Advanced Quantum Mechanics with Applications
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Lecture - 33
Interaction of Radiation with Matter, Landau Levels

So, this particular special lecture it pertains to the time dependent perturbation theory, that is what we have learnt as the interaction of radiation with matter and there is just some small loose ends that I want to clip up.

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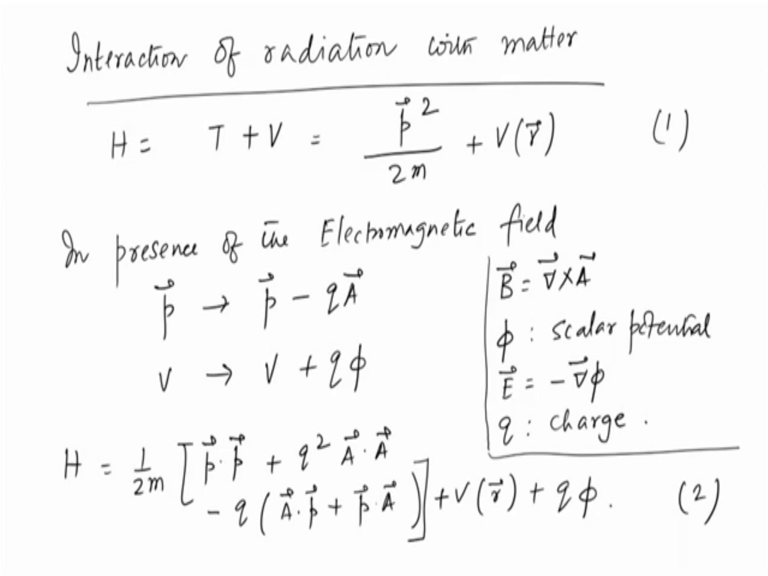

So, this is interaction of radiation with matter. And then we have a Hamiltonian of a system which is written as T plus V, T is in general without any external field this is equal to minus i h cross del square which is p square over 2 m and plus V which is could be a function of all r theta and phi. But in case of that if does not depend upon the angular variables, then it is only a function of the radial variable r.

So, in presence of the electromagnetic field; so, this radiation means the electromagnetic field. So, in presence of that the canonical momentum is written as this which is different than the mechanical momentum. So, this is the mechanical momentum p and now the canonical momentum is written as p minus e A where A is the vector potential. That is corresponds to the magnetic field. So, B is equal to curl of A that is the relation

So, once again the radiation corresponds to the electromagnetic field. An electromagnetic field has both electric and magnetic fields. So, here we are talking about just the magnetic field and we will also talk about the electric field where V now takes a form which is e of phi. And we e we are taking it as a charge and we can play around with the sign of e whether you we really want to talk about. So, we can change this e to q for any charged particle. So, q and this is equal to a q phi where q is the so, phi is the scalar field or the electric field is E equal to minus grad phi and q is the charge.

So, now this change has to be made in the Hamiltonian 1. So, that can be written as so, the Hamiltonian in 1 can be written as it is equal to 1 over 2 m. So, the p again I am writing it in terms of p. So, this is actually equal to p square over 2 m, we may not want to write it like this, but simply writing it as p square over 2 m will do. So, this is p dot p and now it is a q square A dot A and minus q these are the definitions. So, let us just write it below.

So, a minus q A dot p plus p dot A and plus V of r plus q phi. So, that is the Hamiltonian that it becomes if the charge particle is kept in an electromagnetic field, the magnetic field is B the electric field is E which are represented by their corresponding potential quantities. So, we have used a vector potential A where B equal to curl A and we have used a scalar potential or let us write this as a scalar potential instead of a field. So, that has a relation E equal to minus grad phi and q being the charge of the particle.

So, we have arrived at a Hamiltonian 2 written in equation 2. So, we want to now find out a perturbation term such that a perturbation theory can be applied. And in principle this both this B and E can be time dependent fields and we will write down the extra term.

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$$H = H_0 + H'$$

$$H' = \frac{1}{2m}(-q)\left(\vec{A}\cdot\vec{p} + \vec{p}\cdot\vec{A}\right) + q\phi + O(A^2)$$

Perturbation term

<u>Choice of a gauge</u> $\phi = 0,\ \vec{\nabla}\cdot\vec{A} = 0$ (Lorentz gauge)

$$H' = -\frac{q}{2m}\left(\vec{A}\cdot\vec{p} + \vec{p}\cdot\vec{A}\right)$$

$$(\vec{p}\cdot\vec{A})\psi = -i\hbar(\vec{\nabla}\cdot\vec{A})\psi = -i\hbar\left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\vec{A}\psi$$

$$= -i\hbar\left[\psi\underbrace{\vec{\nabla}\cdot\vec{A}}_{=0} + \vec{A}\cdot\vec{\nabla}\psi\right] = \frac{i\hbar q\cancel{2}}{\cancel{2}m}\vec{A}\cdot\vec{p}$$

So, we will write down the Hamiltonian now as H equal to H 0 plus H prime where the Hamiltonian that was written earlier that is equation number 1 is taken as H 0. So, the H prime which is let us call it the perturbation term. So, this is equal to 1 over 2 m and q over a minus q and there is a A dot p plus a p dot A and plus a q phi. So, that is the perturbation term and of course, there is a term which is proportional to A square or of the order of A square which we neglect for the reason that let us consider that we are in the regime, that we can apply the perturbation theory and the electromagnetic field is not too strong. So, that the A square term can be neglected. So, this is the perturbation term.

And of course, we can choose a gauge here. So, this is important the choice of a gauge and this gauge says that we can take phi equal to 0 and the divergence of A equal to 0 as well. So, this is called as a low range gauge. So, if you use this gauge, the perturbation term particularly takes a simple form that is the H prime; then becomes equal to minus q over 2 m A dot p plus p dot A that is the perturbation. And you know I mean let us take the second term and apply it to A wave function as if the perturbation term acts on a wave function which is equal to now minus A i h cross del dot A and then you have a psi.

So, this is equal to a minus ih cross i cap del del x plus j cap del del y plus a k cap del del z and this is going to act on this A psi. So, that tells you that this minus i h cross and we have a psi del dot A plus A dot del psi and by the choice of the gauge this is equal to 0. So, we simply have this as a q i h cross by ih cross q by 2 m.

But then there are 2 terms which are like A dot p. So, there will be a term which is A dot p twice of those terms. So, that 2 will cancel with this 2 and 1 has it in the form.

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$$\vec{A}\cdot\vec{p} = \vec{p}\cdot\vec{A}\,.$$

$$\boxed{H' = \frac{i\hbar q}{m}\,\vec{A}\cdot\vec{\nabla}}$$

$\vec{A}$: Time dependent field.

$$\vec{A} = 2A_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \alpha)$$

So, H prime it is equal to just so, what happens is that your A dot p becomes equal to your p dot A and the H prime becomes equal to i h cross q over m and A dot del. So, if you write it in terms of p then of course, this ih cross will not be there. So, this was a mistake. So, this is this was simply this. So, it is q over m A dot p which is equal to ih cross. Then of course, there will be a minus sign as well. So, it is a q over m A dot p and. So, this is equal to the perturbation term. So, the perturbation term particularly has the form which is A dot del ok. So, how do we handle this perturbation term for a physical problem for a problem that is of interest to us?

So, we can so, A is of course, a time dependent field and which can have a form. So, A can have a form which is 2 A 0 and we can write it as a real part of the exponential and that is this and in a general sense let us have a phase as well. So, general form of this time dependent field can be like this and let us try to simplify this form of the perturbation itself if we can.

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Compute: $[x, H_0]$ $\qquad H_0 = \frac{p^2}{2m}$

$$= \frac{1}{2m}\left[x,\ p_x p_x\right] = \frac{1}{2m}\left[x p_x p_x - p_x p_x x\right]$$

$$= \frac{1}{2m}\left[x \cancel{p_x} p_x + i\hbar p_x - x \cancel{p_x} p_x + i\hbar p_x\right] = i\hbar \frac{p_x}{m}$$

$$= \frac{\hbar^2}{m} \nabla_x$$

$$\nabla_x = \frac{m}{\hbar^2}\left[x, H_0\right] \ ; \quad \text{Similarly } \nabla_y = \frac{m}{\hbar^2}\left[y, H_0\right]$$

$$\nabla_z = \frac{m}{\hbar^2}\left[z, H_0\right].$$

Let us now write the commutation. So, we basically switch a bit of a gear and say commute the or rather compute the commutation of x which is a space variable and H 0 where H 0 can be Hamiltonian which is free particle Hamiltonian or it could contain a potential term, but that potential term has to depend only on x and no other coordinate. So, or rather it can depend upon r if you are taking r and H 0.

So, at this moment let us just talk about just H 0 which is a free particle terms. So, H 0 is of the form p square over 2 m it is a non-relativistic free particle dispersion. And then of course, we can write this as a 1 over 2 m and this is x and there is a p x p x that is there; so, because this is equal to p x square by 2 m. So, this can be simplified as 1 over 2 m x p x p x minus p x p x x this is all known to you and then we can do a bit of simplification again and in which we can write it as x p x p x. So, we can write it as plus i h cross p x minus x p x p x there is one line that I have skipped which you should fill it up there is a i h cross px.

So, this tells you that this is equal to i h cross. So, these 2 will cancel and this will be there 2 of them and that 2 will cancel with this two. So, it is ih cross p x by m and this is if we put the form of p x which is minus ih cross del del x which is equal to h square by sorry this is not 2 this just m and del del x will we write the x component of that. So, if you go back to our perturbation which had a term which is A dot delta. So, it is like A x delta x A y delta y and A z delta z and of course, there is a particular. So, this is a vector or you could take a polarization in a direction that you want. So, delta of x from here is equal to m by h cross square and x H naught. And so, similarly we have y equal to m by h cross square y H naught and z is equal to m by h cross square z H naught and so on.

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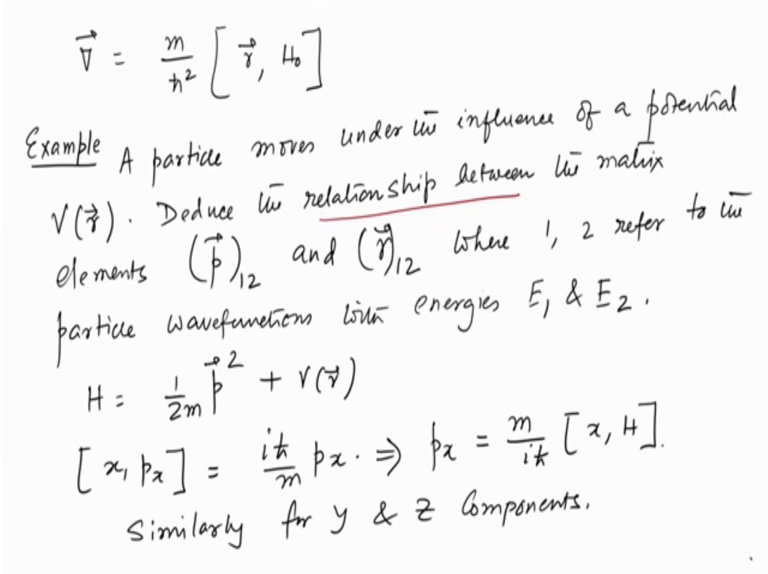

And that tells that the delta is equal to m by H cross square and then it is a r vector which commutes with this. So, we can actually replace in the perturbation term this delta by the A dot and the commutation of this r and H. So, one can actually think of a simple problem that let us assume that a particle moves under the influence of a potential V of r. So, I am not treating it as a vector r, but it only depends on the scalar r the radial variable.

Now, the question is deduce the relationship between the matrix elements p 1 2, I will tell you what 1 2 are which are 2 states maybe an initial and a final states because the time dependent perturbation makes a transition from a state 1 to state 2. We are not talking about the transition at this moment, but just wanted to find out the relationship as it is written here. It is a relationship between the matrix elements of the momentum operator between these 2 states the initial and the final states and r 1 2 if you want. So, this is or you can just the way it is written one can actually have it also as a function of r; so, r 1 2, the relationship between r 1 2.

So, let us write this clearly. So, this is 1 2 where 1 2 refer to the particle wave functions with energies E 1 and E 2. So, the problem is actually simple I mean nothing much needs to be done. So, H is equal to 1 over 2 m and p square plus V of r x of H or rather the just do it component wise which is easier for you to understand. It is ih cross by m and a p x and which also is it gives that p x is equal to i h cross x H similarly for the for y and z components.

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$$(\vec{p})_{12} = \frac{m}{i\hbar}\left([\vec{r}, H]\right)_{12} = \frac{m}{i\hbar}\langle\psi_1|\vec{r}H - H\vec{r}|\psi_2\rangle$$

$$= \frac{m}{i\hbar}\langle\psi_1|\vec{r}E_2 - \vec{r}E_1|\psi_2\rangle$$

$$\boxed{(\vec{p})_{12} = \frac{m(E_2 - E_1)}{i\hbar}(\vec{r})_{12}}$$

So, p 1 2 vector and this is the matrix element of that its m over i H cross r it is H and now 1 2 will have to be taken of this. So, this is equal to m by i H cross. So, it is psi 1 r H minus H r psi 2 and this is equal to m by i h cross psi 1 r E 2 minus r E 1. So, that is by psi 2 which gives that its E 2 minus E 1 multiplied by m divided by i h cross r 1 2. So, that is the relationship between p 1 2 and r 1 2.

So, these are some problems that are often you know important in the context of this time dependent perturbation theory. Of course, we have not done a very rigorous derivation of the Einstein's coefficient, but we have done some derivation where we have taken the a field or the vector potential to have a sinusoidal dependence on time that is a harmonic function of time and then we have calculated the coefficients.

Let us do a one more problem of this special kind it is not about time dependent problem, but it is a time independent problem. But, this problem at times become a very important thing in our application in the quantum mechanics that you learn it is quite a simple. And it has applications in a very important and exciting field in solid state physics called as the Hall effect to the quantum Hall effect. The classical Hall effect is all what you know in your undergraduate texts. It is about deflection of the charged particles in presence of crossed electric and or rather it is when the magnetic field is applied perpendicular to the sample and there is an electric field that biases the sample in the longitudinal direction.

In the transverse direction of voltage is generated and because of that voltage the charges move towards the transverse edges of the sample and that is known as Hall effect. It is a very important tool for calculating the density of charge carriers in a semiconductor and a quantized version of that was discovered just about 100 years later. This was discovered in 1879. The classical Hall effect that you are all aware of from your undergraduate text it is 1980 when the quantum version of the Hall effect was discovered and ever since it has taken a center stage for calculation of various quantities including you know the land out level degeneracy, the various things related to Hall effects such as Hall voltage

And the important thing is that there is a quantization of the plateaus which are seen in the resistivity versus the magnetic field graph and these plateaus are very very robust to all kinds of perturbations including heavy disorder and so on. And so, we will not get into those complexities, we will look at it as a quantum mechanical problem.

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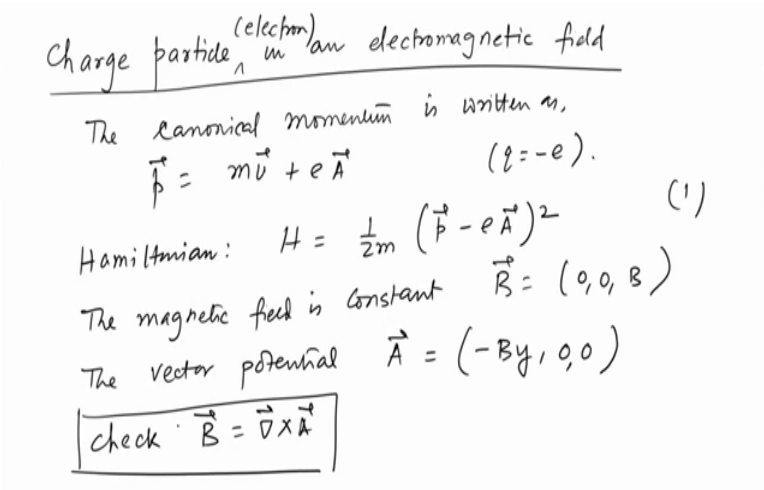

And let us call it as a charged particle again in an electromagnetic field, but we are now talking about constant electromagnetic field. So, we have seen that the canonical momentum is written as so, it is p is equal to mv and a minus a q A. But now let us take that we are talking about. So, this is an electron to make the case more strong and we are talking about q equal to or e equal to minus q. So, we have taken a plus e.

So, the Hamiltonian of the system is written as so, the Hamiltonian H it is equal to 1 over 2 m and p minus e A whole square and of course, we are talking about the low range gauge in which this divergence of A etcetera is equal to 0 rather we will talk about a gauge here. So, here let us take that the magnetic field is constant B is equal to it is in the z direction which is usually the case. So, we have a planar sample. So, we have electrons which are in a plane in a 2 dimensional plane and there is a magnetic field that is acting perpendicular to it which is this B sorry this B has is not a vector quantity. It is just a scalar that is a component of B we have taken that as B itself.

So, the vector potential if this is the form of the magnetic field the vector potential can be chosen as A equal to minus B y 0 0. So, it is only there in the x direction with this position coordinate y and you can check that. So, check B equal to curl A is satisfied because that is an important relation and this you need to check.

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$$H = \frac{1}{2m}\left(p_x + eyB\right)^2 + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} \qquad (2)$$

$$[H, p_x] = 0 = [H, p_z]$$

p_x, p_z and H have simultaneous eigenfunctions.

The eigenstates of p_x and p_z are:

$$\psi_{k_x, k_z} = e^{i(k_x x + k_z z)} \qquad (3).$$

$$\psi = e^{i(k_x x + k_z z)} f(y). \qquad (4).$$

Substituting (4) in (2)

So, if that is true, then let us write down the H this is equal to 1 over 2 m p x plus ey B a square plus a p y square over 2 m plus a p z square over 2 m. And so, the Hamiltonian is cyclic in a y and z coordinate. So, Hamiltonian does not contain any x and z. So, Hamiltonian is cyclic in x and z. So, the corresponding canonical momentum should be conserved. So, which means that H and p x should be equal to 0. So, is H and p z. So, if these are equal to these commute, then we have them as good quantum numbers or we can write down the wave function or we can use this symmetry that they are the correspondingly K x and K z are good quantum numbers

So, one can infer that p x p z and H have simultaneous eigenfunctions. The eigenstates of p x and p z r psi K x K z; so, that is are those are its free particle in those direction sorry y is not there; y there is a variable y in equation let us call this as equation, let us call this as equation 1 and this as equation 2. So, this is K z z and we have this as equation 3.

So, we so, this is only for K x and K z where they propagate like free particles. So, the particle has in the z direction as a propagation like a free particle as well as in the x direction. So, this is your x direction this is your y direction. In y direction it is not like a free particle and it is something else and we are going to see what it is like. So, the total wave function if this is equal to an exponential i K x x plus K z z and f of y.

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$$\left[\frac{p_y^2}{2m} + \frac{K}{2}(y-y_0)^2\right] f(y) = \left(E - \frac{\hbar^2 k_z^2}{2m}\right) f(y) \qquad (5)$$

$$y_0 = -\frac{\hbar k_z}{eB} \qquad \frac{K}{m} = \left(\frac{eB}{m}\right)^2 = \Omega^2$$

Ω : cyclotron frequency $\rightarrow$ Rotation frequency of a charged particle in a uniform magnetic field.

$$\boxed{E_n = \hbar\Omega\left(n+\frac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m}} \qquad (6)$$

Landau levels.

$$f_n(y) = A_n H_n\left[\sqrt{\frac{m\Omega}{\hbar}}(y-y_0)\right] \exp\left[-\frac{1}{2}\frac{m\Omega}{\hbar}(y-y_0)^2\right] \qquad (7)$$

So, if you substitute this 4 in 2, one gets p y square over 2 m plus K by 2 p y is or K y is not a good quantum number because, there is a y in the Hamiltonian and y and p y do not commute; y minus y 0 where y 0 is just a constant which depends on the magnetic field etcetera, we will just write it here. So, it is E minus h cross square a K z square over 2 m and f y. So, we are trying to solve in this equation 4 that is the total wave function the other 2 directions, the particle the charge particle or the electron here propagates like free particles. However, it does not propagate like a free particle in the y direction. We have to know what it what the motion is like.

So, we need to find what is y f of y and your y 0 is nothing, but h cross K x by e B that will take a while. So, what we are trying to figure out is that we are trying to figure out the motion in the y direction where this is a is equal to e B and this K by m which is equal to e B by m whole square which is equal to omega square. We will tell you the physical significance of that so, omega is called as the cyclotron frequency. So, this corresponds to the rotation frequency. So, it is the rotation frequency of a charged particle in an in uniform magnetic field ok.

So, now look at carefully this equation 5, what does it correspond to? It corresponds to a particle that is undergoing a simple harmonic oscillation in the y direction about; it is not above the origin that is y equal to 0, but it is centred around y equal to y 0. So, it is in this direction it is executing simple harmonic motion and this is what the Hamiltonian in 5 or rather the equation in 5 suggests that is the Schrodinger equation. And, it is a undergoing that oscillation with this characteristic frequency which we called is at as a cyclotron frequency.

So, we know now we do not need to solve this equation 5 because, we already know what is the energy spectrum for a particle undergoing a simple harmonic oscillation. So, whose energy is of course, given by it is a h cross omega n plus half that is the oscillator energy and of course, this energy in the z direction free particle direction.

So, it is basically a free motion along the z direction and it is harmonic oscillator in the y direction centered about some y 0 that is there. So, we can take a clue from all the solutions that you have seen for the harmonic oscillator in your quantum mechanics 1 course. So, this is equal to f this is A n H n root over m omega divided by H cross y minus y 0 that is the Harmite polynomial. And then convoluted or rather multiplied with this m omega by h cross y minus y 0 square and so on. So, that is the wave function.

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$$\psi_n = A_n H_n\left[\sqrt{\frac{m\Omega}{\hbar}}(y-y_0)\right] \exp\left[-\frac{1}{2}\frac{m\Omega}{\hbar}(y-y_0)^2 + i k_x x + i k_z z\right] \quad (8)$$

Degeneracy of the Landau levels.

$$\psi(x,y,z) = \psi(x+L, y+L, z+L) \quad (9)$$

$$\psi_k(\vec{r}) = A e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad (10)$$

Put (10) in (9)

$$k_x = \frac{2\pi n_x}{L} \quad \text{where } n_x = 0, \pm 1, \pm 2, \pm 3 \ldots\ldots$$

So, if you want to write down the full wave function then that wave function is equal to the full thing which now let us put a n. We will keep a undetermined constant here which can be determined by the normalization constant and this is equal to root over m omega by h cross y minus y 0 and exponential minus half m omega by h cross y minus y 0 square um. And plus i K x x plus i K z z and that is the full solution of this equation of the of a charged particle in a constant magnetic field. This is not a time varying field, it is a constant magnetic field that we talked about and one can actually get.

So, these are these energy spectrum these corresponds to these are called as the Landau levels. Let us write it in red. The special importance of these Landau levels are they of course, form the energy levels for a particle charged particle or electron in a constant magnetic field. These are enormously degenerate and this degree of degeneracy can actually be found and one can understand the degeneracy of the Landau levels

So, one can note that from the periodicity of the solution which is was suppose, you confine the system in a box of length box which has lengths on all sides as L. So, if you take a psi K of r and which is equal to A exponential i K dot r minus omega t. So, if you put in so, let us call them as 6 and this is 7 and this is 8; so, that is a full solution and call this as 9 and 10. So, if you put 10 in 9, one gets K x equal to 2 pi n x over L with n x equal to 0 plus minus 1 and plus minus 2 and plus minus 3 and so on. So, of course, I mean there is otherwise a continuous spectrum, but it becomes discrete for a confined particle. So, the SHM is centered around y 0.

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SHM in Centered around $y_0 = \frac{\hbar k_x}{e B}$.

$$(y_0)^{max} = L = \frac{h\, n_x^{max}}{e B L}$$

Max. degeneracy

$$n_x^{max} = e B L^2 = e B (\text{Area})$$

$$\frac{\text{degeneracy}}{\text{Area}} = e B \sim B.$$

So, the simple harmonic motion is centered around y 0 equal to h cross K x by e B. So, now, this can only be possible or rather we can find the maximum degeneracy or the limit of degeneracy of this Landau level. Understanding that your y 0 can never go out of the sample direction in the y in the along the y direction.

So, sample has a particular length in the y direction say that is L. So, y 0 max can at the most be L ok; so, it is either it is centered about 0 that is the minimum position for y 0 and the maximum position for y 0 is that the sample length in the y direction. And that is equal to if you take this n x max to be e B L, so, the n x rather the maximum degeneracy n x max equal to e B L square where L square is the area of the sample. So, we call it a e B A, A being A is not the let us just write as area then because, A is also the magnetic vector potential.

So, this is the area which is the planar area of the sample and so, this gives the area of this or rather degeneracy of this Landau levels. And we get the solution as the particle being free in the x direction and a z direction and it executes a simple harmonic motion about the y direction, about some distance which is given which depends upon various things including the magnetic field.

So, as you change the magnetic field this the location about which it executes the simple harmonic oscillation that shifts is well. And there is a particular degeneracy associated with each of these Landau levels which depends on the magnetic field and the area of the sample. So, usually you know one does not want to define the keep the area because that depends on the specific dimensions of the system. So, one can talk about a degeneracy per unit area which is e into B electronic charges of course, constant. So, it depends upon is proportional to B. So, it depends upon the magnetic field that is there.

So, we do not want to extend it beyond this and try to correlate with the quantum Hall effect, but it definitely has a very good relation with the quantum Hall effect that one studies in maybe senior undergrad or the beginning of the postgraduate studies or people do research in that. It especially with you know Nobel prize being declared in 2016 on the correlation between topology and condensed matter physics, these quantum Hall systems and including graphene and other you know systems which are which show Hall effect. They have gained a lot of attention and importance.