Advanced Quantum Mechanics with Applications
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Lecture – 08
Symmetries and Conservational Principles in Quantum Mechanics

So, now we are going to talk about the Symmetries and the Conservational Principles in Quantum Mechanics.

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Symmetries and Conservation principles in Quantum Mechanics.

Symmetries — (1) Discrete Symmetry – Inversion (parity)
Time reversal etc.

(2) Continuous Symmetry – Rotation,
Translation.

Example

(1) If Hamiltonian commutes with displacement operator, then momentum operator is constant of motion ⇒ linear momentum is a conserved quantity.

(2) Hamiltonian commutes with the Rotation operator, then angular momentum is conserved.

So, what do you mean by this is the following. Corresponding to a particular symmetry that is present in a given quantum system there are certain quantities or certain observables which are constant of motion, and corresponding to that constant of motion there are some conserved quantities, which help us to simplify many aspects of a given problem. And these simplifications have helped us in developing various things that we will see along the way.

And so a very this is known to you from your first course of quantum mechanics that, a particular operator commutes with the Hamiltonian then they share the same set of eigenfunctions and they can be diagonalised simultaneously. So, these are basically the symmetric properties of that Hamiltonian and corresponding to that some quantum number which is a good quantum number for that operator would remain conserved.

We will explain that as we go along and so symmetries are usually of two kind. So, one is discrete symmetry or let this so discrete symmetry for example, such as inversion or parity which is known as parity which we will discuss and may be time reversal etcetera. Whereas, there are other examples of continuous symmetry, such as rotation and other things such as you know even like translation etcetera these are. So, corresponding to this symmetry transformations that is the system remains in variant under a particular operation such as say translation or rotation there is there are some generators which would be conserved and it denotes a constant of motion.

So, for example, if a Hamiltonian commutes with a displacement operator then momentum operator is constant of motion of motion or a in other words this implies that momentum is a conserved quantity. So, p or k they can be called as good quantum numbers. So, here of course, we are taking about linear momentum and similarly the Hamiltonian. So, this is example 1 and this is 2. So, Hamiltonian commutes with the rotation operator which also can be read as generators of rotation then the angular momentum is conserved, all right.

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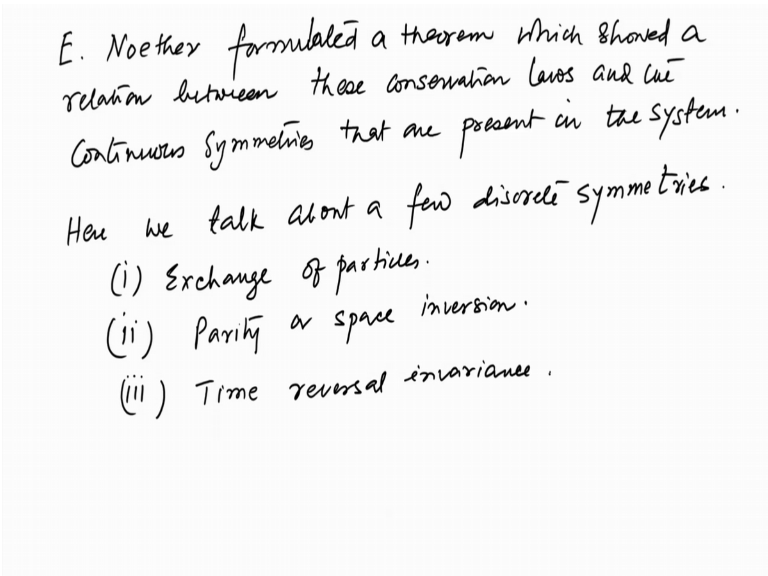

So, Emmy Noether formulated a theorem which showed a relation between these conservation laws, and the continuous symmetries that are present in the system. This you might have also read in classical mechanics in the name of Noether's theorem.

So, here of course, we will talk about a few discrete symmetry principles, symmetries and they are namely say exchange of particles then of course, we will also talk about continuous symmetries and give a few examples which can be given. So, these are the discrete symmetries are exchange of particles and parity or space inversion, and 3 time reversal invariants.

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(1) Exchange of particles.

(a) Fermions : Antisymmetric wave function, $\psi(\vec{r}_1, \vec{r}_2) = -\psi(\vec{r}_2, \vec{r}_1)$

Pauli's exclusion principle.

In a many particle system,

$$\psi(\vec{r}_1, \vec{r}_2 \dots \vec{r}_i \dots \vec{r}_j \dots \vec{r}_N) = \underbrace{-}_{(-1)^p}\psi(\vec{r}_1, \vec{r}_2 \dots \vec{r}_j \dots \vec{r}_i \dots \vec{r}_N)$$

These properties are encoded in Fermi Dirac Statistics.

(Pauli, Fermi and Dirac (1926)) $\rightarrow$ statistical properties of fermions at low temperature.

So, let us see one by one and we will only discuss this briefly. So, one is exchange of particles. So, this the Fermions and the bosons have very distinct properties with regard to the exchange of particles. So, of course, let us talk about the Fermions to begin with. So, they have anti symmetric wave function. What we mean is that we have a psi r 1, r 2, so that is equal to minus psi r 2 r 1. And also there is a Pauli's exclusion principle and in a many particle system a psi r 1, r 2, r i, r j and r n this is equal to minus psi r 1, r 2 and r j r i, r n.

So, as we change the coordinates of a one particle or rather interchange the coordinates of two particles that is r i going to r j and r j going to r i it picks up a negative sign and as we do it twice it picks up another negative sign. So, it becomes equal to positive. So, what it means is that under even number of such transposition or change in the positions of particle you will have a plus sign appearing which is, so this is minus 1 whole to the power. So, this factor is like minus 1 whole to the power p, where p is the number of a changes that are made and so this p is even then we will have a plus sign and p is all we will have a minus sign. So, this is also a as I said that it is also true in many particle sense.

And these properties encoded or rather these properties are encoded in a Fermi Dirac statistics, statistics which are developed by Pauli, Fermi and Dirac this is in 1926. So, which basically describe the statistical properties of Fermions at low temperature.

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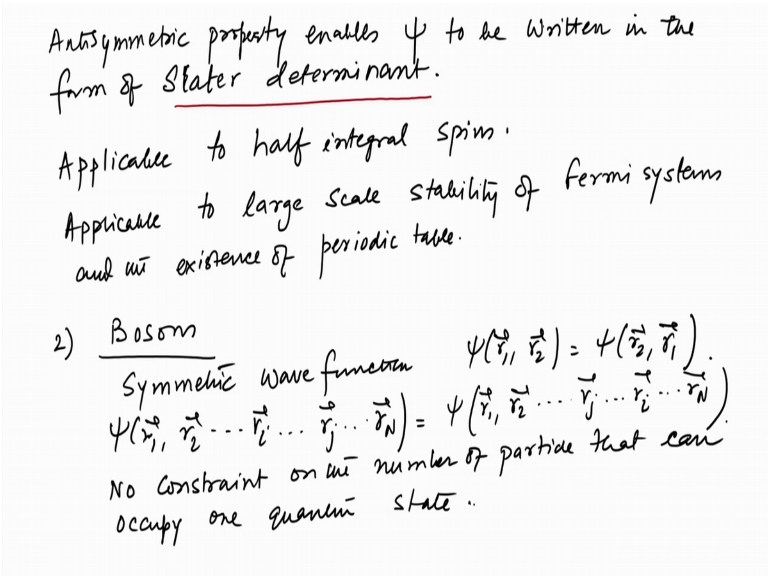

It is important to understand that this anti symmetric property enables writing the wave function in the form of a Slater determinant, psi to be there is a many particle psi that we are talking about to be written, ok. So, what are Slater determinants? They are the following that, basically the wave function is written in terms of matrices and we know that if we change the row or a column interchange a row or a column with another then it picks up a negative sign the determinant picks up a negative sign. And this is exactly what is required for the wave function of a many particle Fermi system and also if two of the rows or the column becomes same identical then the determinant vanishes and this is nothing, but the exclusion principle.

So, these are encoded here and they are of course, for half integral spins, these are applicable to the half integral spins. And the statics of course, has a very great applicability for, and applicable to large scale stability of Fermi systems and the existence of periodic table.

So, let us talk about Boson's which correspond to symmetric wave functions. So, symmetric wave function means the symmetric with respect to the interchange of particles which means that psi r 1, r 2 equal to psi r 2, r 1 and in a many particle sense a psi r 1, r 2, r i, r j, r n equal to psi r 1, r 2, r j, r i, r n. So, and of course, there is no constraint on the number of particles that can occupy one quantum state.

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<u>Important Consequences.</u>

S. Bose and A. Einstein (1924)

Bose Einstein Condensation. (BEC)

<u>Space inversion or Parity.</u>

Think of a Parity Operator, P

$P\psi(\vec{r}) = \psi(-\vec{r})$. : Even parity

$P\psi(\vec{r}) = -\psi(-\vec{r})$: odd parity.

$i\hbar \frac{\partial P}{\partial t} = [P, H] = 0 \Rightarrow$ Parity is conserved.

$\vec{r} \to -\vec{r}$, $\vec{p} \to -\vec{p}$, $\vec{J}$ (or $\vec{L}$) $\to \vec{J}$ (or $\vec{L}$)

And of course, the important consequences are very well known. So, this was by Bose and Albert Einstein in 1924, when they were actually trying to formulate the correct statistics for photons and immediately afterwards a Bose Einstein condensation was proposed below a certain critical temperature for a given bosonic system where all the particles will macroscopically occupy one quantum state. And of course, this is the buildup of particles in the momentum space corresponding to a momentum k equal to 0.

So, that is a lowest momentum value which all the particles go and occupy below certain temperature and this was later verified in or experimentally observed by rubidium atoms and other bosonic atoms in 95. The main impediment was actually achieving a very low temperature which could be done after you know many years of research in getting or achieving a low temperature. So, that one can actually see condensate to form. And this was done by group in Colorado at boulder in 1995. And then soon afterwards large number of groups have made this BEC which is, so this is called as a BEC, BEC possible in a variety of systems, alright.

Let us go over to the next symmetry principle which is let us call it as a parity or the space inversion, so space inversion and parity. So, space inversion or parity. And so basically think of a parity operator P, where P psi r equal to psi of minus r. So, this is called as even parity when the parity operation, so parity operator operating on the wave function does not change the sign of the wave function. So, this called even parity and acting on this gives me minus sign this is called as the odd parity.

So, the equation of motion is i h cross del p del t its equal to P H. So, if the parity operator commutes with the Hamiltonian this commutator is equal to 0, and one says that the parity is conserved um. So, which means that the system has a right left symmetry or it is un has a symmetry if it the special coordinates undergo an inversion or a parity operation. So, under this parity operation r the position vector that goes to minus r, p goes to minus p; however, a J or L goes to J and does not change sign goes to L and simple reason is that L is equal to r cross p they change sign then of course, the angular momentum operator does not change sign. J is of course, L plus S which s is the spin operator.

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In general it is written as,

$$P\psi(\vec{r}) = \epsilon_P \psi(\vec{r})$$

$$P^2\psi(\vec{r}) = \epsilon_P^2 \psi(\vec{r}) = \psi(\vec{r})$$

$$\epsilon_P = \pm 1.$$

$+1$: Even parity

-1 : odd parity.

So, in general P psi of r equal to epsilon P psi of r. P square psi of r if I operate it again by the parity operator it gives me a epsilon P square psi of r. Now, this should be equal to psi of r. So, if you do an inversion once and then again be inverted about some certain fixed point or certain fixed access then the system comes back to its original configuration. So, P squared of psi P square acting on psi r will give me psi r. So, epsilon P will have values which are plus minus 1 and plus 1 corresponds to even parity and minus 1 corresponds to odd parity. So, plus 1 has is called as even parity and minus 1 is called as a odd parity, ok.

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Time reversal Symmetry

Time reversal transformation $t \to -t$.

So it reverses the velocity of particles, but does not affect the position. So if $r(t)$ is a solution of,

$$m\ddot{\vec{r}}(t) = -\vec{\nabla}V(\vec{r})$$

then $\vec{r}(-t)$ is also a solution of this equation.

A term $-\gamma \frac{d\vec{r}}{dt}$ does not validate. OR

In presence of a magnetic field, $q\left(\frac{d\vec{r}}{dt}\right) \times \vec{B}$: time reversed states are not eigenstates.

Under time reversal: $\vec{r} \to \vec{r}$, $\vec{p} \to -\vec{p}$, $\vec{J}\,(\text{or } \vec{L}) \to -\vec{J}\,(\text{or } -\vec{L})$

So, let us talk about the third symmetry property that is called discrete symmetry property that is called as time reversal symmetry. So, time reversal transformation implies that t goes to minus t. So, it reverses the velocity of particles, but does not affect positions. So, if r t is a solution of m r double dot t equal to minus gradient of v which is a force then r of minus t is also a solution of this equation, but the moment we include a damping term which is proportional to velocity this does not hold good.

So, a term say since it is a damping term on writing it with a minus sign, but that is not required for this discussion. So, this is gamma and a dr dt which is basically a velocity term does not validate the fact that r of minus t will is also solution of this. And thus a term like this actually breaks the time reversal in variants and a practical example of this not only a damping term like this, but in presence of a magnetic field or in presence of a magnetic field um. So, this is a. So, it is like q into dr dt cross b again it does not time reversal does not hold. And hence we can say that the time reverse eigenstates time reverse states are not eigenstates of the Hamiltonian are not eigenstates.

And this holds because of the loss of in variants under the reversal of velocities. So, if the velocities are reversed then of course, the time reversal invariants does not hold good. And under this time reversal r of course, remain as r p because its proportional to the velocity it becomes minus p, and because one of them change sign that the j or l they change sign minus j or minus l and so on.

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Time reversal Operator.

The time evolution of a state is represented by,

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi$$

Let a time reversal operator be represented by $\tau \cdot \mathcal{T}$

$\mathcal{T}$ commutes H (τ is an unitary operator)

$$\underbrace{\tau i\hbar \left(\frac{\partial}{\partial t}\right)\tau^{-1}}\tau\psi = \tau H \tau^{-1}\tau\psi.$$

Since $[\tau, H] = 0 \Rightarrow -i\hbar \frac{\partial}{\partial t}(\tau\psi) = H(\tau\psi)$

Since. $\tau\left(i\hbar \frac{\partial}{\partial t}\right)\tau^{-1} = -\left(i\hbar \frac{\partial}{\partial t}\right)$

So, let us see what a time reversal operator is. The time evolution of a state of a state is represented by i h cross del psi del t equal to H psi. Let a time reversal operator be represented by tau um. Now, if this time reversal operator namely tau commutes with H the Hamiltonian of the system, then we can write see we just want to introduce a and tau is an unitary operator. So, i h cross del del t of tau inverse tau acting on psi its tau H, its H tau inverse tau psi. Now, if we left multiplied with a time reversal operator again then this becomes equal to this.

Now, since tau H equal to 0 because they commute so a minus i h cross del del t of tau psi becomes equal to H tau psi, so H. So, if the psi is operated by a tau. So, that still obeys the same equation of motion and since then if we compare between these two. So, a tau i h cross del del t of tau inverse that is this part that is there so this is equal to a minus i h cross del del t.

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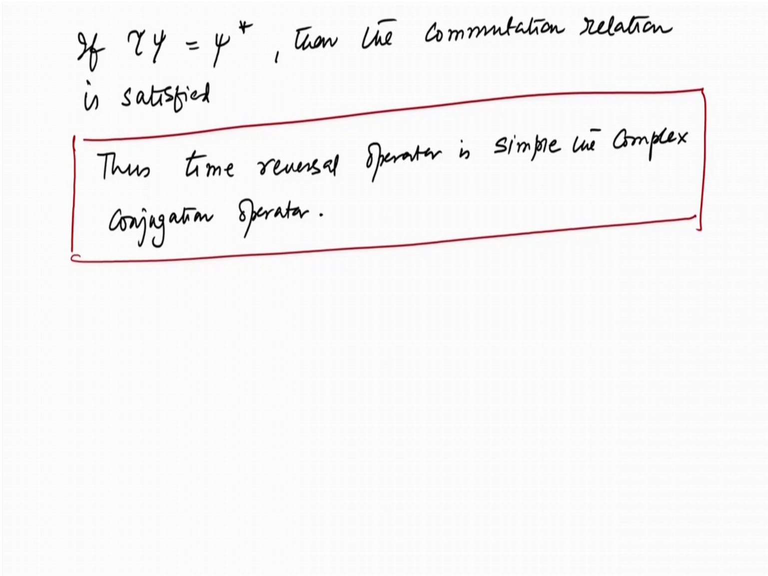

So, that tells that if tau psi is equal to psi star then the commutation relation is satisfied. Thus time reversal operator is simply the complex conjugation operator, ok. We shall in a while a talk about discrete or rather the continuous symmetries we have been talking about discrete symmetries. So, we will talk about continuous symmetries and see some of the examples of continuous symmetries in real systems, and that is what would be the motives or the objective of the course. From now on that will apply these concepts basic concepts to various fields of applied fields which are of scientific importance these days.