Advanced Quantum Mechanics with Applications

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Lecture – 01
Qunatum Harmonic Oscillator, Creation and annihilation Operators

So, today we are going to talk about the Quantum Harmonic Oscillator. And we use the raising and the lowering operators in order to solve the quantum harmonic oscillator problem. In a way this is an alternate way of doing it by without solving a Schrodinger equation with a lot of rigor that we are used to, but this method gives as much of information as we would have gotten by solving that equation by say a power series method and this is quite intuitive and it is quite easy to follow. Today I have prepared the notes, so we will go through them slide by slide slowly.

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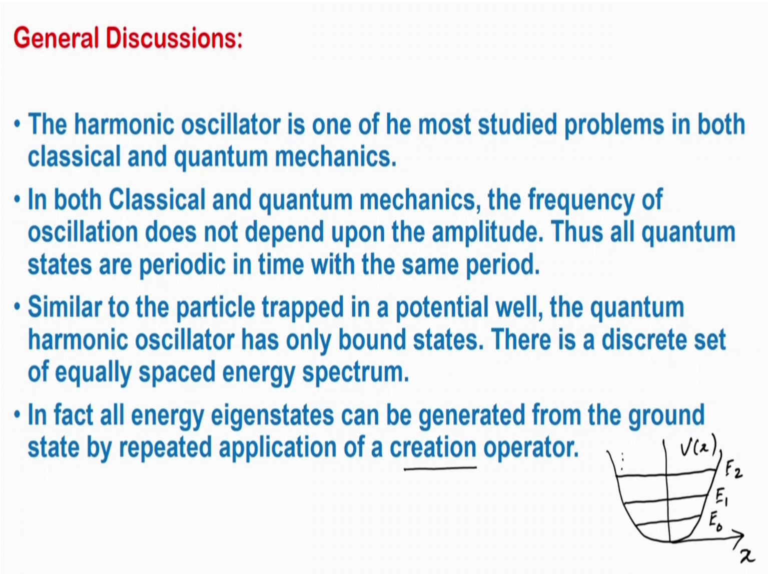

So, to start with some general discussions on the harmonic oscillator it is one of the most studied problem in both classical and quantum mechanics, ok. And in both classical and quantum mechanics the frequency of oscillation does not depend upon amplitude and these are the features of harmonic linear harmonic oscillator problem. So, all the quantum states as a result of this are periodic in time with the same period. So, it is similar to the particle trapped in a potential well the quantum harmonic oscillator has only bound states, ok.

And there is a discrete state set of equally spaced energy spectrum or the given quantum states that are there. So, what we are essentially talking about is that we have an oscillator potential which is like this and this is x and this is V of x. So, all these levels the lowest one we will see that it is equal to half h cross omega, and then it is 3 half h cross omega, and then it is 5 half h cross omega and so on. And we will keep getting these wave functions corresponding to each of these energy levels. So, this is called as ground state energy let us call that as E 0 and then there is E 1, and then there is a E 2 and so on. And this spectrum continues which are equally spaced as I said. So, this is one of the differences with the particle in a box where the energy spectrum is not equally spaced rather it goes as square of the quantum number. So, it goes as n square h cross square pi square over 2 m a square as you know is the width of the well.

So, that way it is different from the quantum harmonic oscillator and in fact, it is because the energy levels are equally spaced one can actually talk about this raising and lowering operator or the creation operator and the annihilation operator they are used interchangeably. So, here we have talked about the creation operator, but generally we will use raising and lowering operator. So, all the eigenstates energy eigenstates of this problem can be generated from the ground state this is what we will see by the end of the discussion that by repeated application of a creation operator.

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Usually the Solution of the Harmonic Oscillator potential in the Schroedinger equation requires significant mathematical rigor to arrive at the solution for the eigenvalues and eigenfunctions. We have to solve an equation of the form,

$$\omega = \sqrt{\frac{k}{m}}$$

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi = E\psi}$$

for the Time independent Schroedinger equation in 1 dimension.

An alternate method to solve this problem (which is much less cumbersome) is to use an operator method that will be outlined below.

So, let us start our discussion. So, the solution of the harmonic oscillator potential in the Schrodinger equation, as I said requires significant mathematical rigor to arrive at the solution for the eigenvalues and eigenfunctions. So, this is the equation that you need to solve and this is the Schrodinger equation time independent Schrodinger equation for a potential which is of the form half k x square where your omega is equal to root over k by m. So, a it is a x square potential which is here. So, this part is the V of x and psi, and there is a kinetic energy term the first term on the left is the kinetic energy and then it is a potential energy and they are operating on psi and giving me the total energy psi E psi.

And as I said that it can be solved by solving this differential equation maybe using something like a power series method which is cumbersome. Here we use a much less cumbersome method and operator method so to say. So, let us highlight this operator method and we will outline that in the following slides.

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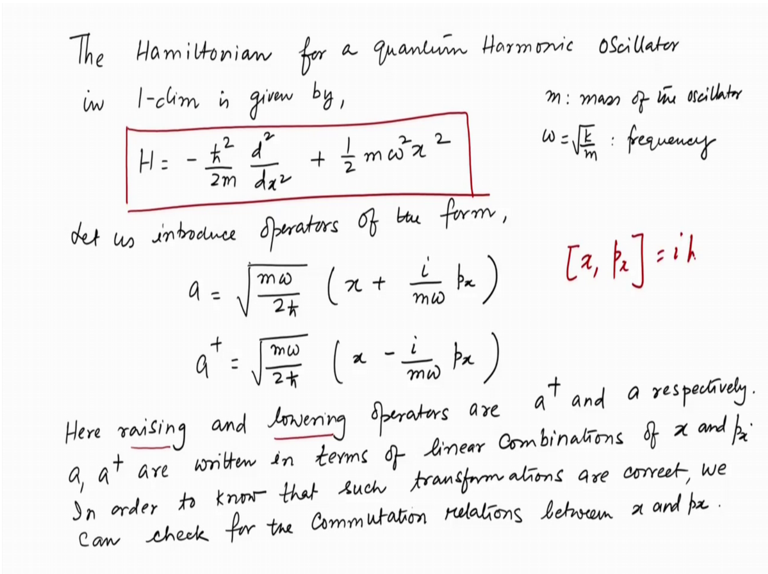

So, to write down the Hamiltonian of a quantum harmonic oscillator which is given by h equal to minus h square over 2 m d 2 dx 2 plus half m omega square x square. So, if we introduce operators of the form which is a equal to root over m omega by 2 h cross a dagger equal to root over m omega by 2 h cross and multiplied by this operator where x is the position operator and p x is the momentum operator, same here with the x plus i over m omega p x and x minus i by i over m omega p x and so we are doing it in one dimension. So, we are having in this particular case variables which are x and p x however, this method can be generalized into 3 dimensions. There is of course, a difference between one dimension and 3 dimensions, so you have degeneracy in 3 dimensions where in one dimension which we do not have any degeneracy.

In any case, so these are called as raising operator. So, a dagger is called as a raising operator and is called as a lower lowering operator. And they carry the same meaning as creation operator and annihilation operators which are represented by or rather denoted by a dagger and a respectively and so here you see that a and a dagger are written in terms of linear combinations of x and p x. Now, in order to know whether this is a valid transformation, because we are taking these operators x and p x and using the linear combinations of that we are writing two more operators which are aand a dagger. So, in order to know, that we need to check the commutation relations and how the commutation relations are preserving the commutation relations of x and p x which is known to be equal to i h cross.

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Using $[x, p_x] = i\hbar$ one can show the commutation relation $[a, a^{+}] = 1$, $[a, a] = 0 = [a^{+}, a^{+}]$

Let us show one of them

$$[a, a^{+}] = aa^{+} - a^{+}a$$
$$= \left(\frac{m\omega}{2\hbar}\right)\left[\left(x + \frac{i}{m\omega}p_x\right)\left(x - \frac{i}{m\omega}p_x\right) - \left(x - \frac{i}{m\omega}p_x\right)\left(x + \frac{i}{m\omega}p_x\right)\right]$$
$$= \left(\frac{m\omega}{2\hbar}\right)\left[x^2 - \frac{i}{m\omega}(xp_x) + \frac{i}{m\omega}(p_x x) + \frac{1}{m\omega}p_x^2 - \left(x^2 + \frac{i}{m\omega}xp_x - \frac{i}{m\omega}(p_x x) + \frac{1}{m\omega}p_x^2\right)\right]$$
$$= \left(\frac{m\omega}{2\hbar}\right)\left[\left(\frac{-i}{m\omega}\right)(i\hbar) - \left(\frac{i}{m\omega}\right)(i\hbar)\right] = 1$$

So, let me just write this. So, this is x p x this is equal to i h cross which means that x and p x are do not commute. In fact, the commutation has a finite value which is given by i h cross and this is very well known. So, let us see that what information we get from the commutation relations of a dagger and so on.

So, if you use x p x equal to i h cross one can show that the a dagger commutation in that order that is first a on the left a and then a dagger will give you one and a commutation of course will give you 0, a dagger a dagger will give 0. So, these are actually the particularly the bosonic commutation relations that we come across which we have seen as well earlier and so we want to see that whether a dagger satisfy the bosonic commutation relations rather this thing comes that is a dagger commutation equal to 1.

So, let us show one them and you should be able to do the other one exactly in the same manner. So, a dagger commutation equal to a dagger minus a dagger a; now, going back to this a and a dagger do you see that we have quadratic in these operators a dagger and a dagger a. So, this term gets multiplied twice which is m omega by 2 h cross the square root goes away. And then we have written the operator corresponding to a and then the operator corresponding to a dagger and then negative of then again a dagger and a. And if we open up the bracket we will have a x square and x p x with a minus i by m omega and there is the other term is i over m omega p x x and plus 1 by m omega p x square.

Now, this minus sign comes here and I have opened a bracket and have again multiplied these two terms the last two terms and so on and then if you simplify and use this commutation relation between x and p x this a a dagger commutation comes out to be equal to 1. So, these are the commutation relations and if I use these commutation relations they actually preserve the well known commutation relations of x and p x, all right.

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The algebra becomes even more simple if we introduce dimensionless variables, $\tilde{x}$ and $\tilde{p}_x$ where $\tilde{p}_x = -i\frac{d}{d\tilde{x}}$ where the dimensionless coordinate $\tilde{x}$ satisfies $[\tilde{x}, \tilde{p}_x] = i$.

In terms of these dimensionless variables, the raising and lowering operators are written as,

$$a = \frac{1}{\sqrt{2}}(\tilde{x} + i\tilde{p}_x), \quad a^{+} = \frac{1}{\sqrt{2}}(\tilde{x} - i\tilde{p}_x)$$

Once again it can be checked easily that,

$$[a, a^{+}] = 1 \quad \text{and} \quad [a, a] = [a^{+}, a^{+}] = 0$$

Besides the Hamiltonian becomes dimensionless.

$$\tilde{H} = \frac{1}{2}(\tilde{x}^2 + \tilde{p}_x^2) = \frac{1}{2}(\tilde{x} + i\tilde{p}_x)(\tilde{x} - i\tilde{p}_x)$$

$$= \cancel{a^{+}a} \quad (?)$$

$$= aa^{+}$$

So, the algebra is becomes actually much simpler if you take care of this m omega and h cross by using dimensionless variables such as x tilde and p x tilde, where p x tilde is equal to minus i d dx tilde and the dimensionless coordinate x tilde satisfy satisfies x tilde and a p x tilde commutation equal to i. So, we do not have to worry about the h cross etcetera. And now, in terms of these less variables or rather these operators the raising and the lowering operator can be written in a much simpler form which is 1 by root 2 x tilde plus i p x tilde where as a dagger is equal to 1 by root 2 x tilde minus i p x tilde.

So, once again one can actually a check very easily that a a dagger just like earlier should is equal to 1 and similarly the commutation relation between a a and a dagger a dagger that will vanish. But importantly what happens is that the Hamiltonian takes a form which is half of x tilde square plus p x tilde square and there is no m there is no h cross etcetera and there is no omega as well, and we can write this of course, in the form of x tilde plus i p x tilde and x tilde minus i p x tilde. So, this x tilde square plus p x tilde square will come. Now, that would give a a feeling that we are actually having because a s are 1 by root 2. So, one a and one a dagger, so this actually becomes a not, so this really becomes a a a dagger and not a a dagger a let me just write that once again.

Because this is a definition of a, so this is actually a a dagger. But however, that is incorrect and the reason that it is incorrect is that that x tilde and p x tilde do not commute.

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But since $\tilde{x}$ and $\tilde{p}_x$ do not commute, we can show that $\tilde{H}$ does not quite become $a a^{\dagger}$ or $a^{\dagger} a$.

In fact in terms of these operators, the dimensionless Hamiltonian is written as,

$$\tilde{H} = \frac{1}{2}\left(a a^{\dagger} + a^{\dagger} a\right) = \frac{1}{2}\left(\tilde{x}^2 + \tilde{p}_x^2\right).$$

But since $a a^{\dagger} = 1 - a^{\dagger} a \quad \because [a, a^{\dagger}] = 1$

$$\boxed{\tilde{H} = \left(a^{\dagger} a + \frac{1}{2}\right)}$$

We can define a number operator $N = a^{\dagger} a$ which when acts on a state yields the number of oscillators in that state.

So, what it should become is actually that h is in fact, a combination of a a dagger or a dagger a in fact, it is half of a a dagger plus a dagger a this you can simply check using the definitions that we have given. So, this is nothing but equal to half of x tilde square plus p x tilde square, ok.

But since a a dagger equal to 1 minus a dagger a that is because your just now what we have seen is a a dagger equal to 1. So, then this can be written as a dagger a the H tilde which is the dimensionless Hamiltonian for the problem becomes a dagger a plus half. So, we can now, define this one as an operator the a dagger a to be an operator which we call as number operator and we will see why we call that because this acting on a energy eigenstate returns me the state itself, and also counts the number of oscillators or the number of bosons if you like to say that is there in that particular quantum state.

So, as I said that so, it is which one acts on a state is the number of oscillators in that state. So, this is my Hamiltonian which is a dimensionless Hamiltonian we have taken care of the dimensions of or rather these variables m h cross omega etcetera are absorbed in the definition of a and a dagger and so on.

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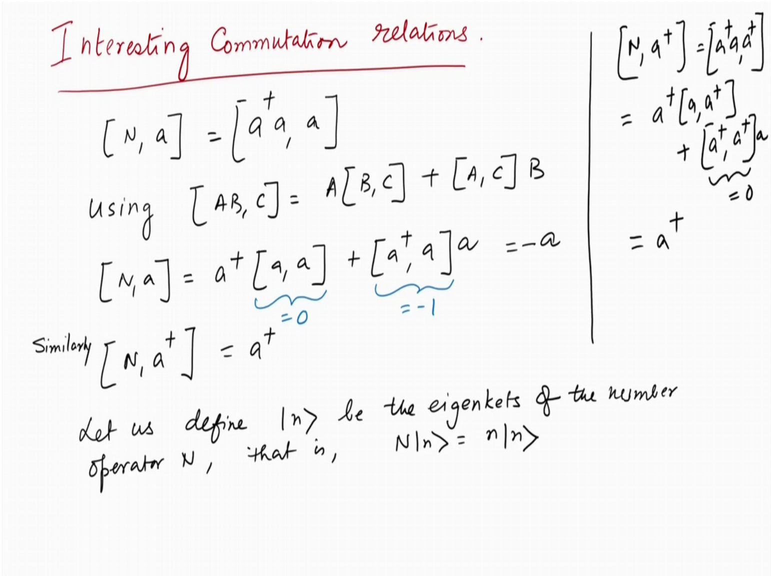

So, there are some interesting commutation relations that exist. In fact, these commutation relations are quite helpful in doing the operator algebra that is involved in this particular case or rather the study of this quantum harmonic oscillator. So, N a commutation N a is a dagger a and commute commutation with a. So, if you use a commutation identity which is A, B and C, where A B and C all are operators. So, that will be a commutation of B C and A C commutation multiplied by B.

So, N and a commutation I will have a to be equal to a dagger, B to be equal to a and C to be equal to a. So, I have a a dagger a a and I have a a dagger a commutation and multiplied by a. So, the first term gives 0 because a a commutation as we say that it is equal to 0 and a dagger a is equal to minus 1, and the reason is that that a a dagger equal to the commutation a a dagger equal to 1. So, a dagger a should be minus 1. So, this becomes equal to minus a.

So, just to remind you that the N the number operator and a the commutation relation yields a minus a which immediately says that a is not an eigenstate of or rather a and n they do not share the same eigenstates. So, that is a does not have an eigenstate which is an eigenstate of n as well. So, the number energy states that we are talking about is not an eigenstate of a, or a dagger in this particular case. We can also show that exactly in the same manner that n a a dagger equal to a dagger. So, let us just repeat that quickly.

So, I have N a dagger. So, this is equal to a dagger a a dagger and this is equal to a dagger if we use this relation that we have written down. So, I have a a dagger and a a dagger and a plus a dagger a dagger and multiplied by a. Now, this is of course, equal to 0 and this is equal to 1, the first term equal to 1. So, this is equal to a dagger which is what we have written.

So now we want to know that what are the eigenstates of N; maybe that they are not the same eigenstate. So, a a and a dagger have different eigenstates or rather these eigenstates are not the eigenstates of a and a dagger, but still we want to know what are the eigenstates of N, capital N, operator which is a dagger a. And let this small n ket small n which is written here E is the eigensket of the number operator N. So, that when N acts on the ket n returns me N and the ket n. As I said that though these ket n is an energy eigenstate and when the number operator operates on the energy eigenstates it returns the same eigenstate and also returns the number of oscillators in that state.

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This brings us to, $\qquad [N, a] = -a$

$$N\left(a|n\rangle\right) = (n-1)\left(a|n\rangle\right)$$

This says that $a|n\rangle$ is an eigenstate of N with an eigenvalue $(n-1)$.

Similarly, $N\left(a^{+}|n\rangle\right) = (n+1)\left(a^{+}|n\rangle\right)$; $[N, a^{+}] = a^{+}$

Thus $a^{+}|n\rangle$ is an eigenstate of N with eigenvalue $(n+1)$

a^{+} is called as raising operator

a is called as lowering operator

Now, what kind of relations do these a or rather if a acts on these N what kind of states emerge even if they are not eigenstates we need to know. And in order to do that we have operated N on a vector or a state which is a multiplied by n and this gives us it is a n minus 1 and acting on these a the vector same vector a act a acting on N. And now this is not difficult to find out because in just the last slide we have calculated that N a is equal to minus a. So, N a minus a n is equal to minus a and this is what we have used here. So, the one so these two terms have combined to give me N acting on the vector a times the n ket which is n minus 1 and operating on the ket a multiplied by the vector n.

So, this of course, says that a n is an eigenstate of n with an eigenvalue and minus 1 and similarly we can also use the other commutation relation that is N a dagger which is equal to a dagger, and by doing that we can take a vector or a state which is a dagger n which went at it by n which gives me a n plus 1 and returns me the same ket. So, this is an eigen ket a dagger n is an eigen ket of or eigenstate of n with an eigen value a n plus 1; a dagger is called as a raising operator and a is called as a lowering operator. So, these are some of the things that we need to know.

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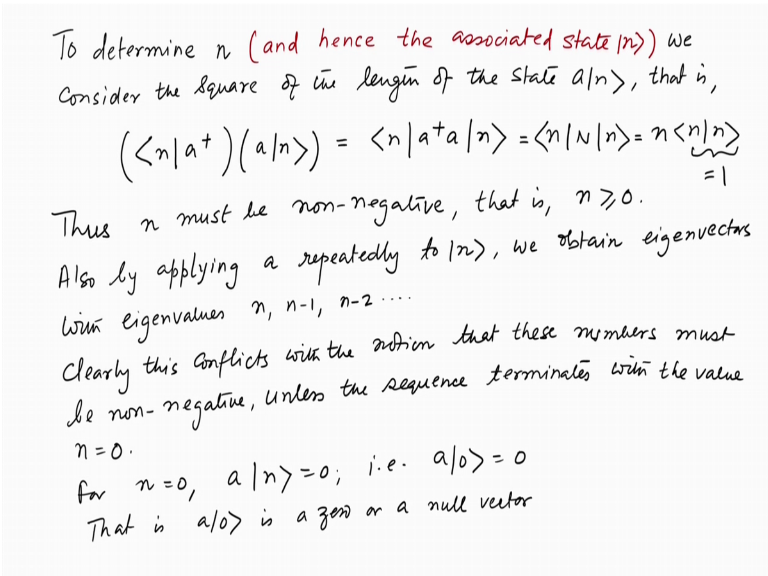

To determine n (and hence the associated state $|n\rangle$) we consider the square of the length of the state $a|n\rangle$, that is,

$$(\langle n|a^+)(a|n\rangle) = \langle n|a^+a|n\rangle = \langle n|N|n\rangle = n\underbrace{\langle n|n\rangle}_{=1}$$

Thus n must be non-negative, that is, $n \geq 0$.

Also by applying a repeatedly to $|n\rangle$, we obtain eigenvectors with eigenvalues $n, n-1, n-2 \ldots$

Clearly this conflicts with the notion that these numbers must be non-negative, unless the sequence terminates with the value $n = 0$.

For $n = 0$, $a|n\rangle = 0$; i.e. $a|0\rangle = 0$

That is $a|0\rangle$ is a zero or a null vector

So, let us try to understand that what kind of these number n. So, what are the restrictions on n or what are the conditions on n that are possible, we are talking both in terms of the quantum number n and the state n, so as to have a better understanding.

Now, consider the length of the vector a operating on n. So, if you want to know the length we have to take the square of the vector and this is the square. So, I take a mod square of this which finally, can be written as prime a dagger and a n. If you open this up it becomes n a dagger a n and so this is equal to n and then capital N and n, capital N acting on small n will give me a n and n this inner product of the ns n vectors. So, it is very clear that this is equal to 1 and because it is a length that we are talking about length of a vector which cannot be negative, so its non negative and so cap this small n must be greater than equal to 0.

So, what it means is that by applying repeatedly a on n we can obtain eigenvectors with eigenvalue n n minus 1 n minus 2 and so on however, this conflicts with the notion that these numbers must be non-negative because it is related to the length as we just talked about unless of course, the sequence terminates with the value n equal to 0. So, if n n minus 1 n minus 2 all are positive and the one that is coming at the end which is simply equal to n that should I mean not that n, but that value of that the last term in this series should become equal to 0 else these cannot be positive definite.

So, what it says is that for n equal to 0 a acting on n should give me 0. So, that is a acting on 0 is the 0 thus that is if a 0 is is 0 or a null vector. So, a acting on a 0 is a null vector which means that this is I mean it has a it is a vector of 0 magnitude.

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Further we have seen that $a^{+}|n\rangle$ is proportional to the state $|n+1\rangle$. That is

$$a^{+}|n\rangle = C_n|n+1\rangle$$

C_n can be calculated as follows:

$$|C_n|^2 = \left(\langle n|a\right)\left(a^{+}|n\rangle\right) = \langle n|a a^{+}|n\rangle$$

$$[a, a^{+}] = 1 \Rightarrow a a^{+} - a^{+}a = 1 \Rightarrow a a^{+} = 1 + a^{+}a = N+1$$

$$|C_n|^2 = \langle n|N+1|n\rangle = (n+1)\langle n+1|n+1\rangle = (n+1)$$

Hence $C_n = \sqrt{n+1}$

$$a^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

So, further we have seen that a dagger n is proportional to the state n plus 1 if we want to understand that how it is related we can simply write that a dagger n its equal to C n n plus 1. And C n can be calculated as follows. I can take multiply it by its conjugate which will give me a a and a bar n a a dagger n and this how I we write. So, there is a C n mod square which is equal to n a and a dagger n. So, this is equal to n a a dagger n

Now, a a dagger is equal to 1. So, a a dagger is equal to 1 plus a dagger a. So, that is equal to 1 plus n and 1 plus n if you put it here then that n what were we what we mean is that capital N that N, mult operating on small n will give me a small n times the ket n and so this C n square is simply nothing but equal to the inner product of the operator N plus 1 between the energy eigenstates which are given by n.

So, this is equal to n plus 1 and there is n plus 1 and n plus 1. So, this is ortho this is normalized. So, this is n plus 1 inner product of n plus 1 and n plus 1 is equal to one. So, C n is equal to root over of n plus 1, which means that a dagger acting on n gives me a state which is n plus 1 which means one oscillator more in that state. So, if I operate a a state with n oscillators I will get one oscillator mode and will return me with a coefficient which is equal to root over of n plus 1.

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Similarly one can find out,

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \quad \text{except for } n=0 \text{ where } a|0\rangle = 0$$

$|0\rangle$: called a vacuum (ground state). A state $|n\rangle$ can be built from $|0\rangle$

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle.$$

: All eigenstates of the quantum oscillator problem can be generated

Remember $H = a^+a + \frac{1}{2} = N + \frac{1}{2}$ {so $|n\rangle$ is an eigenstate of H

Thus $H|n\rangle = (N+\frac{1}{2})|n\rangle = (n+\frac{1}{2})|n\rangle$.

These $|n\rangle$ states are called number states. Also called as Fock states after Russian Scientist V. A. Fock.

And similarly for the other one that is for the a acting on n will reduce a number of the number of oscillators by 1. And so this is root over n into n plus n minus 1 and this is of course, true except for n equal to 0 because you cannot have 0 minus 1 minus 1 to be a eigenstate of this particular problem. So, it cannot be a member of the vector space that we are talking about. So, because a is 0 that is a acting on 0 a is the as I said that it is a lowering operator. So, if you have already 0 oscillators then you cannot lower it any further. So, a 0 will correspond to 0.

So, the 0 is actually called a vacuum and this state n can be built from a a 0. So, any state n can be built from 0 by repeatedly operating by this a dagger. So, a dagger whole to the power n divided by n factorial and the acting on the ket 0; so, this is how all eigenstates of the quantum oscillator problem can be generated. So, this is very intuitive that you just need to know the ground state or the vacuum here and in fact, the better word for 0 is not a vacuum of course, it does not contain any oscillator. So, in that way it can be written as a vacuum, but this is also the ground state.

So, remember H is equal to a plus a dagger a plus half, so which is n plus half. So, if small n ket, ket small n is an eigenstate of capital N it should also be an eigenstate of the Hamiltonian H because this is what they are simply off by a factor of half which should not create any problem. So, the Hamiltonian operating on the ket n gives me N plus half ket n which gives me small n plus half ket n. So, these n states are called as number states they are also called as fock states after this Russian scientist V A Fock.

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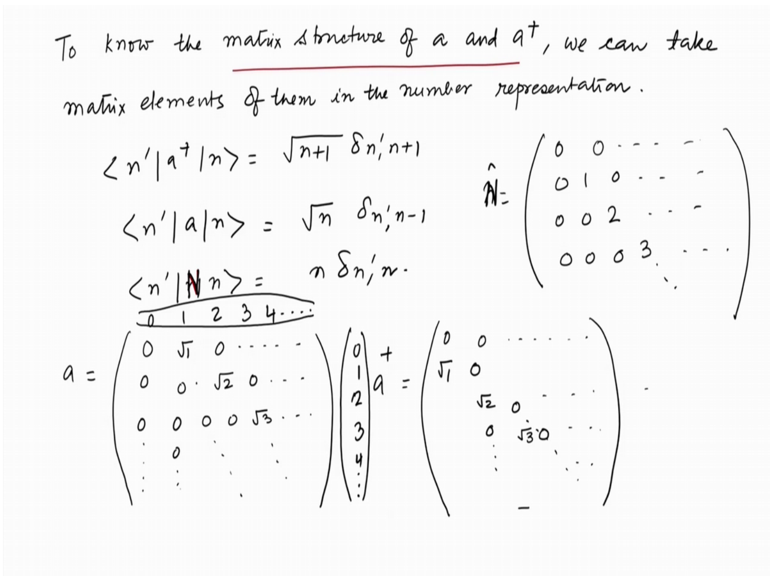

Now, let us try to understand the matrix structure of a and a dagger. Now, it could be possible that and it is it is true that these are not diagonal matrices. they are in fact the diagonal elements are all 0 and these elements which are nonzero they exist on two sides of the diagonal matrix. For example, this relation gives the root over I mean the bar n prime a dagger n which is equal to root over n plus 1 and delta of n prime, so n prime should become equal to n plus 1. And similarly for a it is root over n delta n prime equal to n delta n prime n minus 1 and similarly for the diagonal term which is n prime n. So, this should be capital N which I let me just change it. So, this is capital N let me write it with a black.

And so capital N vector looks like you know it is diagonal with the entries as 0 1 2 3 4 5 6 etcetera etcetera whereas; all the off diagonal terms are equal to 0. While for the a matrix it is the upper diagonal that exists that is basically just the term which is above the diagonal the line above the diagonal that exists and for a dagger the line below the diagonal that exists because of this n prime being n plus 1. So, I write all you know sort of, so I write as if like this. So, I have a 0 1 2 3 4 and so on and 0 1 2 3 4 and so on. So, these are will be the ones that are nonzero are they look like the root over 1, root over 2, root over 3 and so on and similarly here it will be the same, but just lying below the diagonal. So, that is for a and a dagger written in the number basis.

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Combining a and a^+, we can write down the matrix forms of $\hat{x}$ and $\hat{p}$.

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a+a^+)\;;\quad p = \frac{1}{i}\sqrt{m\omega\hbar}\,(a-a^+)$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & \cdots \\ 0 & \sqrt{2} & 0 & \ddots \\ \vdots & & 0 & \ddots \end{pmatrix}\;;\quad p = \frac{\sqrt{m\omega\hbar}}{i}\begin{pmatrix} 0 & \sqrt{1} & & \\ -\sqrt{1} & 0 & \sqrt{2} & \\ & -\sqrt{2} & \ddots & \\ & & & \ddots \end{pmatrix}$$

So x and p operators are infinite dimensional matrices, and hence do not commute.

So, now, one obvious task at hand is to write down the matrix forms for x and p and we just invert the relation we initially wrote down a in terms of x and p. Now, we can write down x in terms of a and a dagger so if that becomes true then its root over h cross by two m omega and then it is all these terms that are coming on both sides. So, it is a basically a plus a dagger. So, whatever is in a will come and whatever is in a dagger will also com. Let me just show that by two different colours. So, the red color is coming from the above the diagonal, and there is a say green color that is coming totally below the diagonal and so on.

And it is as for, so this is for x. So, that is the operator corresponding to the position variable and the operator corresponding to the momentum variable also looks somewhat similar and which are obtained from the linear combinations of a and a dagger. This is just for you to note that look these are infinite operators. So, they are not finite dimensional matrices and in no two infinite dimensional matrices commute and hence x and p they do not commute and so do a and a dagger they are infinite dimensional matrices as well.

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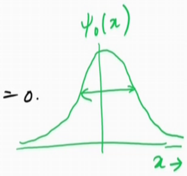
Wave function in position space

We can only talk about the ground state wavefunction because all other states can be generated by successive operation by $a^\dagger$.

Define ground state by $|\psi_0(x)\rangle = \langle x|0\rangle$

$$a|\psi_0(x)\rangle = 0 \Rightarrow \left[\frac{\partial}{\partial x} + \frac{m\omega}{\hbar}x\right]|\psi_0(x)\rangle = 0.$$



$$|\psi_0(x)\rangle = A_0 \exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

Normalization yields $\langle\psi_0(x)|\psi_0(x)\rangle = 1 \Rightarrow A_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$

$$|\psi_n(x)\rangle = \langle x|n\rangle = \langle x|\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle = \frac{\left[\sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\right]^n}{\sqrt{n!}}\langle x|0\rangle$$

So, the wave function in position space that is to connect with what we all know the Gaussian. So, if we define the ground state as a psi 0 x which is equal to x and 0 that the ket the bar x and the ket 0, this notation that we have used earlier. So, a on psi 0 x of course, gives 0 because a annihilates or rather a lowers and this since there is no boson at all or harmonic oscillator at all it cannot lower the number. So, we start from that this is thekinetic energy and this is the the potential energy and then that is acting on the ground state. And if you solve this differential equation the ground state wave function will come as a naught exponential minus m omega by m omega x square by 2 h.

So, if you normalize then a 0 comes out to be m omega by pi h, h cross whole to the power one-fourth and similarly the psi n is equal to x n, and x a dagger by n factorial root over which is on 0 and then we write down this operator which is corresponding to the a dagger. And then operate it on n times to get the nth eigenstate starting from the ground state.

So, as I told earlier that all the excited states of a quantum harmonic oscillator can be obtained from the ground state of this from the ground state, and the ground state as it is shown here is a perfect Gaussian. So, it is a Gaussian, ok. So, this is the psi 0 x and this is x and so on, and this is your the full width of half maximum is actually related to this coefficient m omega by 2 h cross.

So, the whole algebra becomes easy, and this operator method as I said earlier as well that it is possible because the energy spectrum is equidistant from each other and here we have solved the harmonic oscillator problem. And the reason that we introduced this is that we need later to talk about the squeezed and the coherent states which are one of the interesting examples of the oscillator that we are going to do.