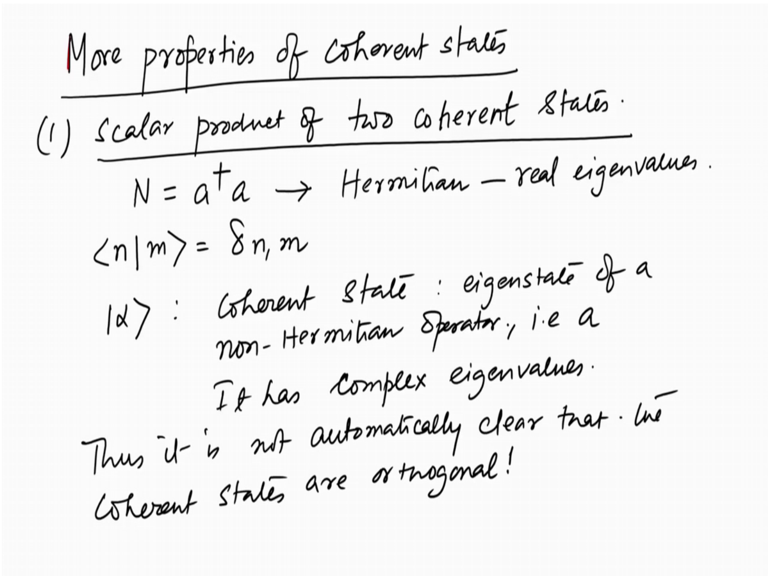
Advanced Quantum Mechanics with Applications
Prof. Saurabh Basu
Department of Physics
Indian Institute of Technology, Guwahati

Lecture – 07
Applications of Coherent States, squeezed states

To continue with our discussion on the Coherent States, let us see some more properties on the Coherent of States. And let us start with the scale scalar product of two coherent states.

(Refer Slide Time: 00:36)



So, let us see what we understand or what we learn from this scalar product. Of course, you know that the Fock states which are the number occupation number states denoted by N which are the eigenstates or eigenfunctions of this operator N, capital N which has a form which is a dagger a this is a Hermitian operator which means that it gives a real eigenvalues and also that this are orthonormal states.

So, one Fock basis is or orthonormal to another Fock basis. And so however, the state alpha which is a coherent state it has complex eigenvalues and it is an eigenfunction of the operator a which is lowering operator which is not a Hermitian operator. So, its eigenstate of non-Hermitian operator let us write it here.

So, please take a note that we have explained this or rather we have stated all of these facts earlier and this non-Hermitian operator is a that is a lowering operator and it has complex eigenvalues. Thus it is not clear it is a not automatically clear, that the coherent states are orthogonal which is and of course, they are distinct than the Fock states as we have seen earlier and so its needs to be ascertained that whether these are orthogonal states.

(Refer Slide Time: 04:34)

Two coherent states $|\alpha\rangle, |\beta\rangle$

$$\langle\beta|\alpha\rangle = \langle 0| D^{+}(\beta) D(\alpha)|0\rangle$$

$$= \langle 0| e^{-\beta a^{+}} e^{\beta^{*} a} e^{\alpha a^{+}} e^{-\alpha^{*} a} |0\rangle e^{-\frac{1}{2}(|\alpha|^2 - |\beta|^2)}$$

$$e^{-\beta a^{+}} = (1 - \beta a^{+} + \cdots)$$

$$e^{-\alpha^{*} a} = (1 - \alpha^{*} a + \cdots)$$

$$\langle\beta|\alpha\rangle = \langle 0|\left(1 + \beta^{*} a + \frac{1}{2!}(\beta^{*} a)^2 + \cdots\right)\left(1 + \alpha a^{+} + \frac{1}{2!}(\alpha a^{+})^2|0\rangle + \cdots\right) e^{-\frac{1}{2}(|\alpha|^2 - |\beta|^2)}$$

$$= \left(\cdots + \langle 2|\frac{1}{2!}\sqrt{2!}(\beta^{*})^2 + \langle 1|\beta^{*} + \langle 0|\right)\left(|0\rangle + \alpha|1\rangle + \frac{1}{2!}\sqrt{2!}\,\alpha^2|2\rangle + \cdots\right) e^{-\frac{1}{2}(|\alpha|^2 - |\beta|^2)}.$$

And in order to do that let us take two coherent states alpha and beta and look at the inner product between them I am just writing it as a beat beta alpha which we know that by definition its D dagger beta D alpha 0.

So, this can be written as, so these 0s are the vacuum states and this D dagger and D dagger beta and D alpha are basically the translation operator that we have learnt earlier. So, this is a exponential minus beta a dagger exponential beta star a exponential alpha a dagger exponential minus alpha star a this is the definition that we have studied for this and along with there is a Gaussian factor which is equal to alpha square minus beta square. So, this is the expectation and our rather the inner product of the two coherence states. Now, it has to be seen that whether these two coherent states are orthogonal.

Now, in order to see that let us see the end states that is this one and this one and they can be expanded in terms of the basically the exponential can be expanded which is exponential minus beta a plus and all that and then you have an exponential minus alpha a rather alpha star a which is equal to 1 minus alpha star a and plus all that. Now, it is very clear that all these terms with a dagger and a in both at the edges will yield 0 when they act on vacuum.

So, the ones that are saved or rather that are survive those survived are the ones and in which case we have we can actually ignore them and we can write this down as a 0 and now, we do the expansion to the other quantities as well which are this and this. So, this is equal to 1 plus beta star a plus 1 by 2 factorial beta star a square plus of all that and then we have a. So, and then we have a 1 plus alpha a dagger plus 1 by 2 factorial alpha a dagger square and so on and then its 0 and of course, these are there.

And of course, we also have a the exponential this factor which will let us write them here exponential minus half alpha square minus beta square. So, this yields, so then this is like 2, 1 by 2 factorial root over 2 factorial. So, this you will get it if you see it carefully by multiplying term by term and a plus um 1 beta star and a plus a 0 which are for different values of these for basically the Fock bases which would generate by successively you know operating the a dagger and things like that a or a dagger and this has to be taken. So, this is really a bracket not an angular bracket.

And then this is multiplied by a 0 plus alpha 1 plus a 1 by 2 factorial root over 2 factorial alpha square 2 and all that plus all this and so on. So, this and of course, this whole thing is multiplied by the exponential half just write it neatly. So, this is alpha square minus beta square and so on. So, that is that is what comes out of this a inner product of two coherent states.

(Refer Slide Time: 09:53)

Using, $\langle n|m\rangle = \delta_{n,m}$

$$\langle\beta|\alpha\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}\Big(\underbrace{1+\alpha\beta^* + \frac{1}{2!}(\alpha\beta^*)^2+\cdots}_{e^{\alpha\beta^*}}\Big)$$

$$= e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2+\alpha\beta^*)}$$

$$|\alpha-\beta|^2 = (\alpha-\beta)(\alpha^*-\beta^*) = |\alpha|^2+|\beta|^2-\alpha\beta^*+\alpha^*\beta.$$

$$|\langle\beta|\alpha\rangle|^2 = e^{-|\alpha-\beta|^2}.$$

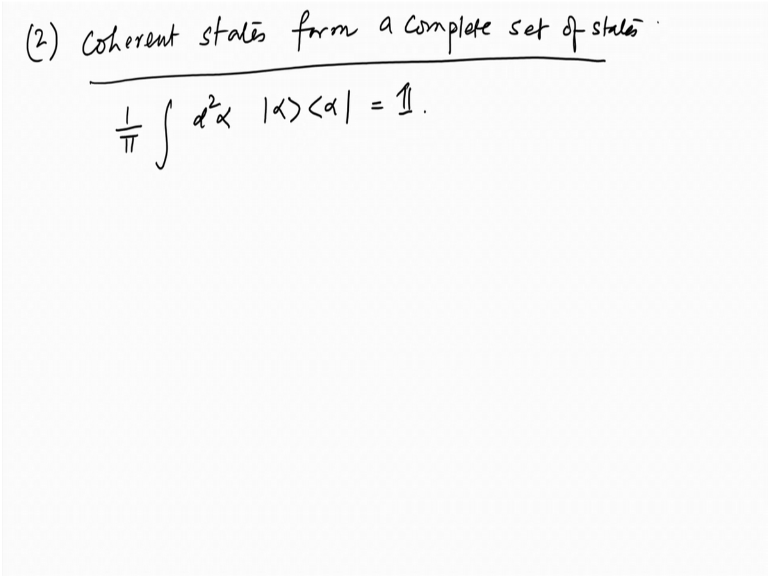
In general Coherent states are not orthogonal. The transition probabilities only vanish if $|\alpha-\beta|>>1$.

And just to remind you that we are trying to find out that whether two coherent states are orthogonal or they form a complete set of states. And so this if you use the orthogonality of the Fock bases that is this equal to delta n m then your beta alpha becomes equal to exponential minus half. So, there is a sign problem that I have introduced from the beginning there is a plus there and there is a plus, there is a plus there.

So, this is alpha mod square plus a beta mod square and a 1 plus alpha beta star plus a 1 by 2 factorial alpha beta star and so on plus. So, this is nothing but exponential that is the expansion of the exponential alpha beta star. So, its exponential alpha beta star, so this is equal to exponential minus half alpha mod square plus beta mod square plus alpha beta star. Now, if you look at this alpha minus beta mod square.

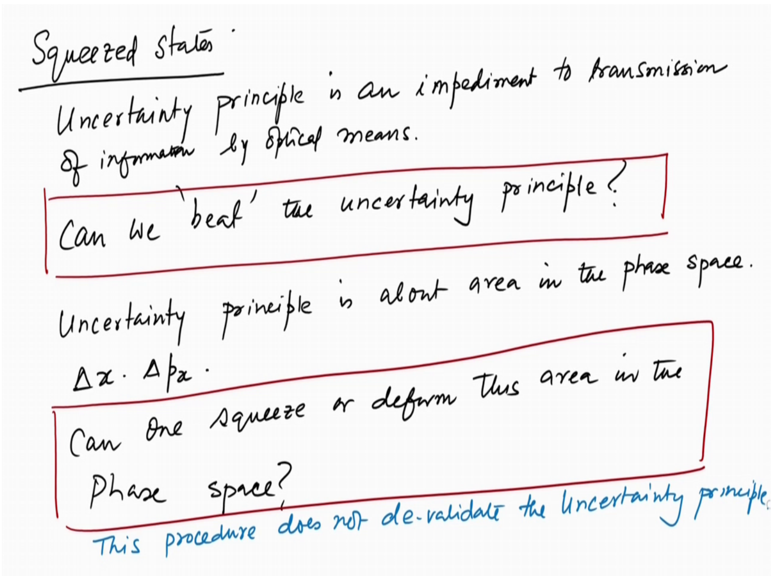
So, that can be written as alpha minus beta and alpha star minus beta star which is equal to alpha mod square if you just open up that is multiplied term by term its alpha square plus beta square minus alpha beta star plus alpha star beta and the. So, this is equal to, so beta alpha mod square is nothing, but equal to exponential alpha minus beta square. So, the so the main thing is that after doing all this algebraic turns out that the coherent states are not orthogonal and in fact, they are only orthogonal when alpha is much greater than beta or alpha minus beta is much greater than 1. So, in general coherent states are not orthogonal. The transition probabilities only vanish if alpha minus beta much greater than 1. So, this is the conclusion for this.

(Refer Slide Time: 13:15)



And, but of course, they form a complete set of states. So, this I will not prove it here, but one can show that which we may actually do it in a tutorial that the coherent states form a complete set of states. So, even if they are not orthonormal to each other they still form a complete set of states and which can be shown by showing that this summing over the coherent states this gives 1 and so on. So, these are some of the properties of the coherent states.

(Refer Slide Time: 14:13)



Let us now go to study what are called as squeezed states, ok. So, this is very well known that the uncertainty principle which Heisenberg had proposed puts an impediment in the in the study of various quantum systems or rather positional measurements of quantum systems. And this of course, is sort of dampens you know the enthusiasm by the quantum engineers where they want to do a lossless transmission of quantum information by optical means. So, just to summarize that uncertainty principle is an impediment to transmission of information by optical means.

So, there are of course, quantum noises. So, quantum noise is an essential ingredient to optical communication. So, the very relevant and pertinent question in this regard is that can we beat the uncertainty principle or can we reduce the effect of the uncertainty principle, and so this is the question that we want to box it and possibly the squeeze states give a clue to this answer. So, it is important to understand that the uncertainty principle is actually a statement about the area in the phase space.

So, is about area in the phase space which what I what we mean by that is that it is this product of a small you know the pixel in the phase space. So, if we can divides the procedure to deform and squeeze this area which can be effectively used for reducing the noise. Such squeezing process is of course, it does not de validate the uncertainty principle, but it is just an engineering that one can think of doing and the possible means of doing it is via squeezed states we will see what that is.

So, the question that one asks is that can one squeeze or deform this area in the phase space, phase space, ok. Again a question that we want to box because it is a very relevant question and to keep in mind that this procedure does not de validate the uncertainty principle that we all are familiar with. So, let us see that how it can be squeezed.

(Refer Slide Time: 19:02)

Consider 2 non-Commuting Hermitian operators A and B which are conjugate to each other. $[x, p]$

$$[A, B] = iC \qquad C \text{ is another Hermitian Operator}$$

The uncertainty principle satisfies,

$$\Delta A \cdot \Delta B \geq \frac{1}{2} |\langle C \rangle|$$

$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$ etc.

The expectation values are computed within a state $|\psi\rangle$.

This $|\psi\rangle$ will be called as a minimum uncertainty state,

if $\quad \Delta A \, \Delta B = \frac{1}{2} |\langle C \rangle|$

Before that let us try to see some basics which we already are quite aware well aware of and consider two non-commuting operators a and b I mean which are which are conjugate to each other each other such that x and p committed, ok. So, let us do this commutator and let us write this as i C, where i C is another. So, these are Hermitian operators just like our p and x are. So, C is another Hermitian operator. So, in general it is a Hermitian operator, but it could be just a C number in some cases that we will just see in a moment.

So, this is the commutation relation and it does not commute, ok, the commutation of these two Hermitian operators give me another Hermitian operators which is i C and the uncertainty principle satisfies delta A multiplied by delta B that should be greater than equal to half of C, where C is the as I told it is an Hermitian operator and we have taken an expectation value of that operator and this expectation value is taken with respect to some state psi. And as usual our definition of delta A is nothing but A square average minus A average square and similarly for B and C, etcetera.

So, as I told that the expectation value of A, B and C are calculated within some given state psi and this state psi the expectation values are computed within a state psi, and this psi will earn a name as a minimum uncertainty state if the one of the observables say delta A will satisfy certain relation we will just write that. So, this psi will be called as or rather will be called as a better word, will be called as a minimum uncertainty state, state provided um. So, if a delta A and delta B they follow an equality rather than an inequality which appears here. So, that is the minimum answer or rather that is a minimum uncertainty and they would follow this relation where an equality exists.

(Refer Slide Time: 23:37)

These states will be called as squeezed if one of the observables, say A, satisfies,

$$(\Delta A)^2 < \frac{1}{2}|\langle C\rangle|^2$$

Alongwith $\Delta A \, \Delta B = \frac{1}{2}|\langle C\rangle|$

This is called as ideal squeezed states.

Thus quantum fluctuations of a squeezed state in one of the observable, say A are reduced to values lower than $\frac{1}{2}|\langle C\rangle|$ at the expense of fluctuations in the other observable.

Now, these states will be called as squeezed if one of the observables say a satisfies a delta A square should be less than half C square of course, along with that the minimum uncertainty condition has to be satisfied. So, a state can actually satisfy a minimum uncertainty condition, but may not satisfy the squeezed condition as its written here. But for a squeezed state to occur this condition has to satisfy it has to be satisfied along with along with what we have just said. So, delta A delta B should be equal to half of a C and this will be called as, so this is called as ideal squeezed states.

So, the whole idea is that the quantum fluctuations of a squeezed state in one observable say A are reduced below these half C, if at the cost of the fluctuations in the other observable. So, let us write this because it is an important statement thus quantum fluctuations of a squeezed state in one of the variables or observables say A are reduced are reduced to values lower than half of mod C at the expense of fluctuations in the other observable. So, let us give some examples in order to validate our claims or at least to check the definition of the squeeze state.

(Refer Slide Time: 27:51)

Say $A = \frac{1}{2}(a + a^{+})$

$B = \frac{1}{2i}(a - a^{+})$

$[A, B] = \frac{i}{2}$ $\quad C \equiv 1$ (constant).

$\Delta A \, \Delta B \geqslant \frac{1}{2}$.

Now let us consider the expectation values of A, B and the variances ΔA and ΔB within the coherent states.

Coherent state : $|\alpha\rangle = D(\alpha)|0\rangle$

$\langle\alpha|a|\alpha\rangle = [\langle A\rangle_\alpha + i\langle B\rangle_\alpha] = \langle 0| D^{+}(\alpha) \, a \, D(\alpha)]|0\rangle = \alpha.$

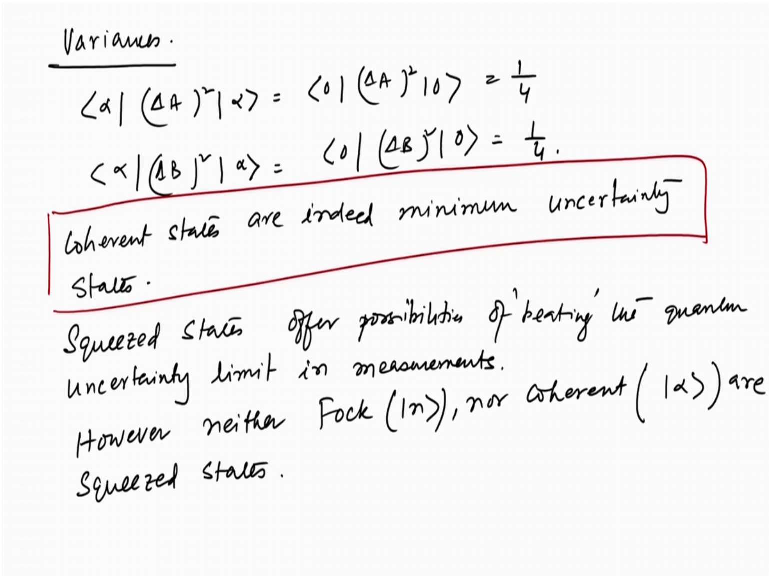
$\langle A\rangle_\alpha = \mathrm{Re}\,\alpha, \quad \langle B\rangle_\alpha = \mathrm{Im}\,\alpha$

So, you remember that we have introduced the canonical variables or observables x and p in terms of a and a dagger and we are going to take those say A equal to half of a plus a dagger and B equal to 1 by 2 i a minus a dagger. So, if you simply so, these are our x and p as we have said that these applies to two canonically conjugate variables which are which could be x and p. So, if I take A and B commutator and want to see that how it behaves this commutator has a value which is i over 2. So, it is very clear that C is equal to 1 here which is a constant.

So, the Heisenberg uncertainty principle takes delta A into delta B should be greater than half, ok. So, that is the uncertainty principle. Now, let us consider the expectation values of A and B and the variances delta A and delta B within the coherent states, ok. So, define so coherent state is given by alpha which is equal to a D alpha and 0, just to reiterate the notation 0 is the vacuum D alpha is the translation operator which produces an a state coherent state alpha that is written with a ket on the left hand side. So, my alpha a alpha that is equal to A alpha plus i B alpha.

So, this is equal to 0 D alpha a D alpha and this is nothing but equal to alpha. So, it is very clear that A alpha is equal to real value of alpha and B alpha is equal to imaginary alpha, alpha. So, one of the canonical variables has an expectation with respect to the coherent state which is real part of alpha. And the other one the imaginary one has got an overlap in the coherent state as imaginary beta sorry I mean some imaginary alpha where alpha is of course, a coherent state.

(Refer Slide Time: 31:38)



So, now, if we want to calculate the variances, so that is equal to alpha delta A square alpha which is equal to 0 or delta A square 0 equal to one-fourth alpha delta B 0 alpha which is equal to 0 delta B 0 which is equal to one-fourth itself.

So, the coherent states are indeed minimum uncertainty states, but whether they are squeezed states or not that we are not sure, but they are definitely minimum uncertainty states. So, these are so. So, now, the squeezed states as we said earlier offer possibilities of beating the quantum uncertainty, the quantum uncertainty limit in measurements. So, let me box this is an important statement. So however, a neither Fock which is n nor coherent which is alpha are squeezed states.

(Refer Slide Time: 34:13)

In fact, $\Delta A = \Delta B = \frac{1}{2}$ for coherent state.

$\Delta A = \Delta B = \frac{1}{2}(2n+1)$ for the Fock states.

A squeezed state can be obtained from a coherent state by applying a squeezing operator,

$$S(\xi) = e^{\xi^* a^2/2}\, e^{-\xi a^{\dagger 2}/2}.$$

$S(\xi)$ acting on a vacuum $|0\rangle$

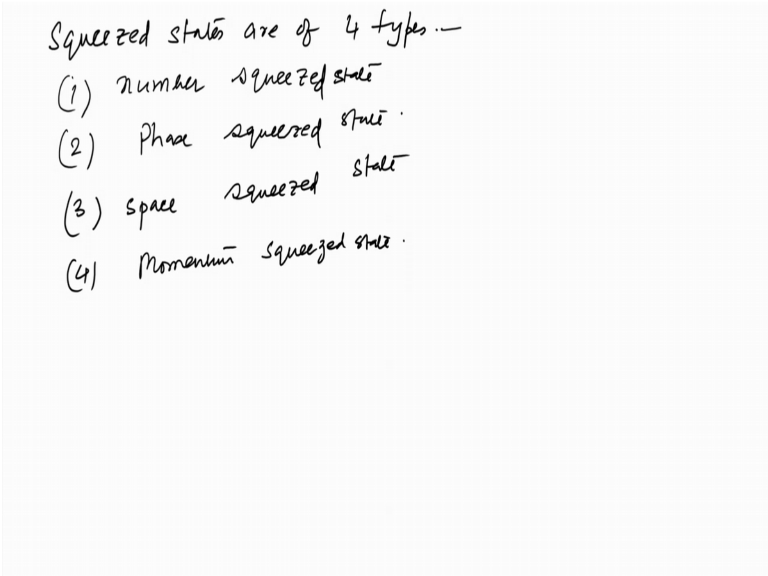
$$S(\xi)|0\rangle = |\xi\rangle.$$

$$S(\xi)|\alpha\rangle = |\alpha, \xi\rangle$$

So, in fact, one can actually calculate that that delta A equal to delta B equal to half for coherent states whereas, delta A equal to delta B equal to half 2 n plus 1 for the Fock. So, this is for the Fock states. So, it is clear that the Fock states are actually the states in the occupation number basis which was quite largely used in dealing with quantum harmonic oscillator. So, a squeezed state can be obtained from a coherent state by applying a squeezing operator which is S by xi, which is xi star a square by 2 exponential minus xi a dagger square by 2.

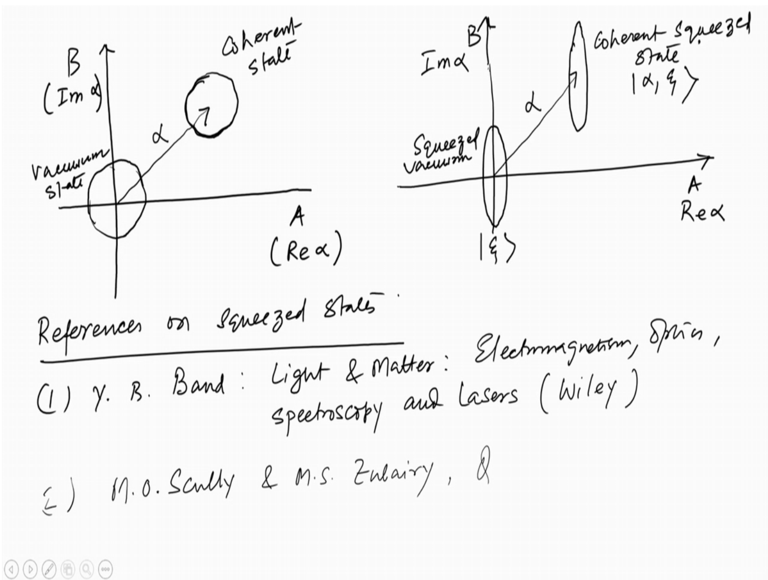
So, S xi, S xi acting on a vacuum which is 0 now it is important to understand that the coherent states are actually built from the vacuum states by the translation operator. So, if the translation operator yields one then coherent state yields a vacuum and there is a special vacuum state, and so if S xi is made to act on 0 that gives me a state coherent state xi. So, this acting on 0 is gives me xi and S of xi acting on alpha gives me another state which is alpha xi. So, these are the so, they show that S xi operating on the trivial coherent state gives a xi which is a coherent state, and a S xi acting on a coherent state gives another coherent state which is alpha xi.

(Refer Slide Time: 37:30)



So, the in fact, the squeezed states are of 4 types, one is the number squeeze states, two is the face squeeze state, and of course, in addition to that we have a space squeeze state and a momentum squeeze states. So, let us give some demonstrations or pictorial representations of these states.

(Refer Slide Time: 38:39)



So, let us look at. So, this is my A axis which is real alpha and this is my B axis which is imaginary alpha, and so there is a vacuum state and this vacuum state can be translated to get a coherent state pardon this shapes that are not coming very well. So, these are intended to be circular. So, these are squeezed. So, there is a vacuum state and this is a coherent state and coherent state which is all we know that. So, there is a D alpha operation that gives a state, so a vacuum state becomes a coherent state and similarly for the squeezing part.

So, there is a this, and then this, this. So, this is A, this is B again, this is real part of alpha, and this is imaginary part of alpha, and so this is of course, a squeezed vacuum. So, we have simply written it as xi and again this is operated upon and one gets coherent squeeze state. And this is given as alpha xi according to the definitions that we have given. So, this is more or less the introduction about the coherent states and squeezed states they are taken up as applications of the quantum mechanics on quantum harmonic oscillators that we have learnt.

And of course, there are many many examples that are used I mean in optical communications, in teleportation and so on and various branches of optomechanics. We will not go into very specific details of these applications, but rather referred to you or, so references on squeezed states are one is a Y B Band. So, it is a Light and Matter and Electromagnetism, Optics, Spectroscopy and Lasers, and it is a Wiley publications.

Number 2 is Scully and M. S. Zubairy, and so this is a quantum optics book we will give you this let us let me just type out this, and this is from the Cambridge University Press, ok. So, these are some of the references on the squeezed and the coherent states which me you may want to look at. And we will move on to another topic.