Advanced Quantum Mechanics with Applications

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Lecture - 32
Examples

So, we are going to look at some of the tutorial problems on this Advanced Condensed Matter Physics with some Applications that we have done so far.

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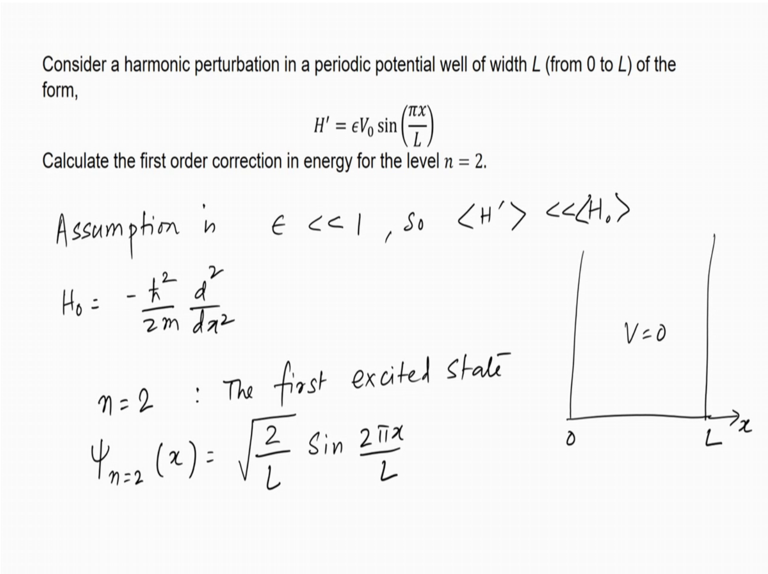

So, this problem is taken from the time independent perturbation theory, where it is said that consider a harmonic perturbation in a periodic potential well of width L it is it extends from 0 to L, it has a form and the word harmonic means it is a sin or cosine which is periodic. So, the perturbation is given to be of the form H prime equal to epsilon V 0 sin pi x by L. And calculate the correction in energy for the level n equal to 2 which is the first excited state.

Now, it is worth mentioning here that the assumption is that epsilon to be much smaller than 1, such that at least of the order of 10 to the power minus 3. So that it is lower than or H prime is much smaller than H 0 which is the unperturbed problem or Hamiltonian for this particular problem where H 0 is nothing, but minus h square by 2 m d 2, dx 2 because there is no potential part for the reason that we are talking about a particle inside a potential well which the potential well itself has. So, V is equal to 0 inside the well and it extends from 0 to L and this is the x direction that is given.

So, n equal to 2 is the first excited state, and by saying that we should be able to understand that it has a form which is root over 2 by L sine of 2 pi x over L, that is the form of the first excited state eigen function for this particular problem. Now, we need to calculate the first order energy correction.

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$$E^{(1)}_{n=2} = \langle \psi_2^{(0)} | H' | \psi_2^{(0)} \rangle$$

$$= \frac{2\epsilon V_0}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right) \sin^2\left(\frac{2\pi x}{L}\right) dx$$

$$= \frac{2\epsilon V_0}{L} \times \frac{16L}{15\pi} = \frac{32\epsilon V_0}{15\pi}$$

$$E^{(0)}_{n=2} = \frac{4\pi^2\hbar^2}{2mL^2}$$

$$E_{n=2} = E^{(0)}_{n=2} + E^{(1)}_{n=2} = \frac{4\pi^2\hbar^2}{2mL^2} + \frac{32\epsilon V_0}{15\pi}$$

Let us write it as E n 1. Now, this n is equal to 2 and this is given by the formula that it is a psi 2 0. So, we can write because it is an unperturbed we can write a 0 here and so this is psi to 0 H prime psi to 0. So, this is the definition of the first order energy correction. The information that is the first order correction comes in this superscript inside the bracket 1 and because it pertains to n equal to 2 so that is written in the subscript. And this is the formula that the matrix element of H prime has to be evaluated between these unperturbed wave functions the first excited state wave function.

So, this is nothing, but 2 epsilon V naught divided by L and because the limit of integration or rather the particle extends between 0 to L. So, the integration will be from 0 to L. In fact, in principle is from minus infinity to plus infinity, but we know that for all region accepting 0 to L the wave function is identically equal to 0. So, this is what we write as the perturbation term and the two wave functions we will put together will give us this and then we do this.

So, this integral is simple. We should be able to solve it and the result is what I am writing. So, it is 2 epsilon V 0 over L that is the term that is there and the value of the integral comes out as 16 L by 15 pi and this is a 32 epsilon V 0. V 0 of course, has the dimension of energy in this case. And so this is the answer.

So, the energy the second eigenstate which has a value, so our E n equal to 2 0 has a value which is 4 pi square h cross square divided by 2 m L square. So, in presence of the perturbation will have the total energy for n equal to 2 which is equal to the unperturbed energy which is this one, that one that we have written just above and the one that we have computed and this would put together will be 4 pi h cross square by 2 m L square and plus 32 epsilon V naught by 15 pi. Of course, the first term has a of course, has a dimension of energy and the dimension of energy in the second term comes from V 0, so that is the total energy of the system in presence of this perturbation.

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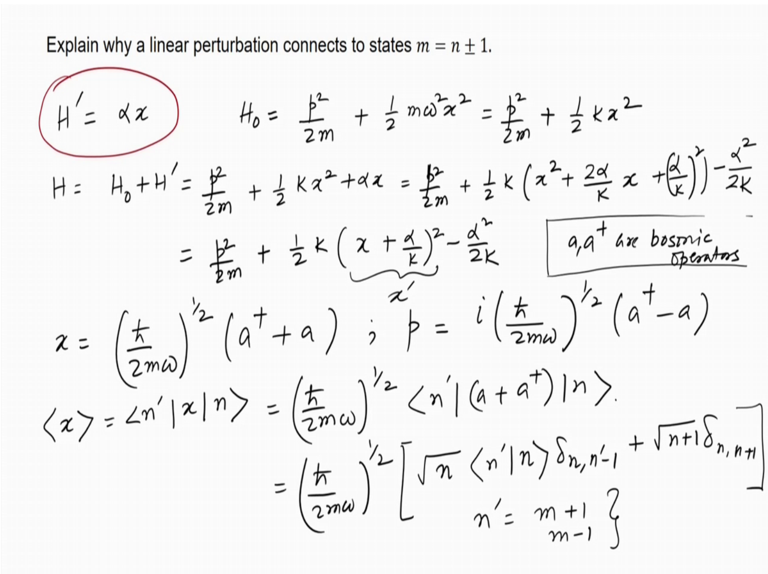

Let us look at another problem which is quite important and interesting in the context of harmonic oscillator that is the one usually asked the question that what is the change in the wave function or change in energy in presence of a linear perturbation for a simple harmonic oscillator.

Now, the problem may be may look quite simple and it has actually has exact solutions. In the reason that if you think of a perturbation of this time its alpha x then for harmonic oscillator which has a H 0 equal to p square over 2 m plus half m omega square x square which can also be written as p square over 2 m plus half k x square. Now, if I write down the total Hamiltonian equal to H 0 plus H prime I have a p squared over 2 m plus a half k x square plus alpha x.

Now, because the perturbation term where of course, the assumption is that the alpha is much smaller than 1 so that it can be treated as a perturbation. But nevertheless this has an exact solution in the form that I can take this its p square over 2 m plus half k I can take common, and it will be x square plus 2 alpha over k x plus alpha by k whole square, and now I have completed the square and now whatever extra I have taken has to be subtracted back. So, I have taken an alpha square by k square multiplied by a half k. So, it is something like a alpha square by so that is alpha square by k square multiplied by half k, so this is like a 2 k. So that has to be subtracted and then we, one can write it as a full square for the potential energy term half k and then one has a x plus alpha k square minus alpha square by 2 k.

Now, this can be defined as a new variable x prime and then it becomes again a harmonic oscillator. But however, here it starts what it asks is that that why a linear perturbation connect to states which are m equal to n plus 1 or n minus 1, and in order to answer that question let us write down the relationship between the x variable and the a and a dagger operators.

So, x is equal to h cross by 2 m omega whole to the power half a dagger plus a and of course, it also has a, i into h cross by 2 m omega whole to the power half a dagger minus a. So that is those are the definitions of x and p in terms of a and a dagger, and this a and a dagger are Bosonic operators. So, we will write it here. So, a, a dagger are Bosonic operators which means that they obey commutation relations and these commutation relations are such that x and p obey the commutation relation that is known which is x and p commutation is equal to ih cross. These are called as canonical transformations and so x which is the first order correction that is, this is equal to a n prime x n.

Now, this has to be evaluated in order to calculate the first order correction due to this term which is, let us circulate here. So, this is the perturbation and so in order to do that one can write this one down as h cross by 2 m omega to the power half, and now we have a n prime a plus a dagger and n and so on. So, this is very easy to see that apart from this factor one has, so when a dagger acts on n it gives rise to a state which is n plus

1 with a coefficient and a acting on n again gives rise to a state which is n minus 1 along with a coefficient and so these coefficients are simply root over n and I have a n prime and n and delta n prime or n equal to n prime minus 1. So, n, n prime minus 1 and also a root over n plus 1 and there is a n prime or rather again n plus 1 and so on.

So, that is the reason that a linear perturbation always connects 2 states which. So, n prime has to be either m plus 1 or m minus 1 for these things to be nonzero which is same as saying that m equal to n plus 1 or n minus 1, for these terms to be nonzero. And that is why the linear perturbation connects 2 states m equal to n plus, plus minus 1 for the harmonic oscillator, all right.

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Consider a perturbation of the form applied to a two-level system (a and b),

$$H' = U\delta(t)$$

Assume U is Hermitian and the diagonal elements of U between the unperturbed states of the two level system vanishes. Further assume the coefficients, $c_a(t)$ and $c_b(t)$ satisfy,

$$c_a(-\infty) = 1 \ \& \ c_b(-\infty) = 0$$

a

Find $c_a(t)$ and $c_b(t)$.

b

We can write down the Equation of motion for C_a & C_b are:

$$\dot{C}_a(t) = -\frac{i}{\hbar} H_{ab}(t) e^{-i\omega_0 t} C_b(t) \qquad (1)$$

$$\dot{C}_b(t) = -\frac{i}{\hbar} H_{ba}(t) e^{i\omega_0 t} C_a(t). \qquad (2)$$

$$H'_{aa} = H'_{bb} = 0 \qquad \omega_0 = \frac{(E_b - E_a)}{\hbar}$$

So, let us now look at a 2 level system. So, consider perturbation of the form applied to a 2 level system let us call those 2 level systems are a and b. So, this is a and this is b where the perturbation term we have written it as H, we can make it H prime in ongoing as per our ongoing notations. So, H prime is equal to U and then the time dependent part of the perturbation is a delta function. So, it acts only at t equal to 0 and it does not act at any other time. So, it is like a pulse that acts at t equal to 0 and then stops. So, it neither exists for t less than 0, nor exists at t greater than 0 and the amplitude of this perturbation is given by U.

Now, assume that U is Hermitian and the diagonal elements of you between the unperturbed states of the 2 level system which are these a and b, they vanish. So, just vanish, further assume that the coefficient C a t and C b t they satisfy the initial conditions given by C a at minus infinity equal to 1 and C b at minus infinity equal to 0.

Now, the question is that find these C a as a function of t and C b as the function of t. So, we can write down the equation of motion. So, the equation of motion are C a dot t which are coming from the time dependent Schrodinger equation which is minus i by h cross H a b t exponential minus i omega naught t and C b, and also a C b dot t which is equal to minus i by h cross and then the Hermitian conjugate of the above term. So, I can I should write it with the H b a t, and since it is a, it says that it is a Hermitian. So, these are going to be same and this is equal to i omega t C a. So, these are functions of t.

So, in addition to that as it is said that the diagonal elements between these states which are this and this are equal to 0, and of course, the h cross omega naught which is introduced here is the difference between the 2 levels E b by E a or one can actually write this omega 0 to be E b minus E a by h cross. So, these are the notations we clear. Let us be very clear about the notations that is this C a dot t is a equation of motion for the coefficient for level a, and that is written as in terms of C b of t, and C b dot t is written in terms of C a t and hence these are coupled equations which need to be solved, all right.

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$H'(t) = U\delta(t) \qquad \delta(t) = \frac{1}{2\epsilon} \qquad -\epsilon < t < \epsilon$

Temporal part $\qquad = 0 \qquad$ Otherwise.

Spatial part $\quad H'_{ab} = \alpha \,, \quad H'_{ba} = \alpha^*$

$$\dot{c}_a = -\frac{i\alpha}{2\epsilon\hbar} e^{-i\omega_0 t} c_b(t) \qquad (3)$$

$$\dot{c}_b = -\frac{i\alpha^*}{2\epsilon\hbar} e^{i\omega_0 t} c_a(t). \qquad (4)$$

Take time derivative of Eq.(4).

$$\ddot{c}_b(t) = -\frac{i\alpha^*}{2\epsilon\hbar}\left[i\omega_0 e^{i\omega_0 t} c_a(t) + e^{i\omega_0 t}\left(\frac{-i\alpha}{2\epsilon\hbar} e^{-i\omega_0 t} c_b(t) \right) \right]$$

Now, there is one thing that you should keep in mind is that the time part of the perturbation that is H prime of t is U delta t. Now, there are various definitions of delta t that can be used. Here of course, we use a rectangular approximation for delta t. So, delta t is equal to 1 by 2 epsilon for minus epsilon less than t less than epsilon, where epsilon is a small number about 0. So, this has as epsilon goes to 0 this becomes large and this is how the delta function is defined in this case and it is equal to 0 otherwise.

So, let the space part. So, this is the time temporal part, let us write that down the different colour, and let us write down the space part the special part rather. So, H prime a b equal to alpha, and C a dot it is equal to minus i alpha by 2 epsilon h cross exponential minus i omega naught t, C b, and the C b dot equal to minus i alpha star which is a complex conjugate of alpha. So, alpha is equal to H ab, H prime ab. So, H prime ba is equal to alpha star and so this is equal to exponential i omega t and C a. So, these are of course, functions of t.

So, in this interval one can so, let us write them down let us call them as equation 1 and equation 2, and let us call this as equation 3 and equation 4. Take time derivative in this interval of course, this interval means minus epsilon 2 plus epsilon take the time derivative of C b of a 4, equation 4. So, C b double dot t it is equal to minus i alpha star by 2 epsilon h cross and i omega naught exponential i omega naught t C a t plus exponential i omega naught t and then I put back where I do a C a dot in the last part of equation 4, I replace equation 3 which is a minus i alpha divided by 2 epsilon h cross exponential minus i omega t and a C b t and so on. So that is the C b double dot.

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$$\ddot{C}_b(t) = -\frac{i\alpha^*}{2\epsilon\hbar}\left[i\omega_0\left(\frac{2\epsilon\hbar i}{\alpha^*}\right)\dot{C}_b - \frac{i\alpha}{2\epsilon\hbar}C_b\right]$$

$$\ddot{C}_b(t) = i\omega_0\dot{C}_b(t) - \frac{|\alpha|^2}{(2\epsilon\hbar)^2}C_b(t) \qquad (5)$$

Eq(5) is a 2nd order, linear, homogeneous different equation for $C_b(t)$. Try out a solution. $C_b(t) = e^{\lambda t}$

$$\lambda^2 - i\omega_0\lambda + \frac{|\alpha|^2}{(2\epsilon\hbar)^2} = 0$$

$$\lambda = \frac{i\omega_0 \pm \sqrt{-\omega_0^2 + |\alpha|^2/(\epsilon\hbar)^2}}{2} = \lambda_{1,2}.$$

$$= -\frac{i\omega_0}{2} \pm \frac{i\beta}{2} \quad \text{where } \beta = \sqrt{\omega_0^2 - |\alpha|^2/(\epsilon\hbar)^2}$$

And one can actually make a simplification of that, so C b double dot t it is equal to minus i alpha star by 2 epsilon h cross and i omega naught. Now, one can actually replace in place of this C a t, one can replace in terms of the C b from equation 4 So, a little bit of manipulation, but this is all very simple and divided by alpha star and a C b dot minus i alpha by 2 epsilon h cross C b this is simplifying. And now this can be written as i omega C b dot minus alpha square 2 epsilon h cross square C b. So, we get a this is equal to C b double dot this is a function of t this is a function of t and this is a function of t. So, this equation if we call it as equation 5 so, equation 5 is a second order linear homogeneous differential equation for C b t.

And one can try out a solution, this is a C b of t which is equal to an exponential lambda t and if one tries out a solution like that one gets an equation in terms of lambda this is just put it into equation 4 and equation 5. And this is equal to plus alpha square divided by 2, epsilon h cross square equal to 0. So that is the equation which was in terms of C b double dot C b dot and consider.

Now, it is an equation for lambda a quadratic equation for lambda which can be solved by the usual route finding method. So, lambda is equal to minus b plus minus root over V square plus 4 a c which is equal to i omega naught and plus minus and then one has this plus alpha square by epsilon h square divided by 2. So, these are the 2 roots of this equation and this equation can be written as minus i omega naught by 2 and a plus minus i beta by 2, where beta is equal to this root over omega naught square minus alpha square by epsilon h cross square.

So, this omega naught just to remind you omega naught is the energy difference between the 2 levels, alpha is the space part of the perturbation term or the matrix element of the perturbation term and of course, epsilon comes from the definition of the delta function which is very important. So, if you see a problem in which there is a delta function type perturbation either in a 2 level system or otherwise one should be able to find out a definition and here this particular definition has been used.

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$$C_b(t) = e^{i\omega_0 t}\left(A e^{i\beta t/2} + B e^{-i\beta t/2}\right). \qquad (6)$$

Use the initial condition, $C_b(-\infty) = 0$

$$C_b(-\epsilon) = 0 \Rightarrow A e^{-i\beta\epsilon/2} + B e^{-i\beta\epsilon/\hbar} = 0$$

$$B = -A e^{-i\beta\epsilon}.$$

$$C_b(t) = A e^{i\omega_0 t/2}\left(e^{i\beta t/2} - e^{-i\beta(\epsilon + t/2)}\right) \qquad (7)$$

Now find $C_a(t)$. use $C_a(-\epsilon) = 1$.

So, we can write down now, C b of t which is because it is taken as exponential lambda t. So, we can take a solution as exponential i omega naught t and A exponential i beta t by 2 plus B exponential minus i beta t by 2. We have taken one of the solutions one can take the other solution also. So, now, one uses the initial condition, which is C b of minus infinity this is equal to 0, this is what the initial thing says that C b of minus infinity equal to 0 and C of minus infinity equal to 1 which can be again the same root can be rerun for the C a coefficient and which I leave it to you just showing it for C b of t.

So, C b at minus epsilon which is equal to 0; so infinity does not have a meaning for the delta function type potential or rather the perturbation So, this is between minus epsilon 2 plus epsilon this is equal to 1 over 2 epsilon, and so we write either minus infinity or minus epsilon does not matter. So, this gives rise to A exponential minus i beta epsilon by 2 plus B exponential minus i beta epsilon by 2 because exponential i omega t or at t equal to minus epsilon is not equal to 0.

So, the that is there inside the bracket of let us call it as equation 6, that should be equal to 0. And so this gives rise to that the fact that B must be related to the other coefficient A in this particular fashion and so C b t should be equal to A exponential i omega naught t by 2 exponential i beta t by 2 minus exponential minus i beta epsilon plus t by 2 and so on.

Now, this gives rise to a form. So, this is the solution 7 is the solution for C b as a function of t. So, this coefficient has got to oscillatory functions. So, one is exponential i omega naught t by 2, and the other is this what appears inside the bracket the first bracket which gets multiplied with the exponential i omega naught t by 2. So, there are 2 kind of distinct oscillations that are going to be there for C b of t, depending on whether omega naught is greater or b or beta is you know greater than that.

So, this is the solution for C b of t. And similarly as I said that one can read on the same scenario for C a of t, that is take a derivative with respect to time of equation 3 and then do all the replacement similar replacements as have been done for this C b of t and so one can actually arrive at a similar relation for C a of t. I simply leave it here and say that, now find C a of t. And in fact, you can take various values of omega naught and beta and epsilon and can plot this using either a MATLAB or of Mathematica and see how the oscillations go.

As I said, that there are two distinct oscillatory functions which get multiplied so one would be able to see this fact there. While, so while you do the C a of t use the condition that C a of minus epsilon is equal to 1 that is the initial condition and so on.

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Find $S_xS_yS_zS_yS_zS_x$ for a $S = \frac{1}{2}$ particle.

$S_x S_y S_z S_y S_z S_x$ Consider $S_z S_x$

$[S_z, S_x] = i\hbar S_y$

$S_z S_x + S_x S_z = 0$

$S_z S_x - S_x S_z = i\hbar S_y$

$S_z S_x = \frac{i\hbar}{2} S_y$

$S_i^2 = \frac{\hbar^2}{4}$

$= S_x S_y S_z S_y \left(\frac{i\hbar}{2}\right) S_y$

$= S_x S_y S_z \left(\frac{i\hbar}{2}\right)\left(\frac{\hbar^2}{4}\right)$

$= S_x \left(\frac{i\hbar}{2}\right) S_x \left(\frac{i\hbar}{2}\right)\left(\frac{\hbar^2}{4}\right)$ Consider $S_y S_z = \frac{i\hbar}{2} S_x$.

$= \left(\frac{\hbar^2}{4}\right)\left(\frac{i\hbar}{2}\right)\left(\frac{i\hbar}{2}\right)\left(\frac{\hbar^2}{4}\right) = -\frac{\hbar^2}{64}$ Ans

So, let us now do this simple problem for spin algebra it says that find this S x, S y, S z, then S y, S z, S x for a spin half particle. And the reason that I am doing it here is that one should know about the spin half algebra and there are certain properties that need to be used and these are quite informative at least in the context of the spin half particles.

So, I write this as S x, S y and S z, S y, S z and S x. So, this is equal to. So, I take this last 2. Now, so, consider this, so consider S z, S x. Now, remember there are two relations which are S z, S x that commutation relation is nothing, but ih cross S y, and particularly the anti commutation is 0 which means S z, S x plus S x, S z that is equal to 0 this is true for the spin half particle one can show that. So, we get a relation that the first one is S z, S x minus S x, S z that is equal to ih cross S y and now this thing can be added and one can actually cancel this and write this S z, S x is equal to ih cross by 2 S y so that is what we are going to write for this one. So, S x, S y, S z, S y and then i h cross by 2 S y.

There is another relation that needs to be utilized here is that the S i square is equal to 1, rather it is equal to h cross by 4 each one is h cross by 2, so it is h cross by 4. So, now, we have a S y and S y so that gives a S x, S y, S z and I write the ih cross by 2 that we have gotten earlier and then we write the h cross by 4 coming from the S y square.

Now, again do this for the S y, S z. Now, it is in the cyclic sense. So, this is now, considered just like earlier consider S y, S z. So, this can be shown that this is equal to ih cross by 2 S x, just like running down repeating the same algebra as we have done. So, this is equal to S x and there is a ih cross by 2 and there is a S x and there is a ih cross by 2 and there is a h cross square by 4. So, this is equal to; now, it just S x square which again is equal to h cross square by 4. So, its h cross square by 4, then its ih cross by 2, and then it is a ih cross by 2 and then it is a h cross square by 4. So, this if you simplify it becomes equal to because there are 2 i. So, this will be minus i square and one gets minus h cross square over. So, there is a 16 and 4, so 64 so that is the answer to this to this simple spin algebra.

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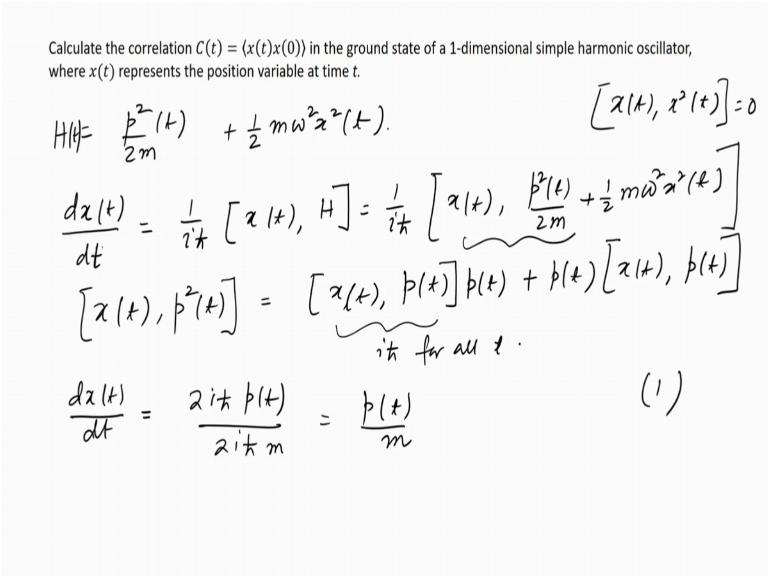

So, let us now calculate slightly more complicated problem which is calculating the correlation of x t and x 0 which is the expectation value for the ground state of a one dimensional simple harmonic oscillator x is of course, it is a position variable at time t. So, our H is given by p square over 2 m and of course, it is a p is a function of t here we have time evolved system and this.

So, we can write down the equation Heisenberg equation of motion which is for each one of the position and the momentum variables whose solution will be x of t that is the only way to get x of t. So, we write down the Hamiltonian which is of course, a function of time and it is a p square with p is a function of time, and x squares, x is a function of time. So, we write down the equation of motion and that is equal to 1 over ih cross and it is a x of t commutation with H. So, it is a 1 over ih cross it is a x of t and it is a p square of t by 2 m plus half m omega square x square t.

Now, x at all t will commute with x square, but the x and p will not commute. So, we need to only see the commutation between this. So, this of course, this commutation; so x t and x square t commutation is identically equal to 0. So, we need to look at this commutation and this is equal to x and p square commutation of course, there is a 1 over 2 m which we are going to include it later. So, this is equal to a x t p t and a p t and plus a p t x t p t. So, this is equal to of course now, x p is equal to ih cross. So, this is equal to ih cross and so on for all t.

So, dx dt it is equal to 2 ih cross p of t divided by 2 ih cross m, so this is equal to p of t by m and let us call this as equation 1. So, the evolution of the space variable for a harmonic oscillator is equal to the momentum divided by the time.

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Similarly, $\frac{dp(t)}{dt} = \frac{1}{i\hbar}\left[p(t), H\right] = -m\omega^2 x(t)$ (2)

$[p, x] = -i\hbar$

Differentiate (1) & (2) once more wrt t,

$$\frac{d^2x(t)}{dt^2} = \frac{1}{m}\frac{dp(t)}{dt} \underset{\text{using (2)}}{=} -\omega^2 x(t) \qquad (3)$$

$$\frac{d^2p(t)}{dt^2} = \frac{1}{m}\frac{dx(t)}{dt} \underset{\text{using (1)}}{=} -\omega^2 p(t) \qquad (4)$$

Solving, $x(t) = A\cos\omega t + B\sin\omega t$ (5)

$p(t) = C\cos\omega t + B\sin\omega t.$ (6)

So, similarly, so dp t dt its equal to 1 over h cross and p t, and then H, again H contains both p and x p will commute with the p term p will not commute with the x term and a little bit of algebra shows that its equal to minus m omega square x of t, using the commutation relation that p x is equal to minus ih cross for all t. So, this is equation number 2. And remember x of t and p of t should be obtained from these equations of motion.

Now, we in order to do that we differentiate both x and t once more, so with respect to t, x and p so, differentiate 1 and 2 once more with respect to t. So, then one gets d to x t, dt 2 the advantage is of course, very well perceived. You see one by taking a double derivative one gets a p dot. Now, p dot is automatically obtained from 2 and by taking a derivative of 2 one gets an x dot x dot is obtained from 1. So, because these are simple you know coupled equations, so we have we can do the or rather this is going to be beneficial for this problem. So, this is equal to 1 over m dp t over dt. Now, using 2 so, this is equal to minus omega square x of t call this as equation 3, and similarly d 2 p t dt 2 its equal to 1 over m dx dt and using one this is equal to minus omega square p t. So, this is equation 4 and so the both of them have the same equations and we know the solution of such equations as harmonic solutions.

So, solving x of t equal to A cos omega t plus a B sin omega t so, this is equation 5, and p of t has a similar solution accepting that we should use different coefficients. So, it is a C cos omega t plus B sin omega t and so these are the solutions of x of t and p of t.

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Initial conditions.

$$x(0) = A \,, \quad p(0) = C$$

$$\frac{d\,x(t)}{dt} = \frac{p(t)}{m}.$$

$$-\omega x(0) \sin\omega t + B\omega \cos\omega t = \frac{p(0)}{m}\cos\omega t + \frac{D}{m}\sin\omega t.$$

$$\text{or,} \quad B = \frac{p(0)}{m\omega} \,, \quad D = -m\omega\, x(0).$$

$$x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t \qquad (7).$$

$$\text{Now:} \quad C(t) = \langle x(t)\, x(0) \rangle$$

Now, the initial conditions so, let us take the initial conditions as x of 0 equal to A that is at t equal to 0 the oscillator is at the maximum position and p of 0 equal to C. So, just by putting t equal to 0 and in the equations 5 and 6 one gets that x of 0 equal to A and p of 0 is equal to t.

So, we should simplify further and in order to compute the B and of course, the D one should notice that dx dt is nothing, but equal to p by t m which is of course, equation 1. And one can put the solutions that we have obtained as so A is equal to x of 0 and this is sine omega t plus a B omega cosine omega t equal to p 0 by m. So, p 0 is, so p 0 by m cosine omega t plus a D by m sine omega t or B becomes equal to p 0 by m omega because of equating the coefficient of cosine terms on both sides and for equating the coefficient of sine terms on both sides one get this is equal to m x, D is equal to m x m omega x at t equal to 0.

So now, all the coefficients are calculated that is A is equal to the value of x at t equal to 0 C is p at t equal to 0, B is equal to p at t equal to 0 by m omega and D equal to minus m omega x at t equal to 0. So, we can using these things we can write down the x of t finally, so, x of t is equal to x of 0 cosine omega t plus p of 0 by m omega sin omega t. So that is the form for x of t. So, what we are asked to calculate is the correlation. So, C of t this is equal to x of t and x of 0. So, I simply multiply this by x of 0 and take the correlation in the ground state of the oscillator.

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$$C(t) = \langle x^2(0)\rangle \cos\omega t + \frac{1}{m\omega}\langle p(0)x(0)\rangle \sin\omega t \qquad (8)$$

$$\langle x^2(0)\rangle = \langle 0|\, \frac{\hbar}{2m\omega}(a+a^\dagger)(a+a^\dagger)\,|0\rangle$$

Do it yourself

$$= \frac{\hbar}{2m\omega}\langle 0|aa^\dagger|0\rangle \qquad 9(a)$$

$$= \frac{\hbar}{2m\omega}\langle 0|1+a^\dagger a|0\rangle = \frac{\hbar}{2m\omega}$$

$$\langle p(0)x(0)\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\sqrt{\frac{\hbar}{2m\omega}}\langle 0|(a^\dagger - a)(a+a^\dagger)|0\rangle$$

$$= -\frac{i\hbar}{2}\langle 0|aa^\dagger|0\rangle = -\frac{i\hbar}{2}. \qquad 9(b)$$

Substitute 9(a) & 9(b) in 1

So, this is so C of t is equal to x square of 0 cosine omega t plus p of 0 x of 0 1 over m omega sin of sin omega t, so that is the correlation that is what we want. And this is very convenient because the right hand side requires only the initial conditions that is x square at t equal to 0 and a product of p at t equal to 0 and x at t equal to 0. So, this now, can be evaluated by the a and a dagger operators and which is what we have learnt in a couple of problems before and so x square at 0, and one takes this the expectation value with respect to the ground state and this is nothing but let us call it as a ground state and h cross by 2 m omega and do not try to square the a plus a dagger term because they do not commute.

So, a plus a dagger and a plus a dagger and one writes it with 0, and there are of course, terms which are a a, but those terms will be 0 because a a will act on 0 will give you a 0 whereas, a dagger a dagger will act on 0 will create two bosons, but when you take overlap with the ground state they are again 0. The only terms that become nonzero or the a dagger a and a a dagger. So, this becomes equal to rather even the a dagger a terms becomes equal to 0 because a acting on 0 will give 0. So, the only term that is nonzero here this you have to work out. So, I gave you hint so, do it yourself in order to gain confidence on this.

So, the only term that is nonzero is h cross by 2 m omega 0, and a a dagger 0 and that is the only thing that is nonzero. Now, we know the commutation relation of a a dagger which is 1 plus a dagger a. So, this is equal to 1 plus a dagger a dagger a and 0. So, this is equal to, so 0 0 will be equal to 1 and again a dagger a acting on 0 will give you 0, because a acting on 0 will give you 0 0 is the vacuum state which has no oscillator. So, if you try to annihilate an oscillator from a vacuum state then of course, it is going to give you 0. So, this is equal to h cross by 2 m omega.

So, the first term in this let us call it equation 8 is calculated, and now we have to calculate the second term of equation a, which is a p 0 and x 0 and this. So, this is equal to m h cross omega by 2 again using the definition of p and x in terms of a and a dagger, so this is equal to h cross by 2 m omega. So, this the first multiplicative factor comes for p and the second multiplicative factor comes for x, and this is equal to 0. So, there is a dagger minus a and a plus a dagger and this is 0 and again use the same procedure as has been said. So, it is a minus i cross by 2 and so 0 a a dagger 0 which is equal to a minus i h cross by 2, because a dagger acting on 0 will create a boson and which will annihilate by the a and we will get normalization which is equal to 1.

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$$C(t) = \frac{\hbar}{2m\omega}\cos\omega t - \frac{i\hbar}{2m\omega}\sin\omega t \cdot$$

$$\boxed{C(t) = \frac{\hbar}{2m\omega} e^{-i\omega t} \cdot}$$

$$\sim \left(\frac{1}{\omega}\right) \times e^{-i\omega t}$$

So, clearly now, we have calculated both the terms, let us call them as 9 a and 9 b. So, substitute 9 a and 9 b in equation 8, and one gets the C of t that is wanted which is h cross by 2 m omega cosine omega t minus i h cross by 2 m omega sine omega t and it can be combined in order to write h cross by 2 m omega exponential minus i omega t.

So, this is a nice expression it is in a very closed form that correlation of x t and x 0, x at time t and x at time 0 can be determined and the time evolution of that is obtained by this expression which of course, it has a variation which goes with an envelope which is 1 by omega and it is an exponential minus i omega t. For a given value of omega you may actually want to see that how these variations of course, the as a t it is a harmonic variation is an oscillatory variation, but there is an envelope which is, which goes as 1 over omega. So, as omega becomes larger and larger the oscillation, the amplitude of the oscillation would go down because this the amplitude contains a term which is 1 over omega.

So, these are some of the problems that are important in the context of this course. Some of these are quite important, and I should work it out yourself. And I have done most of the steps, if there are some steps that I have not done or rather said that one should you should do it, please complete those steps this could be important from the point of view of examinations.