Advanced Quantum Mechanics with Applications
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Lecture - 29
Scattering Theory

So, having done most of the approximate methods in treating Quantum Mechanical problems, we embark on the last one that is Scattering Theory

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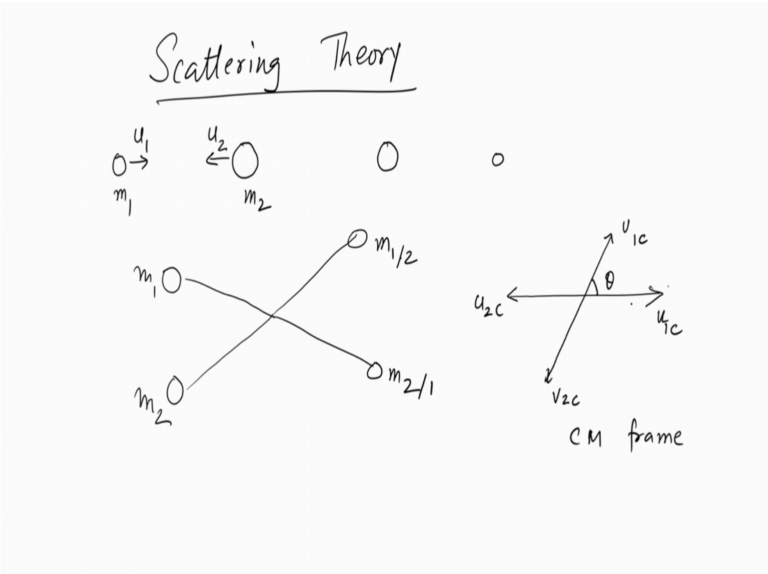

Now, scattering theory is known in some form in the context of classical mechanics, where you have reduced 2 body problem into a one body problem in the center of mass frame. And have defined an effective mass consisting of the 2 masses and then have considered, scattering in the CM frame or the Center of Mass frame, and finally, the cross section was computed in the center of mass frame and then, it was converted back into the lab frame, to be able to establish a connection with experiments.

Here of course, the approach will be quantum mechanical, but very quickly let us see or do a recapitulation of the classical scattering theory that, you have been familiar with in the center of mass frame So, it is about 2 particles, just drawing one slightly bigger than the other. So, m 1 and m 2, they have velocities initial velocities u 1 and u 2. And they scatter and after scattering they go to some. so there are, you know I mean, so if it is a three dimensional scattering problem then, of course we have to consider that, there are three dimensional scattering problems; are slightly difficult, but we still need a lot of input about the initial velocities in order to solve the problem because, if in one dimension of course, you have momentum conservation equations and the energy conservation equation. In 1 d you have 1 momentum conservation equation and 1 energy conservation equation and we are talking about elastic scattering at the moment.

However, in 3 dimensions there are 3 momentum conservation equations and one energy conservation equation. The number of unknowns are, however 6 because, the 3 initial components of or rather the initial velocities and of each of the particles in 3 dimensions are the unknowns, given the initial velocities for the start of the problem that is, before collisions say the velocities are given which are known as, the initial velocities and after collision the velocities are known as, the final velocities

So, our aim is to calculate the final velocities and their angle with respect to the incident directions and so on. And this problem is, it is somewhat complicated in the sense that, number of equations need to be solved and they should be consistent with each other. Now in quantum mechanics, before we go to quantum mechanics of course, the solution of the problem in the CM frame is, somewhat easier.

And what happens in CM frame is that, the velocities actually, so in a 3 dimensional collision problem, so there is a particle, that comes from here and there is a particle that comes from here and they go and they scatter and so, this is say m 1 and this is m 2 and then, they scatter in different directions. It could be m 1 here and m 2 here or the vice versa; I mean this could be 1 or 2 and this could be 2 or 1 and you need to calculate all these final state velocities.

Whereas in CM frame, what happens is that, one usually has a scattering like this, so there is this is say v 1 or rather u 1 c. lets And this becomes u 2 c and this is the so, this is the v 1 c and this is a v 2 c and this angle is theta. So, these are the in the CM frame, the center of mass frame, these are the initial velocities of the particles. So, 1 and 2 correspond to the 2 particles like mass as m 1 and m 2 and c corresponds to the CM frame. And in the CM frame the particles are the basically, the total momentum is 0, in the center of mass frame and the particles actually scatter like this, with the initial with the final velocities v 1 c and v 2 c and it makes an angle with respect to the incident direction, by an angle theta and the problem becomes reasonably simple and it can be solved in the CM frame.

So, of course, we are going to do quantum mechanical scattering theory. So, let us physically discuss that, what happens in a typical collision experiment.

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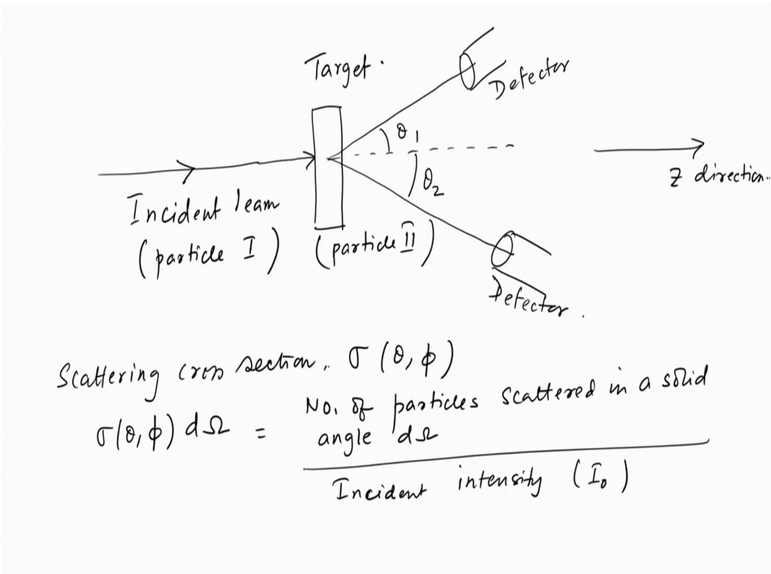

So, what happens is that, there is an incident beam that comes and this incident beam; so, this actually corresponds to the particle 1 that is, mass 1 if you want to say and then it goes and strikes a detect or rather a target ok. So, this target is actually the particle 2 and so this is called as a target and then it gets scattered at various angles and so, this is the incident direction and it can get scattered at this theta 1 and theta 2 and so on. And these are the positions of the detector ok.

So and let us let us, just for convenience call this as a z direction, as we will use this direction right it here, all right. So, this is a typical scattering experiment, such as an alpha particle is being bombarded at the nucleus or say an atom is being bombarded, with some particle and so, the atom becomes the target and the incident particle is the particle, with which it is bombarded and then there are detectors, which could be kept at discrete angles theta 1 and theta 2 and so on or there could be a detector, which is placed at all solid angles and then, it is able to detect the scattered particles within angle between an angle theta and theta plus d theta.

So, the scattering cross section is defined in this case as, we will call it as, sigma theta phi because, in general there will be dependence on the theta and phi angle as well. So, sigma theta phi in a solid angle d omega, so which is given equal to the number of particles scattered, in a solid angle divided by the incident intensity, which usually is denoted by i 0. So, that is the definition of this the scattering cross section, sigma theta phi or it is called as a differential scattering cross section.

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Total cross Section

$$\sigma = \int\int \sigma(\theta,\phi)\, d\Omega = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi\ \sigma(\theta,\phi)$$

for a spherically symmetric potential,

$$\boxed{\sigma = 2\pi \int_0^{\pi} \sin\theta\, d\theta\, \sigma(\theta)}$$

And a total cross section is obtained from this differential cross section by integrating over all theta and phi. So, that is equal to sigma and so, this is equal to a sigma theta phi d omega and this can be written as d theta, sin theta from 0 to pi and d phi 0 to 2 pi and a sigma of theta phi

Now, we will see that, the sigma or that scattering cross sections may not depend upon phi, for spherically symmetric potential. So, most of these potentials are spherically symmetric, so in most of the cases sigma is only a function of theta. And so, that can be written as so for a spherically symmetric potential, sigma equal to 2 pi 0 to pi sine theta d theta and a sigma of theta ok. So, that is the total scattering cross section and we need to calculate this ok.

So, let us just start with the quantum theory of scattering.

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Quantum Theory of scattering.

$t \to -\infty$ incident particle traveling as a free particle.

$t = 0$ Scattering occurs

$t \to +\infty$ scattered particle detected by the detector.

Properties.

(i) Elastic scattering

(ii) We shall not bother about the internal structure of the particle.

(iii) $V(\vec{r}_1, \vec{r}_2) = V(\vec{r}_1 - \vec{r}_2)$.

(iv) $\mu = \frac{m_1 m_2}{m_1 + m_2}$

So, strictly speaking we need a time dependent description of the problem because, the collision event happens as a function of time. So, before collision, there is a time slot, so we can talk about before, times before collision and after collision. So, the wave function or the wave packet, which is a collection of waves, that represents the incident particle as t going to minus infinity, that behaves that, it has passed through, a target and the wave packet corresponding to the final products as, t goes to plus infinity.

So, we have t going to minus infinity, when it is only the incident particle, which is travelling as a free wave, travelling as a free particle maybe and say at t equal to 0, scattering occurs and at t equal to plus infinity that is, at very large times the incident or the scattered particle, rather on the incident particle or maybe a scattered particle, which may include the incident particle, one of the incident particles as well. The scattered particle detected by the detector ok.

So, there are a few conditions that, the scattering would involve. So, these conditions of or rather these properties that, we are going to be particularly interested in is that, we are going to talk primarily about elastic scattering. Just to remind you the elastic scattering the where, the energy is conserved it is called as an elastic scattering; momentum is always conserved both in elastic and inelastic scattering. In inelastic scattering, energy is not conserved and it gets dissipated in the form of maybe heat or light or something ok.

And second is that, we shall not bother or rather take into account about the internal structure of the particle. The scattering potential is has translational invariance. So, v r 1, r 2, that is a 2 body potential, actually depends only on the relative coordinates and that is about it. So, these are some of the properties or rather the constraints of scattering and then of course, we can also talk about the effective mass, which is the which is given by mu, which is m 1, m 2 divided by m 1 plus m 2.

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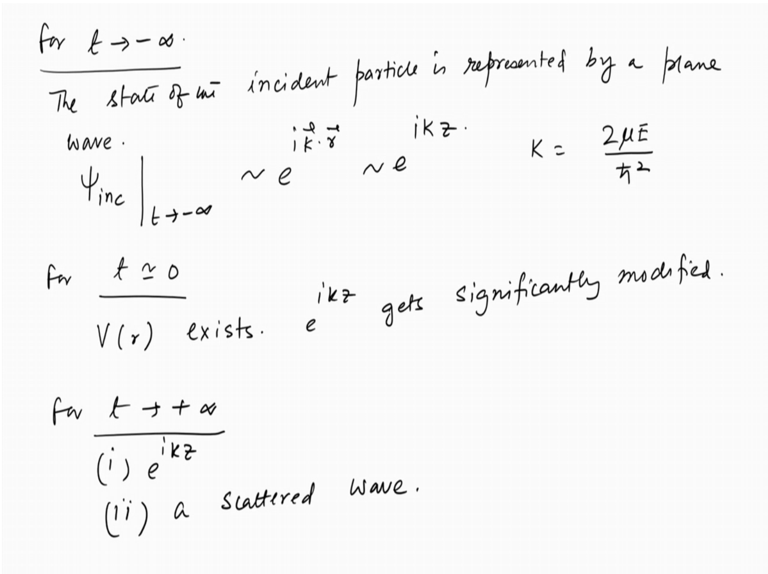

Now, for p going to minus infinity, we can assume that, the potential is equal to 0, so, the particle that incidents on the scatter is free and it is sufficiently far away from the scatter and the state of the incident particle is represented by a plane wave. So, by a plane wave, or a wave packet a plane wave packet and so, we are thinking of that. The psi incident which is at t going to minus infinity is like, exponential i k dot r and since the incident direction was taken as a z direction in the last schematic diagram that, we have shown this can be written as exponential i k z where, k is given by 2 mu e by h cross square mu being the effective mass of the 2 particles.

So, after that of course, for when the scattering occurs and I am just thinking that, that is roughly at the time in the middle of minus infinity to plus infinity. So, at the time when the scattering occurs, then the particle reaches a region where there is a Vr exists. And this exponential i k z gets significantly modified ok and how it gets modified? It is very difficult to figure out because, then 1 has to exactly solve the Schrodinger equation corresponding to a given potential.

Now, the problem with solving that is that, the Vr could be essentially very complicated, in which case and a closed solution of the Schrodinger equation may not exist. But, however, that does not impede us to write down that, for t going to plus infinity that, is the particle has gotten scattered from the target and have reached the detector. It consists of 2 things one is that the plane wave, which of course, it could still remain is a plane wave and not get scattered at all by the detector so and number 2, a scattered wave.

And of course, the first one we have discussed, the important thing is the second one and we need to be convinced of the form of the scattered wave. Because the rest of the analysis is going to crucially depend on, what is the form of the scattered wave that reaches the direct detector as, t goes to plus infinity.

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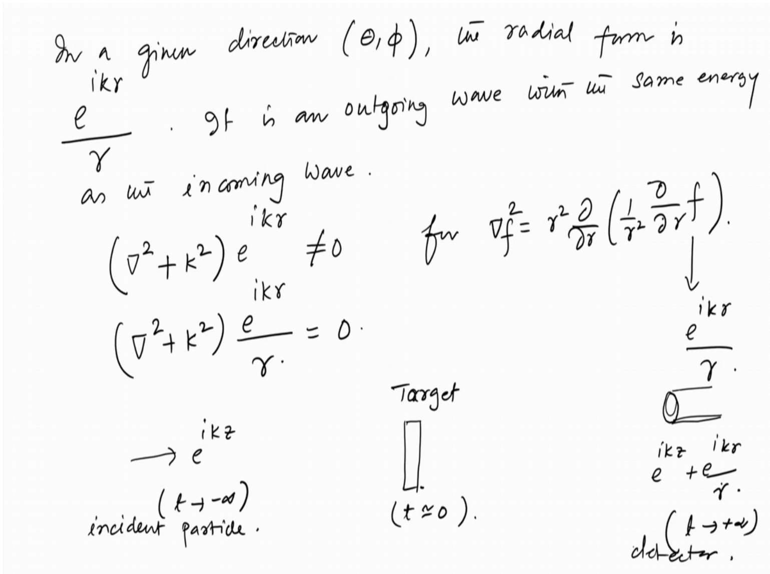

So, in a given direction, theta phi, the radial form is something like exponential i k r by r, so this is called as a spherical wave. So, it is basically, it is an outgoing wave, with the same energy as, the incoming wave. You might then think that, what is this 1 over r in the denominator, as it is shown here, this r in the denominator, where does it come from

Now, it is important to see that, in special, I mean three dimensions rather, 3 special dimensions this quantity which is, the Schrodinger equation, plus k square exponential i k r is not equal to 0, for del square to be of the form r square, del r of 1 over r square, del del r del del r ok.

However this is equal to 0, which you should be able to see very trivially if you simply put here so, we will write del square f and this is f, so if you put exponential i k r by r then, this is equal to 0 ok. So, which means that, the Schrodinger equation is satisfied and this k square is of course, the energy and we are talking about the free particle thing. So, this is the reason, for having a scattered wave packet to be of the spherical form, which is exponential i k r by r.

Now, just to summarize what happened so far is that, a particle of an incident particle which, comes as exponential i k z and so, this is at far away from the detector I mean far away from the target So, this happens at t going to minus infinity, the scattering happens at say t equal to zero and then, the detector detects this wave. So, if this detector detects both free wave plus a spherical wave of the form this, so it is a basically, a sinusoidal function or an oscillatory function with, an envelope which goes as one over r. So, that happens at t goes to plus infinity ok. So, that is being detected by the detector, so this is the target and this is the detector here, let us write it here, and this is the incident particle.

So, we are not bringing in explicit time dependence of these wave functions or rather not we are not trying to evolve them with time, but in our mind we are clear that, far away from the target the incident particle is free and propagates like a free wave. It is like exponential i k dot r because, now the direction of propagation is taken as the z axis.

So, that free wave is like exponential i k z then, it strikes a target and we are unable to solve Schrodinger equation or rather find out any information because, of the complexity of VR, what happens at t equal to 0 and we disregard that fact, but however, we know that when it is detected by the detector at t, goes to plus infinity then, the detector has it has 2 components or rather 2 parts, where one of them is like exponential i k z and the other is exponential i k r by r So, these are the two.

Now, what happens is that, if we simply take this two terms then, we are not having any information about, what scattering takes place there in the target or rather this as if, the target is not there and we simply have a free particle propagating and then, free particle plus a spherical wave being detected at the detector. So, there has to be some information of the scattering that, had taken place in the target and at least that is what, we are interested in.

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$$\psi_k(\vec{r}) \xrightarrow[r\to\infty]{} e^{ikz} + f_k(\theta,\phi)\frac{e^{ikr}}{r}$$

$f_k(\theta,\phi)$ is scattering amplitude.

for spherically symmetric potential $V(\vec{r}) = V(r,\theta)$

$$\psi_k(\vec{r}) \xrightarrow[r\to\infty]{=} e^{ikz} + f_k(\theta)\frac{e^{ikr}}{r}.$$

$$\boxed{\text{Objective : Calculate } f_k(\theta)}$$

obtain $\sigma(\theta) = |f_k(\theta)|^2$

So, in order to bring that, we take the wave function as r goes to infinity ok. So, r goes to infinity, it is a exponential i k z plus a fk theta phi exponential i k r by r where, f k theta phi is called as the scattering amplitude and for spherically symmetric potential, that is V of r simply depending upon V r theta the in which case, we have or rather it is V of it could be simply V of r.

We can simply write this psi k of r as, r goes to infinity its exponential i k z which means, there is an there is an equality, but as r goes to infinity. So, this is i k z plus f k theta and exponential i k r by r. So, that i s the that is the wave that, is detected by the detector and the information about the scattering that had taken place at the scatter of the target, is embedded in this fk of theta, which is called as a scattering amplitude. And now our main objective is to calculate f k of theta ok.

So, in order to calculate, so let us just write the objective so far, now what is fk theta or how is it related to the scattering cross section. Basically the scattering cross section is obtained from fk theta by taking a mod square, which in general is complex quantity. So, we can write that as well. Obtain sigma of theta, which is the differential scattering cross section is fk theta mod square, that is that amplitude square and so, we will have to calculate that. But before that, we need to in order to calculate this, we need to understand a few things and a study of greens function is very important in this regard.

So, let us see how greens function comes in. This we are going to give you a simplified description that is we have to solve an equation.

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Green's function. $\qquad V(\vec{r}) = V(r)$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi(\vec{r}) = E\psi(\vec{r})$$

$$\left[\nabla^2 + k^2 - U(\vec{r})\right]\psi(\vec{r}) = 0.$$

$$E = \frac{\hbar^2 k^2}{2\mu} \qquad V(\vec{r}) = \frac{\hbar^2}{2\mu}U(\vec{r}) = \frac{\hbar^2}{2\mu}U(r) \qquad (1)$$

$$(\nabla^2 + k^2)\psi(\vec{r}) = U(\vec{r})\psi(\vec{r})$$

$$(\nabla^2 + k^2)G(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,') \qquad (2)$$

$G(\vec{r}, \vec{r}\,')$: Green's fn.

$\delta(\vec{r} - \vec{r}\,')$ = Dirac delta fn.

$G(\vec{r}, \vec{r}\,')$ is Green's function of $(\nabla^2 + k^2)$

So, this is a greens function, we are just going to introduce it, not speak too much about it, just say that, how is it relevant for studying scattering theory.

So, we have to solve an equation, which is like this. Del square plus V of r, which is the we can simply take that, V of r to be equal to V of r it is even independent of theta and this psi of r, this is equal to E psi of r and, so this is equal to del square plus k square minus U of r and this psi of r equal to 0. So, where E is equal to h cross square k square over 2 mu and V of r is nothing, but h cross square over 2 mu and U of r ok, or U of r is 2 mu V r by h cross square. In which case we can write down this equation as, the same equation, we are writing it in terms of this u. So, it is Laplacian plus k square psi of r that is equal to u of r psi of r and of course, U of r is nothing, but so, this is this and this is nothing, but h cross square by 2 mu U of r.

Now, let us define a Green's function by this equation. So, as if greens function is a solution of this equation. You can look it up in any mathematical physics book especially (Refer Time: 30:21) gives a very good introduction to this, the greens function. So, greens function is a solution of this equation, which is Laplacian plus k square and acting on G rr prime, which is the greens function and that yields a delta function. So, G r r prime is a greens function and delta r minus r prime is the Dirac delta function ok. So, G r r prime is greens function of the operator this del square plus k square all right.

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We make an ansatz:

$$\psi(\vec{r}) = \psi_0(\vec{r}) + \int d^3r'\, G(\vec{r},\vec{r}\,')\, U(\vec{r}\,')\, \psi(\vec{r}\,') \qquad (3)$$

Also,

$$(\nabla^2 + k^2)\psi_0(\vec{r}) = 0. \qquad (4)$$

$$(\nabla^2+k^2)\psi(\vec{r}) = \underbrace{(\nabla^2+k^2)\psi_0(\vec{r})}_{=0} + \int \underbrace{(\nabla^2+k^2)\, G(\vec{r},\vec{r}\,')}_{\delta(\vec{r}-\vec{r}\,')}\, U(\vec{r}\,')\, \psi(\vec{r}\,')\, d^3r'$$

$$(\nabla^2+k^2)\psi(\vec{r}) = \int \delta(\vec{r}-\vec{r}\,')\, U(\vec{r}\,')\, \psi(\vec{r}\,')\, d^3r'$$

$$= U(\vec{r})\psi(\vec{r}) \quad \rightarrow \text{which is Eq.(1).}$$

Now we propose or make an answers psi of r equal to psi 0 of r plus d cube r prime G r minus r prime U r prime psi of r prime, is the solution of this equation, which we call it as equation 1. Let us call this as equation 1 and let us call this one as equation 2, this one as equation 2 and this one as equation 3, also make this additional requirement that del square plus k square acting on psi 0 of r should be equal to 0. And one can actually see that, the this 3 that is, equation number 3, which is here, which is here, this is a solution of 1, that is the Schrodinger equation, that we have written now all right.

So, after that, we will write down the del square plus k square psi of r that is, we need to check that this is equal to del square plus k square, so we operate del square plus k square on both sides, by this size 0 r. Now we are going to show that, this is 3 is indeed a solution of 1 and plus del square plus k square G of r. So, this can be you know, I mean instead of either you can write it as, r minus r prime or rr prime either of them is ok. So, this is as we wrote r r prime, so this is and then there is a U of r prime and then psi of r prime and d cube r prime. So, that is the volume integral over r prime.

Now, this is equal to 0 by the condition above that is equation 4. And this is the definition of greens function, which gives r minus r prime So, that tells you that this plus this psi of r it is equal to a delta of r minus r prime U r prime psi r prime d cube r prime, now that of course, is nothing, but U r psi r, which is equation 1. So, that means, 3 is indeed a solution for the Schrodinger equation ok. So, that way we are convinced that, this is the solution for that.

Now, let us also try to understand, what the greens function for this particular case is.

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$$(\nabla^2+k^2)\, G(\vec{r},\vec{r}\,') = \delta(\vec{r}-\vec{r}\,')$$

$$G(\vec{r}) \simeq \frac{e^{ikr}}{r}.$$

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(\vec{r})$$

$$(\nabla^2+k^2)\,\frac{e^{\pm ikr}}{r} = -4\pi\delta(\vec{r}) \rightarrow \text{To show}$$

$$\nabla^2\left(\frac{e^{\pm ikr}}{r}\right) \quad \text{to calculate.}$$

$$\vec{\nabla}\cdot\nabla(fg) = \vec{\nabla}\cdot\left[(\nabla f)\,g + f\,(\nabla g)\right]$$
$$= g\nabla^2 f + f\nabla^2 g + 2(\nabla f)\cdot(\nabla g)$$

So, since the equation defining greens function is given by del square plus K square G of r r prime, it is equal to delta r minus r prime, we a priori write down this is a purely from experience that, one can write down the G of r is something like exponential i k r by r ok. And remember this relation, that you might have learnt in your electrodynamics course is that, Laplacian of 1 over r is 0 unless it encloses the origin, in which case it blows up. So, it is minus 4 pi delta r ok.

So, this can easily be seen that a del square plus a K square exponential plus minus i k r by r, so, that is equal to a minus 4 pi delta r, this is to show because then we will be able to show that. So del square of exponential plus minus we the plus and minus both our spherical waves. It is just that, we decide to take a plus because, it is a forward moving wave, exponential minus ikr by r would be a backward moving wave. This since there is nothing to reflect back, after the target after the incident particle has crossed the target, we drop this exponential minus i k r, but mathematically this would be there. So, this has to be computed or to calculate this. And this is nothing, but equal to this is like exponential. So, this is like this.

So, this and f g there are 2 scalar functions. So, there 2 scalar functions exponential i kr and 1 over r. So, this is equal to a divergence of a gradient f into g plus f gradient of g and so on. So, this is equal to a g Laplacian of f plus f Laplacian of g plus 2 gradient f into gradient g.

So, we will apply this to this two functions exponential i k r and r.

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$$\nabla^2\left(\frac{e^{\pm ikr}}{r}\right) = \frac{1}{r}\nabla^2\left(e^{ikr}\right) + e^{ikr}\,\nabla^2\left(\frac{1}{r}\right) + 2\vec{\nabla}\left(\frac{1}{r}\right)\cdot\vec{\nabla}\left(e^{\pm ikr}\right)$$

$$\left(\nabla^2+k^2\right)\frac{e^{\pm ikr}}{r} = -4\pi\delta(r).$$

$$\boxed{\psi_k^+(r) = e^{ikz} + \int d^3r'\, G_+(\vec{r}-\vec{r}')\, U(\vec{r}')\, \psi^+(\vec{r}')}$$

So, a del square exponential plus minus i k r by r equal to 1 over r del square exponential i k r plus exponential i k r r del square 1 over r and a plus 2, 1 over r and this 1 exponential i k r or plus minus i k r and so on ok. So, this is these are the terms. So, if you simplify then, it becomes equal t, so del square plus k square exponential plus minus i k r by r equal to minus 4 pi delta of r. That tells us that we can write down a psi k. Now we will take the plus sign in the in the spherical wave, so we will write it just like a psi k of r, which is a exponential i k z. We could have written it a vector r, but now it depends on scalar r plus a d cube r prime and G plus r minus r prime or rr prime, does not matter, it is this and then it is a psi r prime and so on.

So, that is the answers. See it is a little tricky here because; well I need to put a plus as well here, so this corresponds to the plus sign in this spherical wave. See we are trying to calculate psi k plus r, which is the wave that, reaches the detector as r goes to infinity, which means at t equal to infinity. That is there in the left hand side, as well as that it is there in the right hand side as well. So, one is automatically heading towards an iterative solution. A solution that is self consistent and it has to be iterated upon.

Now, in order to get that, we are left with no choice, but to make an approximation. And this is called as the born approximation.

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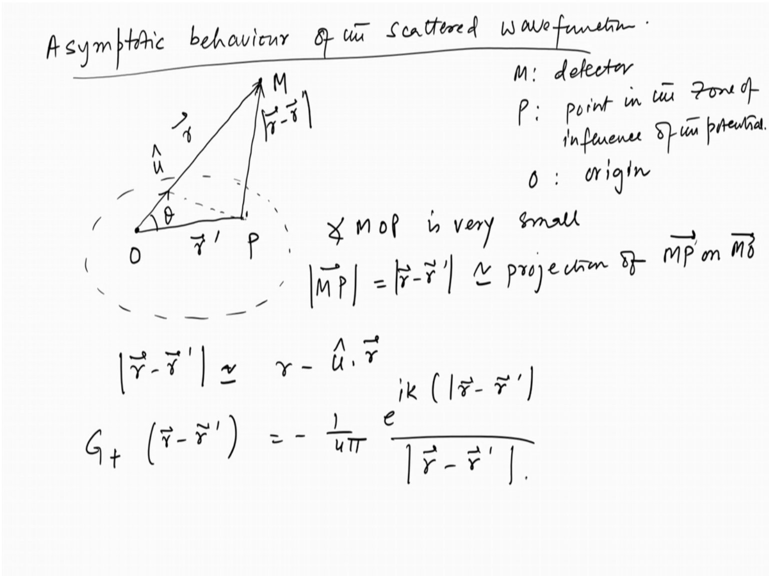

So, before we do born approximation, let us simplify this picture even more and try to understand the asymptotic behaviour of the scattering function, of the scattered wave function

So, there is a point O and there is a point say P, and there is a point M where, the detector is. So, P is a point within the you know, where the potential exists or within the influence of the potential. So, let us call this as r prime and this is r ok. And M is where the detector is, so let us just make these things clear, M is the detector and P is a point in the zone of influence of the potential ok and so of course, this is equal to r minus r prime and the just interested in writing the modulus of that. And let us talk about this direction being u cap and this angle being theta.

So, M is as you understand and O is of course, the origin. M is very far away because, that that is the position of the detector such that, this angle MOP is very small. And because of which we could say that this MP, that is the magnitude of this r minus r prime which is, r minus r prime that is, equal to projection of almost equal to projection of MP on MO ok.

So, r minus r prime, that is equal to r minus u r. So, that is r and minus this thing that, that is that projection, which is u dot r. So, the G plus, corresponding to the plus sign of the spherical wave is r minus r prime, that is equal to minus 1 over 4 pi, exponential i k r minus r prime and divided by r minus r prime this.

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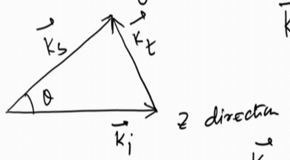
for $r \to \infty$

$$G_+(\vec{r}-\vec{r}\,') \simeq -\frac{1}{4\pi}\frac{e^{ikr}}{r}\, e^{-ik\hat{u}\cdot\vec{r}}$$

$$\psi_k^+(\vec{r}) \underset{r\to\infty}{\xrightarrow{\sim}} e^{ikz} - \frac{1}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{u}\cdot\vec{r}\,'}\, U(\vec{r}\,')\,\psi_k^+(\vec{r}\,')$$

$$f_k(\theta) = -\frac{1}{4\pi}\int d^3r'\, e^{-ik\hat{u}\cdot\vec{r}\,'}\, U(\vec{r}\,')\,\psi_k^+(\vec{r}\,')$$



$\vec{k}_s = k\hat{u}$

$|\vec{k}_i| = k$

$\vec{k}_t$ = transferred wave vector.
$= \vec{k}_s - \vec{k}_i = \vec{K}$

So, for r going to infinity, G plus r minus r prime it is equal to 1 by 4 pi with a minus sign exponential i k r by r. And now I am writing this r minus r prime as, r minus u cap dot r. So, that is r minus u cap dot r u dot r. So the answers for the wave, that reaches the detector. So, this is for r going to infinity is exponential i k z, which was already there and minus 1 over 4 pi exponential i k r by r. Now there is an integral over this and this i k u cap dot r prime u r prime and psi k plus r prime and so on.

So, we have been able to plug in the greens function that, we have obtained into this expression. If you compare this, with this expression, where there is a forward moving free wave and then, there is a exponential i k r by r and then, there is a f k theta, if you look at that and compare with the 1 that, we have just obtained. So, with this, then this fk theta phi simply becomes equal to this entire integral with this minus 1 by 4 pi factor. And this is my minus 1 by 4 pi and d cube r prime exponential minus i k u dot r prime u of r prime psi k plus r prime and so on.

Just a little more simplification in the given present context, if you look at the incident wave vector is in the z direction, this is what was assumed right at the beginning. And there is a scattered wave vector which is in this direction, where the direction in which the detector is placed. And this is the kt which is the transferred wave vector and this is of course, that angle theta. Then k s is actually k u cap because, we have shown that the u cap is in the direction of the detector. So, it is k u cap and of course, your k i the magnitude of that is same as k s and which is equal to k, because of the elastic scattering.

So, k t here is the transferred wave vector and that is equal to k s minus k i equal to k ok. So, that is the k, that we are talking about here in this f k theta that is the transferred wave vector. And now we are almost all set and going to talk about that approximation that, we need to undertake in order to solve this problem that is the problem the problem of finding the wave function or rather finding the scattering amplitude and hence the scattering cross section.

(Refer Slide Time: 47:58)

Born approximation.

$$\psi_k^{+}(\vec{r}) = e^{i\vec{k}_i\cdot\vec{r}} + \int d^3r'\, G(\vec{r}-\vec{r}\,')\, U(\vec{r}\,')\, \psi_k^{+}(\vec{r}\,')$$

change in notation.

$$\vec{r} - \vec{r}\,' \; ; \; \vec{r}\,' \to \vec{r}\,''$$

$$\psi_k^{+}(\vec{r}\,') = e^{i\vec{k}_i\cdot\vec{r}\,'} + \int d^3r''\, G(\vec{r}\,'-\vec{r}\,'')\, U(\vec{r}\,'')\, \psi_k^{+}(\vec{r}\,'')$$

$$\psi_k^{+}(\vec{r}) = e^{i\vec{k}_i\cdot\vec{r}} + \int d^3r'\, G_+(\vec{r}-\vec{r}\,')\, U(\vec{r}\,')\, e^{i\vec{k}_i\cdot\vec{r}_i\,'} + \int d^3r' \int d^3r''\, G_+(\vec{r}-\vec{r}\,')\, U(\vec{r}\,')\, G_+(\vec{r}\,'-\vec{r}\,'')\, U(\vec{r}\,'')\, \psi_k^{+}(\vec{r}\,'')$$

So, the Born approximation, so a psi k plus r equal to exponential i k i dot r plus, d cube r prime G r minus r prime. I am once again writing this r prime psi k dagger, not dagger this is plus, I am sorry, so this plus and this so, this was the original answers that we had. So, a simple change in notation, change r to r prime and r prime to r double prime, in which case psi k plus r prime it is equal to i k i dot r prime plus d cube r double prime G r prime minus r double prime u r double prime psi k plus r double prime and so on.

Now, you see I have got the psi k psi k plus r prime, which I can put it in the first. So this one can be actually replaced here or rather can be plugged in here and we get a equation, which is a psi k plus r equal to exponential. Now I can if you seen that, I have started writing the k i dot r instead of k z, but; that means, the same thing. So, it is i k i dot r so, that is the part of the incident wave itself and this is prime and this is a G plus r minus r prime U r prime exponential i k i dot r i prime plus a d q r prime d q r double prime and G plus r minus r r prime u r prime G plus r prime minus r double prime u r double prime and psi k r double prime and so on.

So, you see that if you compare between the first line on the slide that is, the last term that is here and the 2 terms or rather three terms that we have written here. This one in the first approximation has been replaced as the plane wave. And there is a second term and each term comes with an additional factor of U. See this is a function of or rather depends only on one power of U, this depends on u and u prime or rather two powers of U U and U r prime and U r double prime. And a sec at third approximation or third iteration we will get in another U and so, on

So, let us consider that, U to be small and drop all terms that are quadratic in u and onwards. So, in which case we only keep the first 2 terms in this expression and that is called as born approximation.

(Refer Slide Time: 52:20)

Using Born approximation implies.

$$f_k^{B}(\theta) = -\frac{1}{4\pi}\int d^3r'\, e^{-i\underbrace{k\hat{u}}_{\vec{k}_s}\cdot\vec{r}\,'}\, U(\vec{r}\,')\, e^{i\vec{k}_i\cdot\vec{r}\,'}.$$

$$= -\frac{1}{4\pi}\int d^3r'\, e^{-i(\vec{k}_s-\vec{k}_i)\cdot\vec{r}\,'}\, U(\vec{r}\,').$$

$$= -\frac{1}{4\pi}\int d^3r'\, e^{-i\vec{k}_t\cdot\vec{r}\,'}\, U(\vec{r}\,').$$

k_t : transferred wave vector.

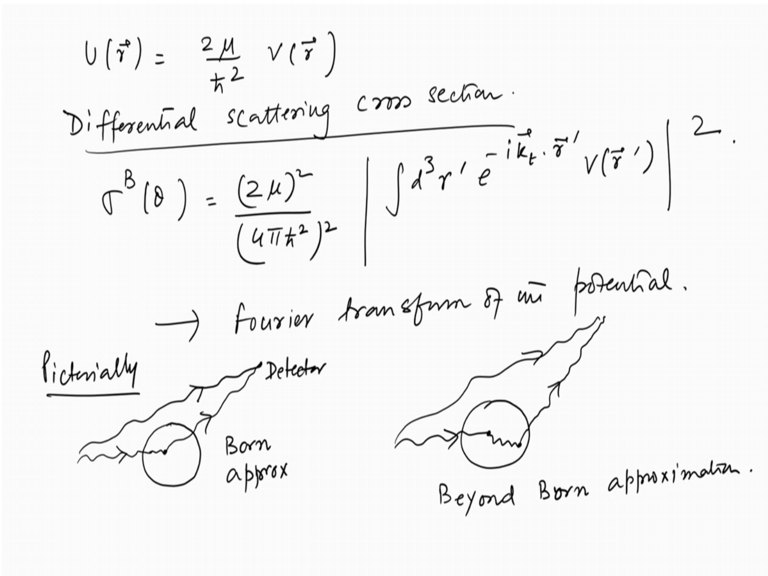
$$\sigma^{(B)}(\theta) = |f_k^{(B)}(\theta)|^2$$

So, just one power of that and so, using born approximation implies that, we stop at the first order. And in which case, we can extract out the scattering amplitude from this second term and then, f k theta and in the born approximation which so, we write a p there in the superscript and this and an exponential minus k u dot r prime, U r prime exponential i k i dot r prime and so on.

So, that now a little bit of adjustments in notation it is a d cube r prime. And because this is k s this term is k s. So, then we can write it as minus i k s minus k i dot r prime U r prime and so on. So, this is equal to minus 1 over 4 pi d cube r prime exponential minus k t dot r prime u of r prime. So, k t is the transferred wave vector, we have already told.

So, sigma the scattering cross section is, as we have told, it is the mod connected sorry. So, this is that B theta mod square so, what comes is that, it is simply the scattering amplitude is simply, the fourier transform of the potential. So, once we know the potential, we can do the fourier transform, that should give us immediately the scattering amplitude and from the scattering amplitude, one can easily calculate the scattering cross section or the differential scattering cross section.

(Refer Slide Time: 54:51)



And given that our U of r equal to 2 mu by h cross square V of r sigma B, that is a scattering cross section, So let me just write that, just to remind you once more differential scattering cross section.

So, this is equal to sigma B theta and so, this is equal to 2 mu square by 4 pi h cross square whole square and there is a I mean, there is a mod of d cube r prime exponential minus k t r prime V r prime mod square and so on. So, this is equal to so, it is basically, the scattering or so, the scattering cross section is a fourier transform of the potential.

So, the potential, that exists at the target, that is influencing the incident particle. So, the physical meaning of the greens function is that that, it represents the amplitude at a point r of a wave that is radiated by a point source situated at r prime. So, pictorially. if you want to understand what born approximation mean is that, so, there is an incident so, here and so, there is an incident particle it gets into the target and there is a, it interacts with the point in the target and here is the detector, so it goes and reaches the detector, carrying the information about the scattering and there is also another free wave that, reaches the detector, so, this is that detector. So, this is Born approximation and what we are neglecting is that, there are multiple scattering events that, could take place in the so, there is one level more than the born approximation is that, there is this and then this reaches the detector and there is another one reaches the detector ok.

So, there are 2 scattering events and there are multiple scattering events, which are beyond Born approximation. So, this is beyond Born approximation and this is quadratic in the potential ok. So, Born approximation finally, gives a very important and a very simple result that, the scattering cross section is simply equal to the mod square of the fourier transform of the potential. So, if the potential is supplied, suppose the potential is coulomb like and I mean, you have to consider a screen coulomb for the reason, that you would understand when you try to do that problem.

Then it is simply term, which is you have to calculate the fourier transform of the potential and that is about it multiply this by the 2 mu square, mu whole square and 4 pi h cross square and all that. And that is most of the things, that are needed for scattering theory, the quantum scattering theory in the born approximation. Of course, one can go beyond the born approximation; however, if the potential is weak then those terms which are square in potential and cubic in potential etcetera etcetera, they will start contributing lesser and lesser ok.

So, that pretty much completes the approximate methods in quantum mechanics, just a very brief recap, we have done the perturbation theory, the time independent perturbation theory, where we have talked about the degenerate and non degenerate perturbation theory and then, we have talked about the variational principle, where a variational state is chosen, which when minimized the energy being minimized with respect to that tunable variational parameter, one gets an upper bound to the ground state energy.

And then, we have also done WKB approximation, which is for a slowly varying potential. We have learned how to compute the connection formula of the you know, the solution connecting the two sides. Of course, these solutions are only valid asymptotically there is far away from a turning points and from the connection formula one could actually calculate, m what is called as the Bohr Sommerfeld quantization condition.

And then, we have also done the time dependent perturbation theory, where we have calculated transition between different levels or a different states, the transition probabilities and in case, the final state falls into a continuum of states, one needs to invoke the density of states for the final states and then, use the Fermis golden rule in order to calculate the transition probability.

And here, we have learned how to deal with scattering problem in 3 D, where the an incident particle comes from minus infinity, where it is far away from the range of the potential, behaves like a just like a free wave and then it reaches the detector also again at far away locations from the position of the scatter or the target. And there one gets actually a spherical wave, but in addition to that one also gets a factor which is called as a, scattering amplitude associated with the spherical wave such that, the spherical wave has the knowledge of the scattering that it has the incident particle has gone through in the scatter. And this, there is a way, that we have shown to calculate this scattering amplitude and hence the scattering cross section within an approximation called as born approximation.

In this context, we have also included the definition of greens function and for this particular problem shown what the greens function can be it is actually nothing, but the spherical wave that we have been talking about. So, using that and an iterative solution; however, terminating the iteration at the first order, one can get the expression for the scattering cross section, which is nothing, but this fourier transform of the or rather it is a mod square of the fourier transform the potential.