Advanced Quantum Mechanics with Applications
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Lecture – 06
Coherent States and their Properties

Now, we shall take up the discussion on Coherent States. This coherent states are very central to the discussion of quantum optics, and was introduced by Glauber, R. J. Glauber in 1963. And it gives a very convenient description of precision, measurements and as well as correlation spectroscopy in quantum mechanics quantum optics rather. And in 2005, Glauber was awarded the Nobel Prize because of these coherent states, discovery of the coherent states and its relation to the optical coherence.

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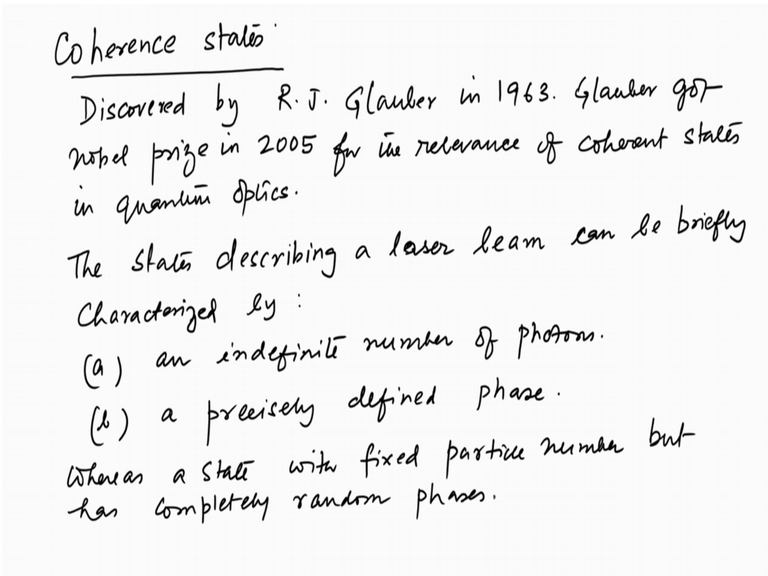

So, we will start with defining what are coherence states and some of the properties of coherent states and then we will talk about some of the applications that could be given in the context of coherent states. So, let us start saying that; so, just to write those things which I have already told, so it was discovered by R. J. Glauber in 1963 3. So, Glauber got Nobel Prize in 2005 for the relevance of coherent states in quantum optics. So, how is it introduced and what are the sort of definitions of a coherent state let us see that.

So, the states describing a laser beam can be briefly characterized by, one is an indefinite number of photons, two is that precisely defined phase. Whereas, the normal light which is incoherent has fixed particle number, but has a completely random phases, ok. So, this is the difference between laser and normal light where we have complete coherence between the phases of the transitions that take place in a laser beam whereas, in an unpolarized light or a normal usual regular light we do not have that kind of a phase coherence.

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Uncertainty relation: $\Delta N \cdot \Delta \Phi \geq \frac{1}{2}$

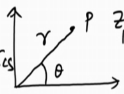
Definition of Coherent states:

A Coherent state $|\alpha\rangle$ (also known as Glauber state) is defined as an eigenstate of the amplitude operator, which is the annihilation operator a with an eigenvalue $\alpha \in \mathbb{C}$,

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

Since a is a non-hermitian operator, α must be a complex number, $\alpha = |\alpha| e^{i\phi}$

α corresponds to a complex wave amplitude in optics



$z_P = re^{i\theta}$

And this is also because there is an uncertainty relation that exists between the particle number fluctuation and the phase fluctuation, and this is written as it is a something like half of its half. So, delta N delta phi is should be greater than equal to half, and if you want to have a very precise control of one then you have to give up the other, which is what happens in normal light in which the particle number is constant or nearly constant which means delta n is nearly 0 which means delta phi that is the fluctuation in the phase is large and just the other way round happens in a laser beam.

So, let us talk about the definition of coherent states. So, a coherent state, alpha we will denote it by alpha all the while it is also called as a Glauber state is defined is defined as an eigenstate of the amplitude operator which is the annihilation operator a that we have seen in the previous discussion on harmonic oscillator, with an eigenvalue alpha belonging to a complex space. So, we will write that a C, so that a alpha is equal to alpha and returns me the ket alpha.

So, this is the definition of a coherent state that it is a state upon which the annihilation operator operates and it is an eigenstate of the annihilation operator returns me back an eigenvalue which is alpha and returns that state the coherent state alpha itself. So, just to remind that these a is the annihilation operator that we have seen in the harmonic oscillator discussion which when acts on a number state N or an occupation number basis given by N, it reduces the particle number by 1. Whereas, the creation operator or which we also have called as a raising operator while this one was also called as a lowering operator. So, a raising and lowering and also a creation and annihilation are equally acceptable terms in this context.

So, a dagger which is a creation operator or the raising operator when acts on a occupation number basis will increase the state the occupation of the state by 1. And just to remind you that an occupation number state N could be built from a vacuum 0 by repeated operation of this a dagger on the vacuum. So, now here of course, we know that a is a non Hermitian operator which means that it is not the same as its transpose and of course, the complex conjugate or the transpose of the complex conjugate and so these are known Hermitian operators both a and a dagger.

So, here we are only talking about a. So, alpha must be a complex number. So, this is what we say it because the eigenvalue in which case is a complex number, so that is why we have said that it belongs to a complex number. And all of you are very well aware that a complex number is written with an amplitude and a phase of this form this should remind you of this z to be a complex number which is equal to r exponential i theta, where theta.

So, we are trying to describe it the polar coordinate. So, r is the magnitude of the distance of the point that we are trying to consider and theta is the angle that that point makes with the x axis or rather the line joining this origin to that point with the x axis. So, just that, so this is your, so if this is a point p then this is equal to r and this is equal to theta and so the point z corresponding to this or the variable z corresponding to this point p is written as re to the power i theta.

So, it is in the same spirit that alpha is equal to alpha and exponential i phi where phi is the phase. And here of course, alpha corresponds to a complex wave amplitude in optics and classical optics what we mean. So, let us just talk about some of the very key properties of the coherent state.

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Properties of the coherent state

1) A vacuum $|0\rangle$ is a coherent state with $\alpha = 0$.

2) Mean energy of the Harmonic oscillator can be obtain between the coherent states,

$$\langle H \rangle = \langle \alpha | H | \alpha \rangle = \hbar\omega \langle \alpha | (a^{\dagger}a + \tfrac{1}{2}) | \alpha \rangle$$

$$= \hbar\omega \left(|\alpha|^2 + \tfrac{1}{2} \right)$$

$\hbar\omega |\alpha|^2$ is the classical wave intensity.

3) Phase shifting operator : $U(\theta) = e^{-i\theta N}$ $\quad N = a^{\dagger}a$

$$U^{\dagger}(\theta)\, a\, U(\theta) = a e^{-i\theta}.$$

$U(\theta)$ is an unitary operator which yields a phase shift to a.

So, just before we begin let us remind ourselves that a vacuum 0 is also a coherent state with the eigenvalue alpha equal to 0. And the mean energy of the harmonic oscillator; so this is say 1, 2 harmonic oscillator can be obtained between the coherent states as you have H is the Hamiltonian. So, these are the coherent states. So, I am taking an expectation value of the Hamiltonian between the coherent states and we know that the coherent state with the Hamiltonian is written as a dagger a plus half with the h cross omega there and so on.

Now, this is equal to because a acting on alpha gives me alpha times alpha and a dagger acting on alpha will give me alpha star and will return me back this state. So, this will be like h cross omega and alpha square plus half. And this is so this to remind you that this half comes because of the 0 point energy of a quantum harmonic oscillator that we are aware of. And so if you want to draw a parallel then this h cross omega alpha square is the classical wave intensity.

Now, we will introduce in fact two operators one is called as a phase shifting operator which shifts the face of a coherent state and also a displacement operator which displaces a quantum state and in fact, the formalism of this coherent states are mainly built up by these phase translation or a phase shifting operator and a displacement operator.

So, third one is a phase shifting operator. And this is defined as u theta which is equal to exponential minus i theta well this theta is the phase we have called that as a phi here. So, let me change this, we have called it phi as well here, but what we mean is by phi or theta is that we want to talk about the phase, but so we can we can carry on with a theta for now and so this i exponential minus i theta n where n equal to a dagger a.

So, we can have a relation that a u dagger theta and this a and acting on u theta should be equal to a exponential minus i theta, where a u theta is an unitary operator which yields a phase shift to a. So, let us see some of the properties or rather try to prove this, with this definition of u theta that we get u dagger theta a u theta gives me a shifting phase for the operator a let us prove that.

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Proof of $\underline{U^\dagger(\theta)\, a\, U(\theta) = a e^{-i\theta}}$

$$\frac{d}{d\theta}\Big(\underbrace{U^\dagger(\theta)\, a\, U(\theta)}_{A}\Big) = i\, U^\dagger(\theta)\,[N, a]\,U(\theta) \qquad \Big[\, [N,a] = -a \,\Big]$$

$$= -i\, \underbrace{U^\dagger(\theta)\, a\, U(\theta)}_{A}$$

$$\frac{d}{d\theta}\Big(\underbrace{a e^{-i\theta}}_{B}\Big) = -i\, \underbrace{a e^{-i\theta}}_{B}$$

Since A and B obey the same differential equation.

$$A = B. \Rightarrow U^\dagger(\theta)\, a\, U(\theta) = a e^{-i\theta}$$

4) The phase shifting operator $U(\theta)$ shifts the phase of the coherent state

$$U(\theta)|\alpha\rangle = |\alpha e^{-i\theta}\rangle$$

So, in order to do that let us take a dd theta of the left hand side. So, that is equal to u dagger theta a u theta which is equal to a i u dagger theta. Since we are taking a derivative with respect to theta, so this will be i u dagger theta and this will be a n a which is u theta. So, we are using the Heisenberg equation of motion for the operator a which is now, you see that we usually use the Heisenberg equation of motion by taking a d dt and it is with the Hamiltonian of the system. Now, it is with two canonically conjugate variables. So, we are taking it derivative with respect to theta and the resultant is written as a commutation of N and a, where N is a number operator. And this we know that N a is minus a, we have done that in the last discussion that. So, N a commutation is minus a and if that is true then this is equal to a minus u dagger theta a u theta.

And now, if you come to the right hand side if you take this d d theta of a e to the power minus i theta. So, this is equal to minus i a exponential minus i theta. So, if you see that these two obey the same differential equation that is we have this is an operator let us write it with a different color maybe. So, this is an operator let us call that as operator A and so I get a minus i into the operator A whereas, if I call this as an operator B and then I get the same equation of motion.

So, we will write that since A and B have same equation of motion the two operators A and B obey the same differential equation then of course, A must be equal to B and that implies that my u dagger theta a u theta is equal to a exponential minus i theta. So, that is the rationale behind this and let us talk about the fourth one which is a little tricky. So, the phase shifting operator u theta, u theta shifts the phase of the coherent state which means you theta acting on a coherent state alpha it gives me it shifts the phase of the coherent state by a theta. And let us try to prove this.

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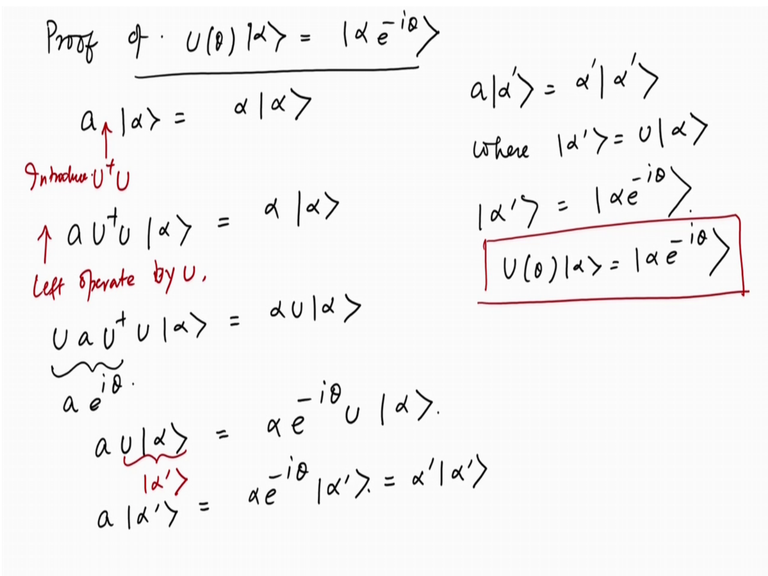

So, proof of u theta alpha, alpha exponential minus u theta. So, what I do is that I write down a alpha equal to alpha alpha. So, this is the defining equation for a coherent state. Now, let us introduce here a u dagger u, so introduce. So, which means that a u dagger u acting on an alpha gives me a alpha and alpha. Now, I if left multiplied by u. So, let us again write this, so that you follow left multiply or operate rather left operate because these are all operators by u. So, that gives of course, which will have to do it for both sides. So, it is au a u dagger u alpha it is equal to alpha u alpha, and this was shown in the last proof that this is equal to a exponential i theta. So, that tells that a u alpha its equal to alpha exponential minus i theta u and alpha.

So, if you call this you alpha to be a state say call it alpha prime. So, it is a alpha prime its equal to, so a alpha prime is equal to alpha exponential minus u theta and an alpha prime, ok. Now, you see the relationship defining relationship of the this coherent state is e alpha equal to alpha alpha. So, now, I can make and this everything to be alpha prime where alpha prime equal to u times alpha.

So, this must be then equal to alpha prime alpha prime and then alpha prime is of course, then equal to alpha exponential minus i theta which gets me to prove that you theta acting on an alpha equal to gives me a state which is shifted in terms of phase by this exponential minus i theta. So, that is the proof of this statement that we have made. Now, let us try to also learn the coherent state a little more in depth as so that we can make use of that. And in this context we will introduce a displacement operator, ok.

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5) Introduce a displacement operator – This operator acts on vacuum and generates a coherent state (Similar to $a^\dagger \to$ acting on $|0\rangle$ generates $|n\rangle$)

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} \quad \text{where } \alpha = |\alpha| e^{i\phi} \in \mathbb{C}$$

The Displacement operator is an unitary operator, $D^\dagger D = \mathbb{1}$.

$$D(\alpha) = e^{A+B} = e^{A} e^{B} e^{\frac{1}{2}[A,B]} \quad \text{where } A = \alpha a^\dagger, \; B = \alpha^* a.$$

$$[A,B] = [\alpha a^\dagger, \alpha^* a] = |\alpha|^2 \underbrace{[a^\dagger, a]}_{=-1} = -|\alpha|^2$$

$$\boxed{D(\alpha) = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a}}$$

And so introduce, so if I see it is number 5, introduce a displacement operator which is similar to the creation operator or the raising operator which when acts repeatedly on the vacuum that sort of generates state in the occupation representation. So, this displacement operator will be similar to that and using this displacement operator on the vacuum we should be able to generate coherent state.

So, this operator acts on vacuum and generates a coherent state. So, this is a similar to similar to dagger acting on 0 or vacuum generates n, ok. So, this is defined as. So, D alpha where alpha refers to a coherent state is defined as exponential alpha a dagger minus alpha star a, where of course, we have defined that alpha is equal to alpha exponential i phi that belongs to C. So, the displacement operator is an unitary operator that is the D dagger D is equal to 1.

And now, let us take this each of these terms to be say A and B. So, a D alpha can be actually written as exponential A minus exponential B and this is equal to. So, this is exponential A exponential or let us write it with a plus sign exponential B and exponential A B. In operator quantum mechanics you will find this relation if they are simple C numbers or classical numbers then of course, exponential x plus y becomes equal to exponential x into exponential y, but if their operators one has to check the commutation between that and the commutation, if only the commutation becomes 0 they can be used as classical numbers, but otherwise not.

So, let us have let us write this as, so where A equal to alpha a dagger and B equal to a alpha star a, and so A B apart from a minus sign associated with this. So, its alpha star a and so this is alpha square and a dagger a and we know that a dagger a equal to 1 this is what we have proved earlier. So, this becomes equal to simply equal to alpha square. So, I think there is a factor of two which I have missed here there is a factor of two.

So, then the D alpha becomes equal to exponential. So, this is equal to a minus 1 rather, so this is equal to a minus 1. And so this is equal to a minus alpha squared. So, this is equal to a minus half alpha mod square exponential alpha a dagger exponential minus alpha star a, and so this is the form for the displacement operator let us just box it so that we can. So, this is the form for the displacement operator which as we said that would generate coherent state starting from a vacuum.

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6. The Displacement Operator $D(\alpha)$ generates a coherent state from vacuum

$$\boxed{|\alpha\rangle = D(\alpha)\,|0\rangle}$$

7. Properties of the displacement operator $D(\alpha)$

a) $D^{\dagger}(\alpha) = D^{-1}(\alpha) = D(-\alpha)$

b) $D^{\dagger}(\alpha)\, a\, D(\alpha) = a + \alpha$.

Proof of $|\alpha\rangle = D(\alpha)|0\rangle$

Use a negative translation to $|\alpha\rangle$

$$a\, D(-\alpha)|\alpha\rangle = \underbrace{D(-\alpha)\, D^{\dagger}(-\alpha)}_{\text{introduced}}\, a\, D(-\alpha)|\alpha\rangle$$

$\underbrace{D^{\dagger}(-\alpha)\, a\, D(-\alpha)}_{a-\alpha}$ from (b).

These so, let us write 6 as the displacement operator D alpha generates a coherent state from vacuum. So, what I mean is that there is a coherent state and this d alpha that acts on a vacuum this of course, is a reminiscent of that equation that which we have talked about that starting from a vacuum, by repeated operation of the creation operator or raising operator we can get a occupation number representation or a rather a state with an occupation number n.

So, if we need to prove that we need to understand properties. So, let us call it 7, properties of the displacement operator D alpha. So, these properties will simply be stated and not proved, but the proofs are not too difficult. So, D dagger alpha which is D minus alpha which is equal to D minus alpha which means it is an inverse translation rather if you translate it forward. So, reminds you of translating it backward. And so this is A and B is a D dagger alpha a D alpha its equal to a plus alpha.

And so, if will use this in order to prove this statement which is circled here. So, we have, so we use a negative displacement to alpha proof of alpha equal to D alpha and 0. So, so use a negative translation to alpha. So, what is meant is that, if I take a D of minus alpha and acting on alpha and now, I operate it by a which is the annihilation operator that we have talked about. So, this is equal to D of minus alpha, a D dagger minus alpha which I simply introduce it because of its unitary. So, this is introduced and this is equal to a and D a and D minus alpha acting on and acting on a state alpha, ok.

So, I took a state which is a D minus alpha acting on alpha I have introduced this D minus alpha d dagger which is equal to 1 and then written the rest of the states. Now, I need to look at this part which gives me minus alpha from this property number 2, from b.

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$$a\,D(-\alpha)\,|\alpha\rangle = D(-\alpha)\underbrace{(a-\alpha)\,|\alpha\rangle}_{=0} = 0.$$

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle$$

$$\underbrace{D(-\alpha)\,|\alpha\rangle}_{\Downarrow\ \text{a vacuum state.}} = 0$$

$$a\,|0\rangle = 0$$

$$\boxed{|\alpha\rangle = D(\alpha)\,|0\rangle}$$

So, I have used both a and b, so if you see that then it becomes equal to a D minus alpha acting on a state coherent state alpha which is equal to a D minus alpha a minus alpha alpha. Now, that is equal to 0 let us see why that is equal to 0 because my a alpha is equal to alpha alpha. So, if on this side I see this part. So, this is equal to 0 because a acting on alpha will give me an alpha times alpha, so a minus alpha operating on alpha would give me 0.

So, that is reason that my, so a D minus alpha acting on alpha should give me 0 which means that this is actually a vacuum state because only when a acts on a vacuum it gives me 0. So, because is acting on D of minus alpha and the state alpha then if this is a vacuum then only this the right hand side could be equal to 0, so which means my alpha equal to a D alpha on 0. So, this is reminiscent as I told you earlier that generating a number state n from the vacuum. So, this displacement operator does exactly the same job of doing it in the or doing it for the coherent state. So, it acts on a number operator and it acts on a vacuum and gives me a coherent state. So, we can now, go to the coordinate representation or the occupation basis representation of the coherent state.

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Coherent state in the number representation

$$|\alpha\rangle = \sum_{n=0}^{\alpha} \left(|n\rangle\langle n| \right) |\alpha\rangle$$

$$= \left(\sum_{n=0}^{\infty} |n\rangle \langle 0| \frac{a^n}{\sqrt{n!}} |\alpha\rangle \right)$$

Operating, $\frac{a^n}{\sqrt{n!}}$ on $|\alpha\rangle$ $\qquad a|\alpha\rangle = \alpha|\alpha\rangle$

$$= \langle 0|\alpha\rangle \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = |\alpha\rangle.$$

$$\langle 0|\alpha\rangle = \left(\sum_{m=0}^{\infty} \langle 0| \frac{(a^{+})^m}{\sqrt{m!}} |\alpha\rangle\langle m| \right) \left(\sum_{n=0}^{\infty} \langle 0| \frac{(a^{+})^n}{\sqrt{n!}} |\alpha\rangle\langle n| \right)$$

$$= 1$$

So, the coherent states in the number representation is written as, so this coherent state is expanded in the number basis as using the completeness of the number states because they form a complete state is this. And this as I as we know that it is equal to n equal to 0 to infinity and n and now, we will use a 0 and a to the power n by root over n factorial and on alpha. So, that is our representation.

Now, so this was operating this operator on alpha. So, a to the power n divided by n factorial on the coherent state alpha that gives me equal to a 0 alpha and n equal to 0 to infinity, and alpha to the power n divided by n factorial and operating on n because remember that a alpha equal to alpha alpha. So, this is equal to a state alpha.

Now, this 0 alpha that is written here is written as a 0 alpha is determined as, so there is a m equal to 0 to infinity we are using a dummy index here a dagger to the m root over m factorial and alpha m and similarly for another dummy variable n. So, this is equal to a 0 a dagger n by root over n factorial alpha n and so on. So, this is equal to 1 which you can check.

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Thus we find $|\langle 0|\alpha\rangle|^2 = \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!}\right)^{-1} = \exp(-|\alpha|^2)$

Finally, the coherent state can be written as,

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$

Thus a coherent state is a superposition of number states (Fock states).

The probability of getting a Fock state $|n\rangle$ in a coherent state is given by,

$$P_n(\alpha) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{\alpha^{2n}}{n!}$$

Poisson distribution with a mean $|\alpha|$

So, we actually find this a 0 alpha mod square equal to nothing but square nothing, but equal to a n equal to 0 to infinity alpha to the power n rather 2 n divided by n factorial and inverse of that and which is equal to nothing but the expansion of minus alpha square, ok. So, finally, the coherent state can be written as a alpha are equal to exponential minus alpha square by 2 and n equal to 0 to infinity alpha to the power n n factorial root over and n. So, this is the expansion of a coherent state in terms of the number states. So, coherent state is a superposition of number states or which are also known as Fock states.

Now, the probability of getting a Fock state n in a coherent state in a coherent state is given by P n, no this P n alpha which is equal to n alpha square mod square which is equal to minus alpha mod square alpha to the power 2 n divided by n factorial. So, this is a nice relationship because it gives you a transforming from one basis to another. So, this the probability of finding number occupation number state which is also called as Fock state in a coherent state is given by this distribution which is a Poissonian distribution. So, it is a Poisson distribution, and with a mean alpha square. So, this is the probability of a Fock state in a coherent state.