK} + \omega = 0$

$\frac{E_m - E_k}{\hbar} + \omega = 0 \Rightarrow E_m = E_k - \hbar\omega \Rightarrow E_m < E_k.$

H'_{mK} carries away energy (Absorption)

And the Hamiltonian the perturbation Hamiltonian is written as its H prime m k which is a function of t and then we have a Amk sine omega t that is the perturbation that we have been talking about. K is the initial state where the system was before the perturbation was switched on and m is say an excited state or a different state that than compared to K and we are going to talk about computing this coefficient. So, that is related to the probability of making a transition for the system to make a transition from an initial state K to a final state m, we could simply say that it is a final state and if we are resorting to a first order perturbation theory we will put a one here and the that is going to be the thing or the quantity to compute and this is what we have found yesterday or rather the last discussion that we were having that is. So, this is equal to i A m k and 2 h cross and exponential i t naught omega m k plus omega and minus 1 and omega m k plus omega and the exponential i t naught omega m k minus omega minus 1 and omega m k minus omega.

And. So, it consists of 2 terms the first term is called as the absorption term, which we will see immediately the second term is called as the emission term and we will talk about these separately. In fact, we are going to concentrate mostly on the second term which as i said that its related to the emission that is transition from one state to another.

So, let us look at each one of the terms; so, the denominator of the first term. So, one is first term in RHS that is inside the bracket. So, then the poles of that pole spins were the denominator becomes 0 that is this is the condition and omega m K as we have said that it is E m minus E K over h cross plus omega equal to 0. So, that immediately tells us that E m equal to E K minus h cross omega which means that E m is less than E K and. So, E K is the initial state or the unperturbed level before the perturbation was switched on. So, what it means is that the H prime m K which is of course, related to the A m K and so, on it carries away energy and so, its related to the absorption. So, the field is absorbing energy from the system. So, that is why its called as a absorption term and let us look at the second term.

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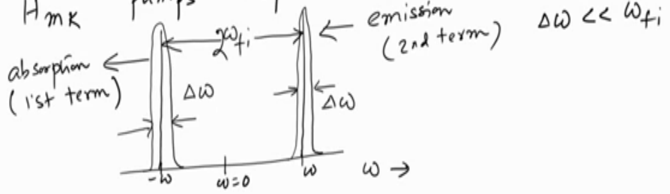
(ii) Second term $-\frac{iA_{mK}}{2\hbar}\left[\frac{e^{it_0(\omega_{mK}-\omega)}-1}{\omega_{mK}-\omega}\right]$

$$= \frac{A_{mK}}{\hbar}\left[e^{\frac{it_0}{2}(\omega_{mK}-\omega)}\right]\left[\frac{\sin\left\{\frac{t_0}{2}(\omega_{mk}-\omega)\right\}}{\underbrace{\omega_{mK}-\omega}_{\to\,\alpha}}\right]$$

Denominator

$\omega_{mK} = \omega \Rightarrow E_m - E_K = \hbar\omega \Rightarrow E_m > E_K$. Emission.

H'_{mK} pumps energy into the system —



And that term is of course, with a negative sign it is a minus i Am k by 2 h cross and I have an exponential i t naught omega m k minus omega minus 1 divided by omega m k minus omega and so, on we can write this we can simplify this we can write the numerator as a sign by taking exponential i t naught by 2 omega m k minus omega out. So, that gives this one the second term with one exponential minus i t naught omega m k minus omega by 2.

So, if we do that simplification then we will and that way we can get rid off this i as well because the sign in terms of the exponential comes with a definition which has a 2 i in the denominator, and once we do all that which i leave it to you its a simple step to complete. So, this is h cross and we will keep a i t naught by 2 omega m k minus omega and. So, this is exponential. So, this becomes really a sin. So, its a sin and then there is a t naught by 2 omega m k minus omega and divided by omega m k minus omega and we just come back to this term.

Now look at the denominator the denominator is of course, the denominator makes sense when or rather it has a pole, when omega m k equal to omega which means that Em now is a E m minus E k its equal to h cross omega and that tells us that the E m is greater than E k which means that the H prime m k or the perturbation term is it pumps energy into the system, and this is related to emission

So, now; so, basically we will more concentrate on this rather than on the first term because this term makes a large contribution close to the resonance that is known omega m k becomes equal to omega, and in any case both terms are equally important let me just with a diagram let me explain that. So, this is your omega axis and this is omega equal to 0, now I have plus omega here and a minus omega here. So, these are. So, there is a plus omega and minus omega. So, there is a a line which are. So, this corresponds to the emission line or the second term and this corresponds to the absorption and that is the first term.

Now, of course, these are the spectroscopic lines or these are the lines that will show up in experiments, and these lines cannot just be a you know very sharp lines which would of course, and the width of the line would depend on the basically the least count of the measuring apparatus or the energy range over which it can measure sort of these or rather these see this line.

So, let me give a little bit of a weight to these lines. So, these will be the spectral lines that will be seen for each one of them and let us just talk about these things to be delta omega and delta omega here as well and this distance or rather the the 2 lines are separated by a twice of omega fi which is. So, in this particular case as it the way the figure is drawn, it is very clear that we need to consider one line at a time in the limit that your delta omega is much much smaller than omega fi or 2 omega fi.

So, if this condition is satisfied, then only we need to look at one of the lines at a time and maybe at this moment let us just consider that we are concentrating on the second term which is the resonance term. The other term is called as the which is in the negative minus omega so, that is called as the anti-resonant term. So, they are symmetrically placed about 0 and at this as I said that we are just going to talk about the resonant term at this moment. And let us tell this call this quantity let us just write with a different color. So, let us call this quantity as alpha.

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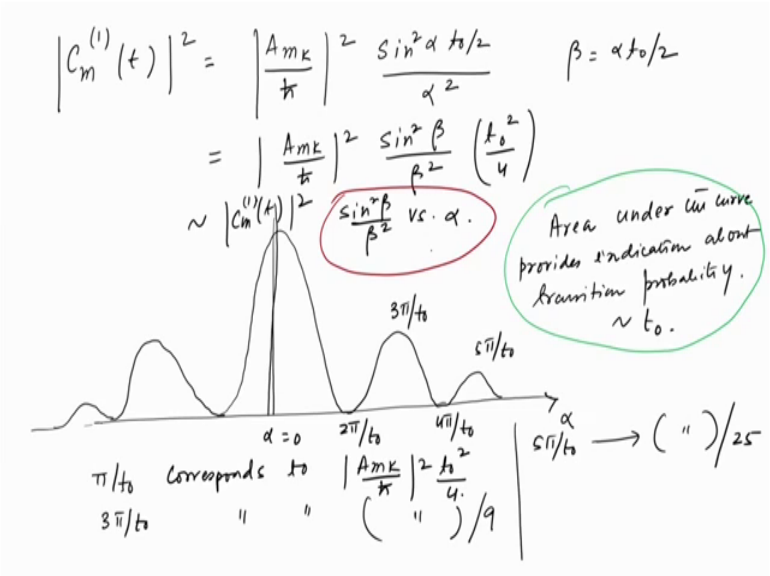

$$|C_m^{(1)}(t)|^2 = \left|\frac{A_{mk}}{\hbar}\right|^2 \frac{\sin^2 \alpha t_0/2}{\alpha^2} \qquad \beta = \alpha t_0/2$$

$$= \left|\frac{A_{mk}}{\hbar}\right|^2 \frac{\sin^2 \beta}{\beta^2} \left(\frac{t_0^2}{4}\right)$$

$\sim |C_m^{(1)}(t)|^2$ $\frac{\sin^2\beta}{\beta^2}$ vs. α.

Area under the curve provides indication about transition probability. $\sim t_0$.

π/t_0 corresponds to $\left|\frac{A_{mk}}{\hbar}\right|^2 \frac{t_0^2}{4}.$

$3\pi/t_0$ " " $(\;"\;)/9$

$5\pi/t_0 \longrightarrow (\;"\;)/25$

And in which case the so, the C m 1 t mod square its equal to A m k by h cross square and there is a sin square alpha t naught by 2 divided by alpha square.

So, that is the form of this and let us in addition takes beta equal to alpha t 0 by 2. So, then this thing would be written as Amk by h cross square sin square beta by beta square and since we have taken a t 0 by 2 whole square common, that had to be multiplied here. So, we will multiply it. So, t t 0 square by 4 and that will be there ok.

So, this is our form of the transition amplitude, which is related to the probability of transition from initial state k to a final state m, and this quantity sin square beta by beta square is called as the sinc function and the sinc function or the square of the sinc function sin theta by theta is called as a sinc function. So, this is if we plot it it looks like that you know the square will never become negative. So, it will look like this resembles the diffraction pattern that you may have seen in optics is nicely symmetric I might not have drawn it accurately, but it is it has to be you know I mean a little sort of all right.

So, this is; So, I am plotting a sin. So, this is my C m C m 1 t square and this is as a function of you know. So, alpha in fact, this is like sine square beta by beta square apart from these factors or you can take it to be proportional to this. So, this is proportional to this is actually sin square beta by beta square versus alpha that is being plotted. So, this is at alpha equal to 0 and this is at 2 pi over t 0, this is at 4 pi over t 0, this is at 3 pi over t 0 and this is at 5 pi over t 0 and so on so, forth all right.

So, this is a plot that we are plotting sin square beta by beta square versus alpha and so, pi by t 0 corresponds to this entire thing Amk by h cross square t 0 square by pi square t 0 square by 4 and. So, 3 pi by t 0 will corresponds to this entire thing divided by 9 and at 5 pi by t 0 it is. So, at 5 pi by t 0 it is it corresponds to this entire thing divided by 25 and so, on.

So, if you see this the area under the curve which is of course, related to the probability of transition, it is proportional to t 0 because the height goes as 1 by t 0 and this alpha is and then of course, it is a there is a t 0 square that is there and there is a 1 over t 0. So, this multiplied by 1 over t 0 will give a t 0. So, that is the transition probability and the. So, area under the curve we will write it here. So, area under the curve transition probability ok.

So, let me. So, that is the transition probability which goes as of course, as t 0 all right. So, let us just give a physical picture that what is happening.

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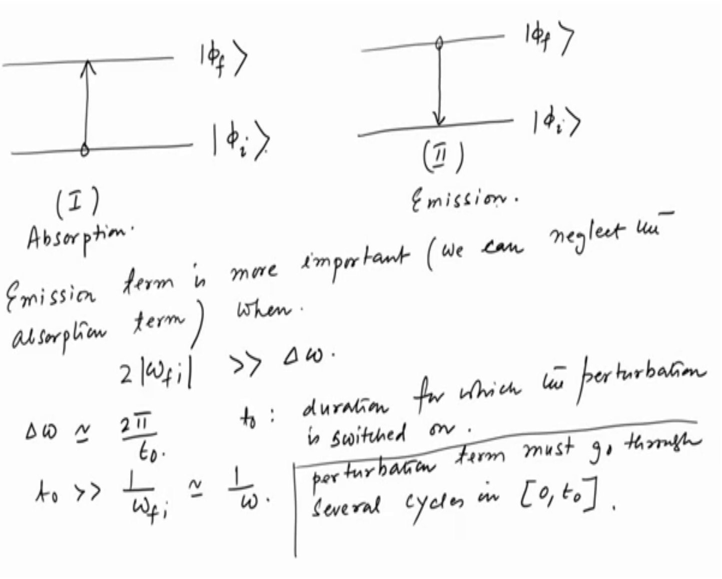

Emission term is more important (we can neglect the absorption term) when.

$$2|\omega_{fi}| >> \Delta\omega.$$

$\Delta\omega \simeq \frac{2\pi}{t_0}.$ t_0 : duration for which the perturbation is switched on.

$t_0 >> \frac{1}{\omega_{fi}} \simeq \frac{1}{\omega}.$ | perturbation term must go through several cycles in $[0, t_0]$.

So, you have 2 level system here call this as a phi f which is nothing, but that m by the index m and this is a phi i and so, this is that term number 1 that is the absorption term or the anti-resonant term and for the other one it is the phi f and a phi i and so, there is a particle that makes a transition from phi f to phi i and that is the second term, which is related to the emission. So, that is the physical picture for these 2 terms and of course, we are more interested in talking about this the emission term, that we are talked about.

So, of course, the emission term is more important that is we can neglect the absorption term when as I said earlier that 2 of omega fi we can put it inside a mod sign that is it does not matter which one is bigger the as the final state energy is a bigger of the initial state energies are bigger, this has to be much greater than delta omega.

Now delta omega that is the line width depends on the time duration for which the perturbation is switched on. And if we simply go by the order of magnitude using a or an estimation rather using Heisenberg's uncertainty principle, then this can be simply written as 2 pi over t 0. So, just to remind you t 0 is the duration for which the perturbation is switched on ok. So,; that means, that the t 0 has to be much greater than 1 by omega fi which of course, at resonance its equal to 1 over omega which means that the additional restriction of neglecting the anti-resonant term is that, the perturbation term must go through several cycles in this interval 0 to t 0 ok. So, that is the the additional restriction that we have.

Now, if the t 0 is of the order of 1 over omega, we more we have a situation where the perturbation is linearly varying with time and in which case it is not possible to neglect or rather separate out these 2 terms, such that we can concentrate on 1 at the expense of another. So, we will take this to be true that that this the perturbation term must go through several cycles. So, it is rapidly oscillating during the time that the perturbation is switched on all right.

Now, this has been told earlier it could happen that during the course of transition the particle or the atom goes on to a state, the final state is a part of a continuum which means that it is not a single well defined state it is a sort of collection of many states and its a continuum; which forms a continuum by a having many energy levels having very close by values all right. In which case we cannot talk about a probability of this transition and rather we should talk about a probability density of transition, and this probability density should be integrated over in order to get the total probability, integrated over all possible energy states that line that you know in that range.

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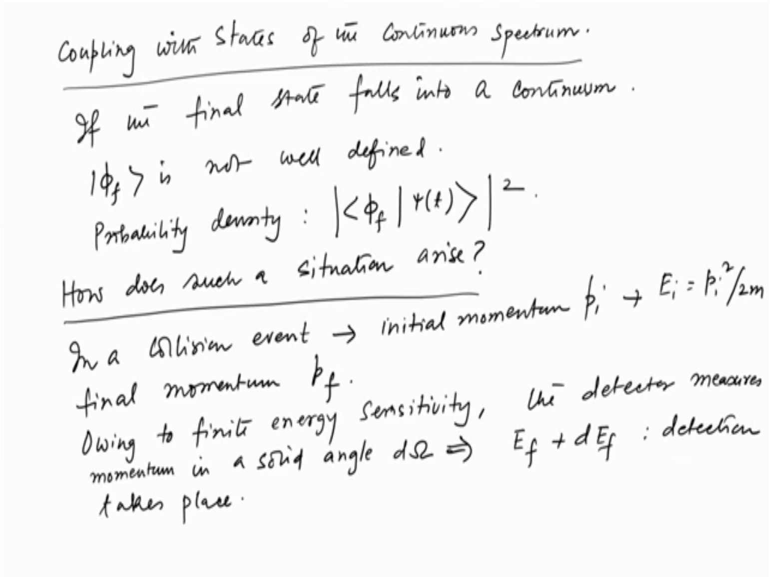

Coupling with States of the Continuous Spectrum.

If the final state falls into a continuum.

$|\phi_f\rangle$ is not well defined.

Probability density : $|\langle\phi_f|\psi(t)\rangle|^2$.

How does such a situation arise?

In a collision event → initial momentum $p_i \rightarrow E_i = p_i^2/2m$

final momentum p_f.

Owing to finite energy sensitivity, the detector measures momentum in a solid angle $d\Omega \Rightarrow E_f + dE_f$: detection takes place.

So, coupling with states of the continuous spectrum and so, if the final state falls into a continuum and so, phi f is not well defined. So, the probability density will be given by phi f and psi t mod square, and this probability density has to be integrated over and how does it arise? It arises say consider a collision event, in which a particle with an initial momentum p i which. So, p i which means that the energy E i equal to p i square 2 m we are talking about a free particle and this particle after the collision gets scattered on to a final momentum pf.

And now the detector by which it is being captured has an has a finite angular aperture, and the energy selectivity of the sensitivity may not be perfect, and in which case we talk about the solid angle of the detector which or other detects this particle and this gives. So, this going to finite energy sensitivity, the detector measures energies in a solid angle or momentum rather we talked we should talk about momentum in a solid angle d omega and that gives. So, basically the final state energy. So, which means that its between some E f and E f plus d E f is the detection takes place, in which case we will have to actually count the number of states which falls into that continuum of states between E f and Ef plus dEf and so, the probability P of say phi f t it should be sum over all these final states and then it is a phi f psi t and square.

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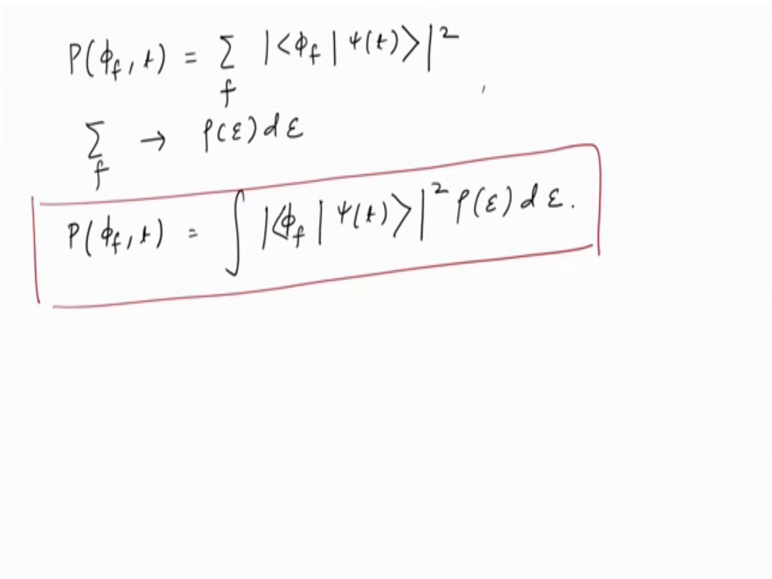

$$P(\phi_f, t) = \sum_f |\langle \phi_f | \psi(t) \rangle|^2$$

$$\sum_f \rightarrow \rho(\varepsilon)\, d\varepsilon$$

$$\boxed{P(\phi_f, t) = \int |\langle \phi_f | \psi(t) \rangle|^2 \rho(\varepsilon)\, d\varepsilon.}$$

Now this sum over f that is the energy states that line that range Ef to Ef plus dEf is nothing, but the density of states corresponding to that final state energy, hence this p of phi f and t has to be integrated over and one gets a phi f and psi t mod square and rho epsilon d epsilon, and this is the final result for that probability for system to make a transition from an initial state to a final state in presence of a time dependent perturbation.

So, we now equipped to talk about the Fermis golden rule and let us see what it means in the present context that is including this density of states for the final states and so, on.

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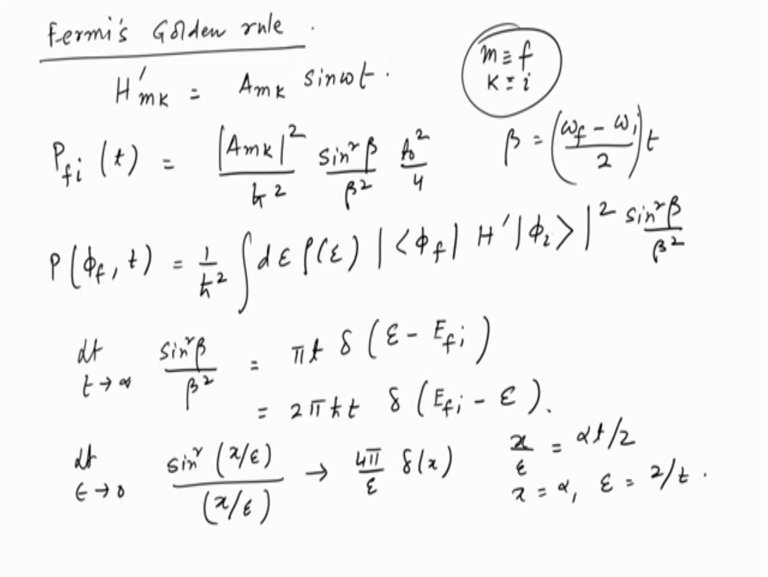

Fermi's Golden rule.

$$H'_{mk} = A_{mk} \sin\omega t.$$

$m \equiv f$
$k \equiv i$

$$P_{fi}(t) = \frac{|A_{mk}|^2}{\hbar^2} \frac{\sin^2\beta}{\beta^2} \frac{t_0^2}{4} \qquad \beta = \left(\frac{\omega_f - \omega_i}{2}\right)t$$

$$P(\phi_f, t) = \frac{1}{\hbar^2}\int d\varepsilon\, \rho(\varepsilon)\, |\langle\phi_f| H' |\phi_i\rangle|^2 \frac{\sin^2\beta}{\beta^2}$$

$$\underset{t\to\infty}{\mathrm{Lt}} \frac{\sin^2\beta}{\beta^2} = \pi t\, \delta(\varepsilon - E_{fi})$$
$$= 2\pi\hbar t\, \delta(E_{fi} - \varepsilon).$$

$$\underset{\epsilon\to 0}{\mathrm{Lt}} \frac{\sin^2(x/\epsilon)}{(x/\epsilon)} \rightarrow \frac{4\pi}{\varepsilon}\delta(x) \qquad \frac{x}{\epsilon} = \alpha t/2$$
$$x = \alpha, \quad \varepsilon = 2/t.$$

So, let us. So, let us take again a sinusoidal perturbation of the form that we had taken. So, we have taken this H prime to be or rather H prime m k to be equal to a mk sin omega t that is what we have been discussing

And now we let us call it as a P fi t just the same thing as what we have written as the transition probability its nothing, but Amk a mk is in general Hermitian; so, which means Amk equal to a km star and so, on. So, this divided by h cross square and a sin square beta by beta square and at t 0 square by 4 and beta equal to omega f minus omega i divided by 2 into t. So, it is alpha into t and hence this is what we need to calculate and then we if you wish. So, this is that P fi t now this P final state phi f r t is equal to 1 by h cross square d epsilon rho epsilon and phi f H prime phi i square and a sin square beta by beta square which is just again writing this Amk, where just try to connect that m is equal to the final state and k equal to the initial state ok. So, this is implicit here and now we have to perform this integral.

Now in order to perform this integral it could be quite complicated because there are. In fact, 3 terms which all depend upon the energy, that is the term which is shear the term which is the density of states, which depends upon energy which is usually a smooth function in most of the cases that we are accepting in some pathological cases where this density of states show a divergence close to some bandage or some such thing this is the matrix element square and this is that sin square beta by beta square of the sinc square function that we have obtained because of the time dependent perturbation

Now, it is not too difficult to understand that both these quantities are somewhat slowly varying quantities. The only quantity that varies very rapidly for the resonant term we are talking about the just the resonant term is this quantity that is the sin square beta by beta square. And this sin square beta by beta square if we just go on to see your the plot that we have drawn earlier is this. So, if you see this you see the major contribution of this probability density would come from this central peak and the auxiliary peaks are of course, there but they are you know by down by a factor of 9 or down by a factor of 25. So, they are not as important as this and because of this close to resonance, we can sort of replace it by a delta function with the width which is very small and that width of course, we will talk about that width.

But the fact that the sin square beta by beta square can be replaced by your you know by a delta function, that one has to be convinced about first and this is what i am trying to impress upon, that in this particular this term or this integral that we are talking about the one that is most important is the sin square beta by beta square.

So, if we are trying to replace it by a delta function c what are the conditions corresponding to that. So, let us take this limit t tending to infinity sin square beta by beta square is nothing, but a delta function up to a constant factor, which is E minus Ef i or we can simply write it as a epsilon minus which is a general energy and this is 2 pi h cross t delta of Ef i minus E. So, this is just like that limit epsilon tending to 0 sin square x over epsilon divided by x over epsilon is can be replaced by a 4 pi by epsilon delta of x and just you try to connect that x by epsilon is equal to alpha t by 2 in our case where x is equal to alpha and epsilon equal to 2 over t ok.

So, the main or the summary of this discussion is that, that we are going to replace this sin square beta by beta square by delta function which is the singularly most I mean rapidly varying function, the other functions are somewhat slow varying and can nearly be taken as constant in the vicinity of resonance ok. And this delta function has got a width which is given by 2 pi h cross over t and so, in this width we are going to talk about the density of states in that energy width how many energy states are there, that need to be incorporated in calculating the transition probability. So, that is actually encoded into our rho epsilon d epsilon.

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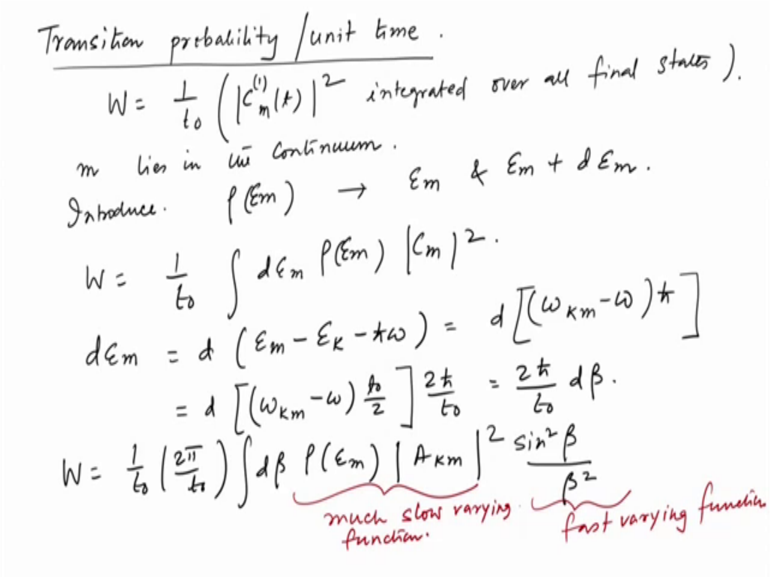

Transition probability/unit time.

$$W = \frac{1}{t_0}\left(|C_m^{(1)}(t)|^2 \text{ integrated over all final states}\right).$$

m lies in the continuum.

Introduce $\rho(\varepsilon_m) \rightarrow \varepsilon_m$ & $\varepsilon_m + d\varepsilon_m$.

$$W = \frac{1}{t_0}\int d\varepsilon_m\, \rho(\varepsilon_m)\, |C_m|^2.$$

$$d\varepsilon_m = d(\varepsilon_m - \varepsilon_k - \hbar\omega) = d\left[(\omega_{km}-\omega)\hbar\right]$$

$$= d\left[(\omega_{km}-\omega)\frac{t_0}{2}\right]\frac{2\hbar}{t_0} = \frac{2\hbar}{t_0}\, d\beta.$$

$$W = \frac{1}{t_0}\left(\frac{2\pi}{\hbar}\right)\int d\beta\, \underbrace{\rho(\varepsilon_m)\, |A_{km}|^2}_{\text{much slow varying function.}}\, \underbrace{\frac{\sin^2\beta}{\beta^2}}_{\text{fast varying function}}$$

So, let us just introduce that we are in interested in talking about the transition probability per unit time which is equal to 1 by t 0 and then we have a C k t mod square let us just write C m because this is what you are C m t square and if we are talking about the first order perturbation theory we will have a one on in the here and then its integrated over all final states and of course, these m lies in to in the continuum.

So, we will introduce as we have said that we will introduce density of states, we can write it as a rho m which is between epsilon m and epsilon m plus dE m ok. So, w will be 1 over t 0 d E m rho m c m mod square now the dE m equal to d of epsilon k or rather epsilon m minus epsilon k minus h cross omega this is as we said that is for the resonant term and this is equal to d of omega k m minus omega h cross and this is e equal to d of omega k m minus omega and multiplied by a t naught by 2 just to make it look like a beta that we have introduced and there is a 2 h cross by t 0. So, this is equal to a 2 h cross by t 0 tb d beta.

So, the transition probability per unit time becomes equal to 2 pi by t naught and d beta rho m or we can I mean instead of writing it as rho. So, we can write it as rho epsilon m and a km mod square Amk and A k m are same as I said that they are mostly Hermitian. So, its sin square beta by beta square. So, this is much slow varying function as compared to this this is a fast varying function. So, this is what it is. So, these are the transition probability per unit time and then of course, we can write this down as.

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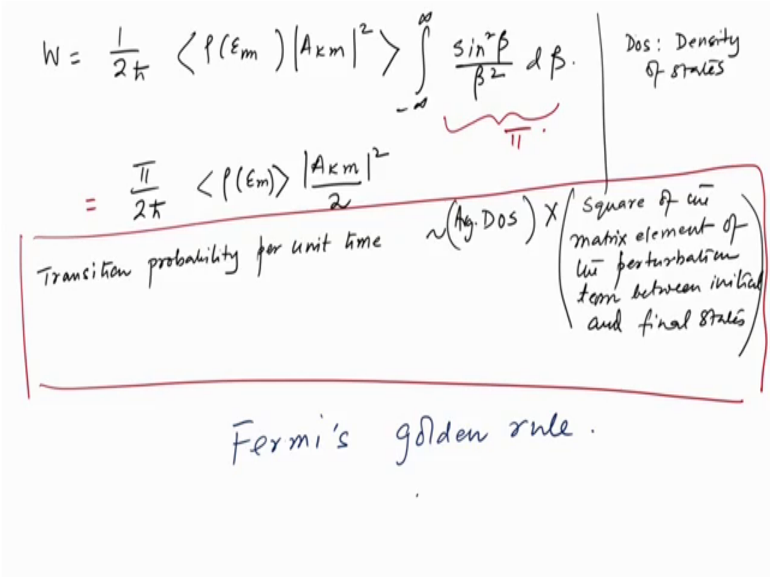

$$W = \frac{1}{2\hbar}\left\langle \rho(\epsilon_m)|A_{km}|^2\right\rangle \int_{-\infty}^{\infty}\underbrace{\frac{\sin^2\beta}{\beta^2}d\beta}_{\pi}.$$

Dos: Density of states

$$= \frac{\pi}{2\hbar}\left\langle \rho(\epsilon_m)\right\rangle \frac{|A_{km}|^2}{2}$$

Transition probability per unit time ~ (Avg. Dos) × (Square of the matrix element of the perturbation term between initial and final states)

Fermi's golden rule.

So, this W is equal to 1 over 2 h cross we take this thing out and take its average value since this is a much slowly varying function as compared to the second one then we take this and take an average value of these 2 terms and take them out, and let us write them as rho epsilon m and A km square and a minus infinity to plus infinity that is sin square beta by beta square and there is of course, a d beta.

Now, this has a value this integral has a value that value is equal to pi ok. So, this becomes simply equal to pi over 2 h cross and this rho now I can I can also split this 2 terms and so, this is A km mod square by 2. So, the average value of this matrix element can be written as a half of the; of these A km square and. So, this is equal to simply equal to. So, if we write it in words, its equal to the transition probability per unit time is proportional to the density average density of states average density of states we write it with a dos. So, Dos means Dos is for density of states for the final continuum of states. So, its average density of states a multiplied by the square of the matrix element of the perturbation term between initial and final states ok.

So, write it with this thing and let me just box it once again this is called as a Fermi's golden rule all right. So, this is the Fermi's golden rule and its applicable to a variety of situations especially say, the ionization of the hydrogen atom in the ground state. So, when hydrogen atom or the electron in the hydrogen atom is in the ground state, and if it is somehow given an energy which is equal to 13.6 electron volts, that is the magnitude of the perturbation of the time independent part of the perturbation there is a matrix element Amk, if that is of that order 13.6 electron volt and then this electron will be ejected or this hydrogen ion will be ionized, and this electron will go on to final continuum of states and the final those density of states of those final continuum of states will have to be taken into account in order to calculate the transition probability by unit time and it has also to be multiplied by the average density of states of those final states

So, instead of doing that which is ionization of hydrogen atom let us give a short in introduction to the lasers or which is a part of a big or rather discussion called as a interaction of radiation with matter, we will not do that very rigorously, but we will just introduce the a b coefficients and we will tell you that how to calculate these coefficients Einsteins A, B coefficients ok

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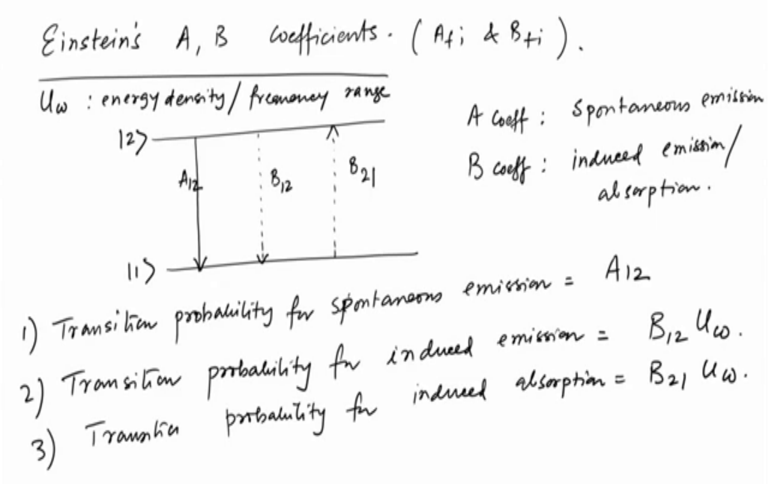

Einstein's A, B coefficients. (A_{fi} & B_{fi}).

U_ω : energy density / frequency range

A coeff : Spontaneous emission

B coeff : induced emission/ absorption.

1) Transition probability for spontaneous emission = A_{12}

2) Transition probability for induced emission = $B_{12} U_\omega$.

3) Transition probability for induced absorption = $B_{21} U_\omega$.

So, let us again define a 2 level system this is called level one and this is called level 2 and this is called the A12 and this is called I will write it with a dotted line which is called as a B 12 and we will also write with another dotted line which is called as a B 2 1. So, it is just the final state is written first and then the initial state is written. So, these are called as the Einstein's A B coefficients. So, where so, these are called A fi and B fi. So, there A f i and Bf i. So, the A coefficient it stands for spontaneous emission and the **b** coefficient stands for induced emission also absorption we will tell you what and in which case the first term that we are neglected in that original expression we will have to be brought back.

So, the transition probability for induced emission for spontaneous emission say first spontaneous emission is equal B 12 u omega let me just write that u omega is the energy density or. So, let us have u omega is the energy density per unit frequency range frequency range, that is between or you can say u omega d omega to be the energy density between omega and omega plus d omega which is a better representation. So, we will talk about u omega d omega together which is an energy density in the range in the frequency range omega and omega plus d omega.

So, the transition probability for spontaneous emission is this I am sorry this is not for spontaneous emission spontaneous emission does not depend upon the energy density its simply equal to A12. Now the transition; so, this is number o1; number 2 is transition probability for induced emission its equal to B 12 u omega and third is transition probability for induced absorption. Once again to remind you that this is the first term that we have talked about earlier; so, this is B 21 u omega

So, these are the nomenclatures of these ab coefficients and let us see that at steady state.

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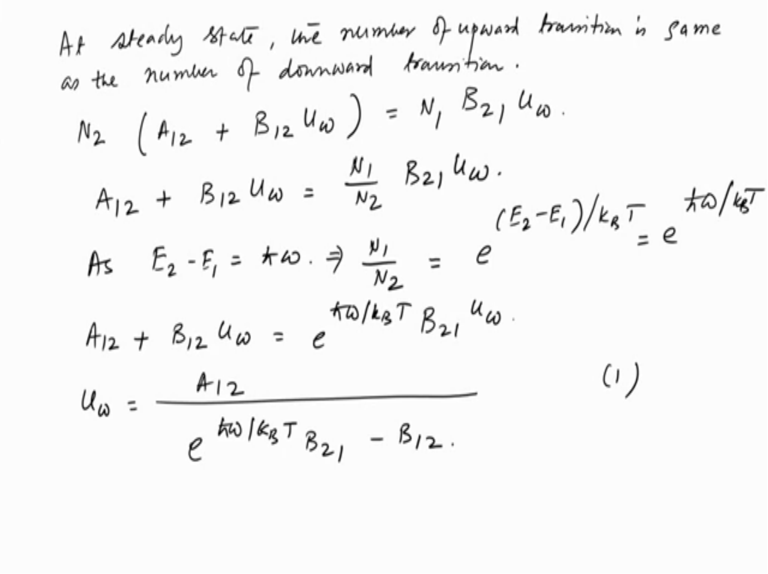

At steady state, the number of upward transition is same as the number of downward transition.

$$N_2 \left(A_{12} + B_{12} u_\omega \right) = N_1 B_{21} u_\omega .$$

$$A_{12} + B_{12} u_\omega = \frac{N_1}{N_2} B_{21} u_\omega .$$

$$\text{As } E_2 - E_1 = \hbar\omega . \Rightarrow \frac{N_1}{N_2} = e^{(E_2 - E_1)/k_B T} = e^{\hbar\omega/k_B T}$$

$$A_{12} + B_{12} u_\omega = e^{\hbar\omega/k_B T} B_{21} u_\omega .$$

$$u_\omega = \frac{A_{12}}{e^{\hbar\omega/k_B T} B_{21} - B_{12}} . \qquad (1)$$

The number of upward transitions is same as as the number of downward transition. So, that tells that N 2 multiplied by A 12. So, that is the number of atoms making a transition from 2 to 1. So, that is n 2 and n one is the number of atoms making a transition from 1 to 2. So, it is A 1 2 plus B 1 2 u omega this is equal to n 1 B 2 1 u omega. So, i can write that A 12 plus B 1 to u omega equal to N 1 plus N 2 B 2 1 u omega. So, as minus E 1 equal to h cross omega one can write down from the Boltzmann distribution N 1 by N 2 is equal to exponential by e one beta or by k t which is equal to exponential h cross omega by k t. So, that is N 1 by N 2. So, A 1 2 plus B 1 2 u omega equal to exponential h cross omega by k t B 2 1 u omega. So, u omega is equal to A 1 2 divided by exponential h cross omega by k t b 21 minus B 21.

Now, in order to find these coefficients we can do a comparison with a Planck's law of blackbody radiation.

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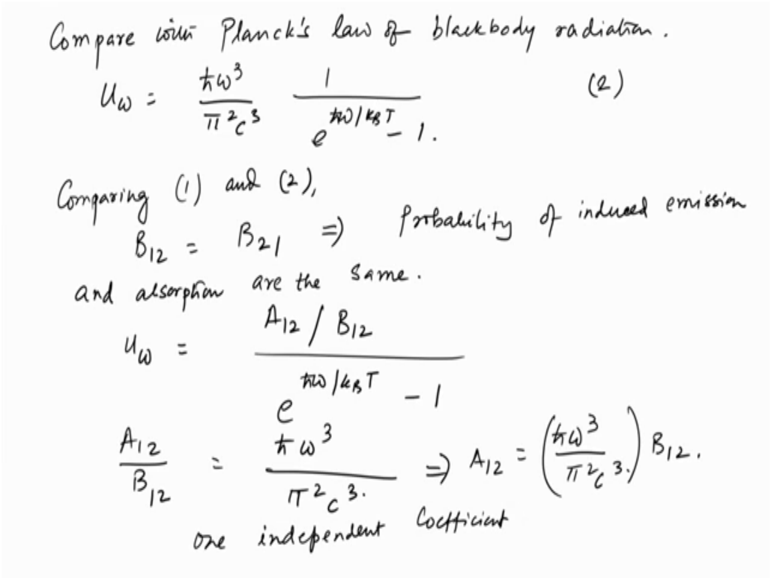

Compare with Planck's law of blackbody radiation.

$$U_\omega = \frac{\hbar\omega^3}{\pi^2 c^3}\,\frac{1}{e^{\hbar\omega/k_B T}-1}. \qquad (2)$$

Comparing (1) and (2),

$B_{12} = B_{21} \Rightarrow$ probability of induced emission and absorption are the same.

$$u_\omega = \frac{A_{12}/B_{12}}{e^{\hbar\omega/k_B T}-1}$$

$$\frac{A_{12}}{B_{12}} = \frac{\hbar\omega^3}{\pi^2 c^3.} \Rightarrow A_{12} = \left(\frac{\hbar\omega^3}{\pi^2 c^3.}\right)B_{12}.$$

one independent coefficient

Radiation and where u omega is given by some constant quantities such as this and the both distribution follows here and so, comparing let us call this as 1 and 2. So, comparing 1and 2 B 12 is equal to B 21 which says that the probability of of induced emission and absorption are the same.

So, from one can get u omega equal to A 1 2 divided B 12 now since they are same one can simply write it as h cross omega by k t minus 1. So, A 1 2 divided by B 1 2 its equal to h cross omega cube by 4 pi square c cube. So, that tells that A 12 a one 2 equal to h cross omega cube by pi square c cube and into B on12. So, there is a actually one coefficient out of the 3 which is independent because where the 2 of them are equal and one of them has a relationship with another; so, one independent coefficient ok.

So, the last thing that one can see here is the following that if we if we want to prove that these B 1 2 equal to B 2 1 how that comes about in a more rigorous way.

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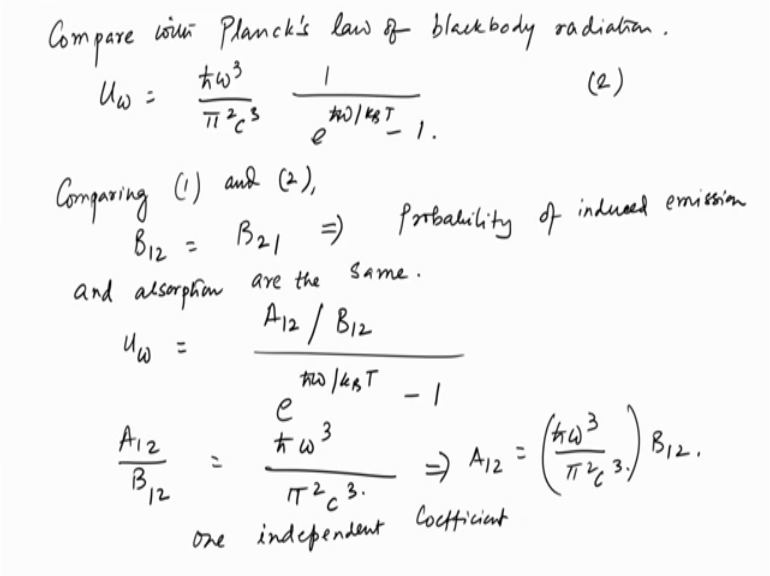

Compare with Planck's law of blackbody radiation.

$$u_\omega = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{e^{\hbar\omega/k_BT}-1}. \qquad (2)$$

Comparing (1) and (2),

$B_{12} = B_{21} \Rightarrow$ probability of induced emission and absorption are the same.

$$u_\omega = \frac{A_{12}/B_{12}}{e^{\hbar\omega/k_BT}-1}$$

$$\frac{A_{12}}{B_{12}} = \frac{\hbar\omega^3}{\pi^2 c^3.} \Rightarrow A_{12} = \left(\frac{\hbar\omega^3}{\pi^2 c^3.}\right)B_{12}.$$

one independent coefficient

Then let us talk about the absorption term in presence of the time dependent perturbation. So, absorption; so, which means that m equal to A 2 k equal 1 and. So, c 2 t in our original language is equal to minus i A 2 1 by 2 h cross exponential i t naught omega 2 1 minus omega minus 1 divided by omega 2 1 minus omega. So, that is the c 2 and the emission term is m equal to one and k equal to 2 and. So, that is the c 1 t is equal to a plus a 1 by 2 h cross and exponential i t naught omega one 2 plus omega minus 1 omega 12 plus 0 So, we will prove that c 1 t mod square equal to c 2 t mod square. So, c 1 star t can be written as.

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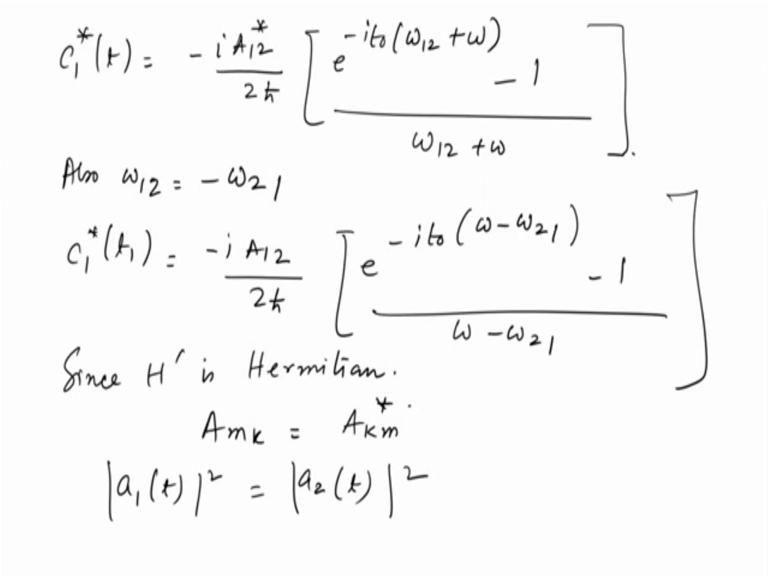

$$c_1^*(t) = \frac{-iA_{12}^*}{2\hbar}\left[\frac{e^{-it_0(\omega_{12}+\omega)}-1}{\omega_{12}+\omega}\right].$$

Also $\omega_{12} = -\omega_{21}$

$$c_1^*(t_1) = \frac{-iA_{12}}{2\hbar}\left[\frac{e^{-it_0(\omega-\omega_{21})}-1}{\omega-\omega_{21}}\right]$$

Since H' is Hermitian.

$$A_{mk} = A_{km}^*.$$

$$|a_1(t)|^2 = |a_2(t)|^2$$

So, c 1 star t can be written as minus i A 1 2 by h 2 h cross exponential minus i t naught omega 1 2 plus omega minus 1 omega 12 plus omega that is also omega one 2 equal to minus omega 2 1. So, that makes the c star t is equal to minus A 1 2 star ok. So, I should make it start as well knowing that these are our mission we will we can change it 2 h cross and exponential minus i t naught omega minus omega 2 1. So, that is the and minus 1 divided by omega minus omega 22 1.

Since H prime is Hermitian A m k is equal to a km star which has been already told. So, it is if one can see that now a one t mod square equal to a 2 t mod square.

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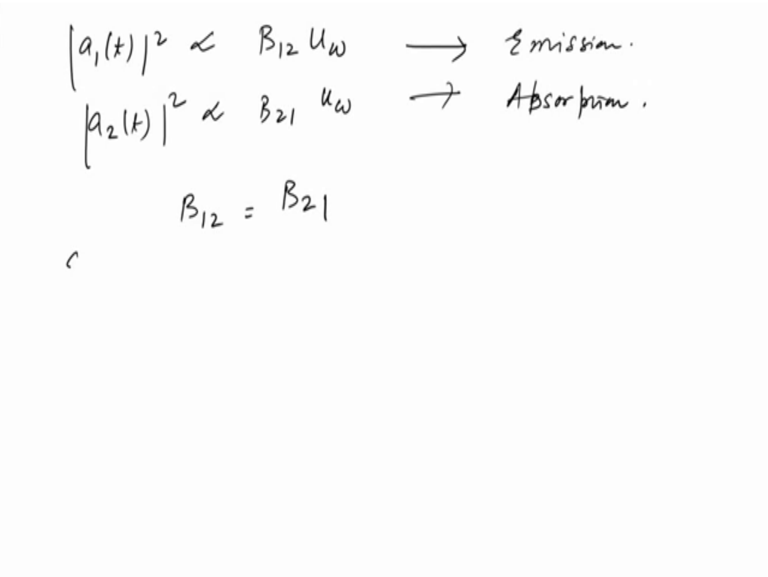

$|a_1(t)|^2 \propto B_{12}u_\omega \longrightarrow$ Emission.

$|a_2(t)|^2 \propto B_{21}u_\omega \rightarrow$ Absorption.

$$B_{12} = B_{21}$$

c

And since a 1 t mod square is proportional to B 12 u omega which is the emission term and a 2 t is B 2 1 u omega which is the absorption term; so, B 1 2 equal to B 2 1.

Now, microscopic derivation of these coefficients we have not derived this coefficient, but those coefficients are very easy to derive from whatever we have done and each of these, we know now that there is just only one coefficient that is unknown and that coefficient can be determined exactly in the same way that we have calculated this these x these transition probabilities the only thing that we need to know is that the H prime which will come from the interaction of radiation with matter and we will do it in a tutorial problem this interaction of radiation with matters

And what is the Hamiltonian for that what you need to know is that a priori without going into the details now the canonical momentum now changes from p to p minus e a where a is the vector potential, which gives the magnetic field which be equal to curl a and of course, there is a gauge choice of gauge that could be also discussed there, but other than that we now have all the necessary tools to calculate the transition probability which is only a feature of the time dependent perturbation, as has been stressed many times for and.

So, as I told that it can also be used to calculate the ab coefficients, now we just know need to know one coefficient and the rest will follow or we can also calculate the ionization of hydrogen atom by using this time dependent perturbation theory.