Quantum Technology and Quantum Phenomena in Macroscopic Systems
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Lecture – 12
Quantization of Standing EM Waves;Quantum States of Radiation Fields-I.

Hello! Welcome to lecture nine of this course. In the last class we learned how to quantize a propagating electromagnetic wave. In this class today, we are going to learn how to quantize a standing electromagnetic wave. It will be followed by discussion on quantum states of harmonic oscillator particularly the so-called coherence states.

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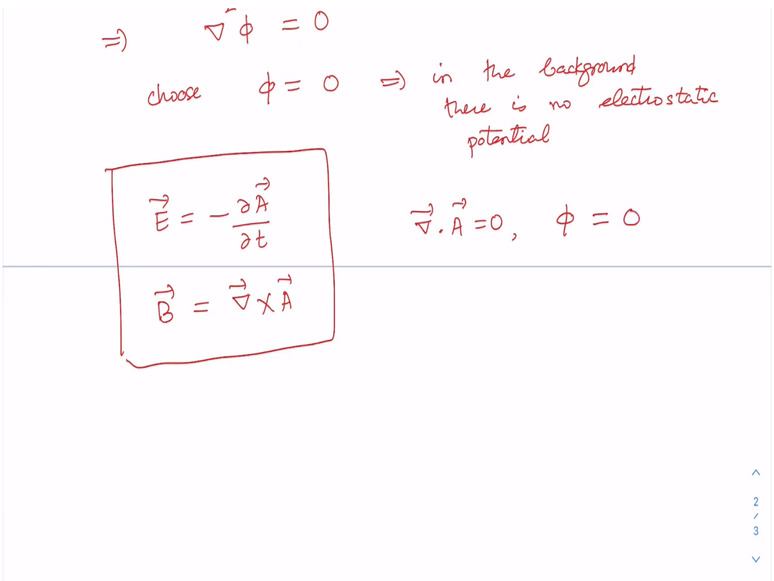

Let us begin our discussion by writing the Maxwell's equations first. Maxwell equations in the absence of charge and current and these equations are: divergence of E is equal to 0, divergence of B is equal to 0 then we have curl of E is equal to - del B del t and curl of B is equal to 1 by c square del E del t. Now because divergence of B is equal to 0 we can write B as curl of A and you know that A is called the vector potential, it is called vector potential.

Moreover, please note that this vector potential is not unique because I can have another vector potential defined as A dash which may be different from this vector potential A by some say gradient of a scalar quantity say chi. Then if I take curl on both side of this equation. So, I will have curl of A dash is equal to curl of A + curl of gradient of the scalar quantity and you know that this is equal to 0 and curl of A is B.

So, a vector potential is not unique and these actually help us to make a source. We can choose the vector potential such that that divergence of A is equal to 0. This choice is known as Coulomb's guage and this is going to be very useful for our treatment. Also, another thing that because curl of E is equal to - del B del t right here. So, I can also write this expression curl of E is equal to - del B del t. So, instead of B I can write curl of A and from here I can write curl of E + del A del t is equal to 0.

And therefore, I can write E + del A del t to be some kind of a scalar quantity say grade of phi or say - grade of phi therefore I can express the electric field as - del A del t gradient of the scalar potential phi. So, electric field I can express like this and the magnetic field B I have curl of A and now from this equation divergence of E is equal to 0. If I put this expression there I will have divergence of A. You will just have - del t divergence of A.

I just have to put this equation here and you will immediately see that you are going to have delta to phi is equal to 0 and because under Coulomb's guage divergence of A is equal to 0 I have delta 2 phi is equal to 0. Now what we can also have another choice. We can choose this phi to be equal to 0 because in the background there is no electrostatic potential if we can always take it like that.

In the background there is no electrostatic potential. This is what this choice means electrostatic potential and then what we are going to have is this electric field. I have as - del A del t and B is equal to curl of A. This is going to be useful for our treatment of course under these choices that is divergence of A is equal to 0 and this scalar potential electrostatic potential phi to be equal to 0. Under this we can express electric field and magnetic field in this form.

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$\longleftarrow L \longrightarrow$

$$\vec{A}(z,t) = \vec{\eta}\, \underline{\underline{A(t)}} \sin kz$$

$$\vec{E}(z,t) = \vec{\eta}\, \underline{\underline{E(t)}} \sin kz$$

$$\vec{B}(z,t) = \vec{k} \times \vec{\eta}\, A(t) \cos kz$$

$$\vec{k} = \hat{z} k$$

$$\vec{E}(0,t) = \vec{E}(L,t) = 0$$

$$\Rightarrow \quad \sin kL = 0 = \sin n\pi$$

$$\Rightarrow \quad \boxed{k_n = n\frac{\pi}{L}}$$

So, now let us discuss quantization of standing wave. Quantization of standing electromagnetic wave, standing EM wave. Let us do that. Consider a set of 2 plane mirrors separated by a distance L and this setup supports a standing wave and the electric field at the mirrors are 0. So, the electric field would have, say, this kind of a structure and we are considering only one mode of the standing wave and the standing wave is described by the vector potential say A. So, this is say o z it is directed along o z direction.

So, it would be A z t is equal to, it is linearly polarized along say eta direction and we have this A of t sine kz and electric field E z t is equal to eta E of t sine kz and the magnetic field is B z t because B is equal to curl of A. So, you can immediately see that it would be k cross eta A of t cos of kz. So, this defines the standing wave inside this mirror 2 mirrors.

And A of t and E of t are real quantities, real field amplitudes, and k is the wave vector along z direction and the boundary condition imposed by the mirrors leads us to the electric field at 0 and the electric field at the other end of the mirror, that is, at L it has to be 0. And from here you can immediately see that you will get sine k L is equal to 0 and which I can write it as sine into some integer into pi. So, from here I have k is an integral multiple of pi by L. So, k, the wave vector becomes discretized because of the boundary conditions.

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$$= -i\omega \frac{1}{2N}\left(\omega A + i\dot{A}\right), \quad \vec{E} = -\frac{\partial \vec{A}}{\partial t}$$

$$= -i\omega \frac{1}{2N}\left(\omega A - i\,E(t)\right)$$

$$\dot{\alpha}(t) = -i\omega\,\alpha(t)$$

$$\Rightarrow \boxed{\frac{d\alpha(t)}{dt} = -i\omega\,\alpha(t)}$$

Now we have curl of B is equal to 1 by c square del E del t and B is equal to curl of A. Now if I utilize this equation here, so, curl of curl of A and that is equal to 1 by c square del E del t. If I open it up then I will get del of divergence of A - delta 2 A is equal to 1 by c square del E del t. Now because we are using Coulomb's gauge. So, this divergence of A is equal to 0 and therefore I have, because we are directed along z, so, only z component will come into the picture here delta 2 A delta z 2. So, this one and here I have 1 by c square del E del t and also you see that if you because we have already taken A to be like this, so, from here I can immediately write that this would be k square A is equal to 1 by c square del E del t, all right, or I can just write the time dependence of this electric field would be equal to k square c square A of t which also I can write it as omega square A of t where omega is equal to c into k.

Now let us define a dimensionless variable. So, define a dimensionless variable say alpha of t is equal to 1 by twice of some normalization factor N, omega A - i E of t. This normalization factor if you look at carefully because alpha of t is dimensionless, this normalization factor N should have the dimension of electric field. Now if I take the time derivative of this expression alpha of t then I have 1 by 2 N omega A dot - i E dot t and this I can further write as 1 by twice N omega A dot i E dot.

So, if I just use this expression here I can write it as omega square A of t and which further I can write it as - i omega 1 by twice N omega A + i A dot. Now recall that E is equal to earlier we wrote it as -del A del t. I can utilize this here and then I will be able to write it as i omega

1 by twice N omega A i E of t. So, this quantity is nothing but this whole thing is nothing but alpha of t. So, I have alpha dot t is equal to - i omega alpha of t.

So, what you see is that the standing wave is fully determined, its dynamics is completely determined by this dimensionless variable alpha of t. I am sure you can see the signature of the treatment that we have done in the last class regarding the propagating waves or the traveling waves.

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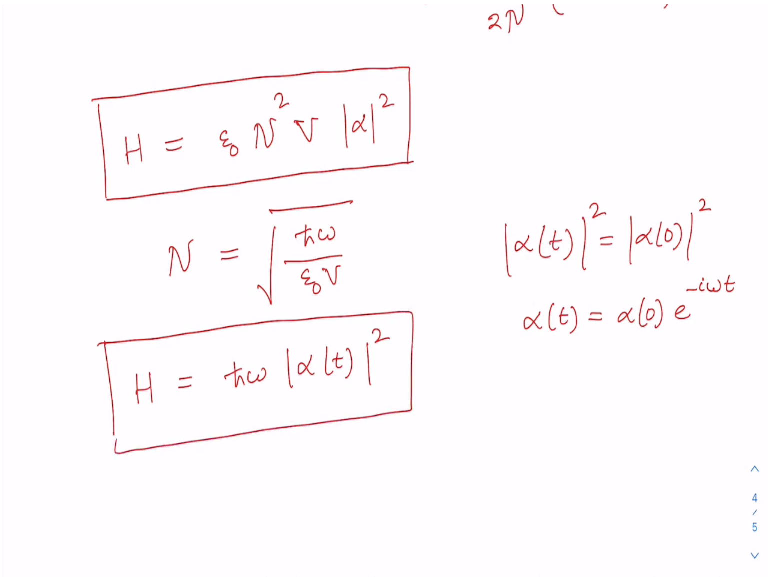

Now the next job is that to find out the energy of the mode. Energy of the mode is as usual it is epsilon 0 by 2. So, you have this electric field E of r t square + c square B of r of t square and this is the volume integral you have to do. So, say d 3 r. So, this is what we have to calculate and let us do that. The integration is evaluated over the quantization volume V is equal to L into L perpendicular square where L is the length of the cavity and L perpendicular refers to the size of the quantization volume along the transverse direction with respect to the mode axis.

Now if we put down the expression for the electric field and the magnetic field in the expression then we can write H is equal to epsilon 0 by 2. We have E of t square sine square k z + omega square A square cos square k z and the volume d 3 r and you can easily actually work it out into the same way. You just have to evaluate 0 to L sine square kz dz and there to turn out to be L by 2 and similarly, for the cos square kz that would be again for cos square kz it would be the same cos square kz dz.

And finally, you should be able to get this expression. I am leaving it as an exercise for you or you can simply do it. You should be able to get this expression E of t square omega square A square t into the volume, quantization volume. Now you know that alpha we have defined as 1 by twice N omega A - i E. So, using this I can write my final expression as epsilon 0 N square V into mod alpha square. Please verify it. This is what you should be able to get.

Now exactly the way we have done earlier for the propagating waves we can now choose our this constant N as say h cross omega divided by epsilon 0 V under root then I can write my Hamiltonian H as h cross omega mod alpha of t whole square and please note that alpha of t mod square is equal to alpha of 0 mod square because you remember alpha of t we can write it as alpha of 0 e to the power - i omega t.

So, this is the expression for the Hamiltonian the only difference that we have, what we had in the earlier class, the constant we had that factor 2 in the propagating wave but here we are with this factor 2 is not there. This difference is actually due to the fact that the energy of the standing wave is preferentially located close to the nodes of the cavity mode instead of being uniformly distributed as in the case of plane wave.

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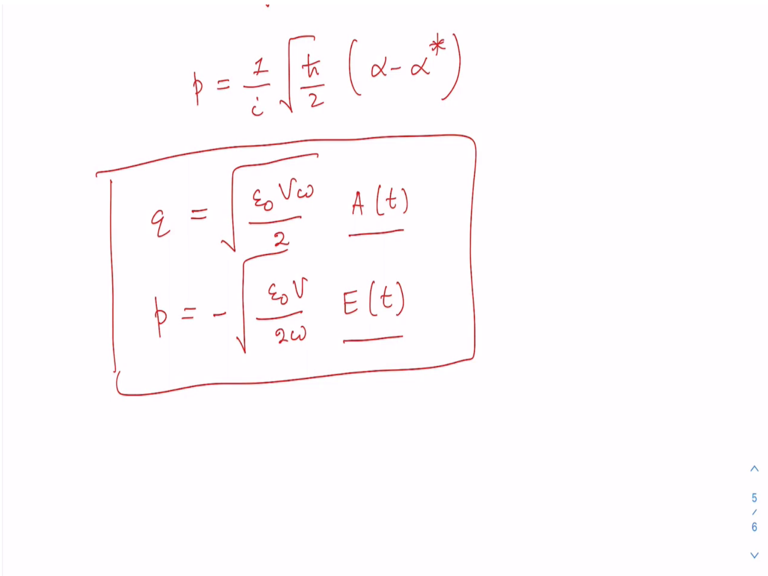

So, we can say that a photon in a standing wave a photon in a standing wave occupies on average only a half of the geometrical volume of the cavity. So, this is what is different from the so-called travailing wave for the propagating wave. Now let us define a pair of canonically conjugate variable q and p as follows, this treatment is now going to be the similar as we have done earlier q + i p.

And then we can write down the Hamiltonian H h cross omega mod alpha t square as simply omega by 2 q square + p square and in analogy with our discussion with plane wave we can claim that q and p are canonically conjugate variables. We can have q is equal to h cross by 2 under root alpha + alpha star and p is equal to 1 by i h cross by 2 alpha - alpha star. And if we do the calculations in fact you will be able to show that q is equal to epsilon 0 V omega by 2 under root A of t and p is equal to - epsilon 0 V by twice omega E of t.

So, within normalization factor A of t and E of t are respectively actually similar to position and momentum of a massive harmonic oscillator.

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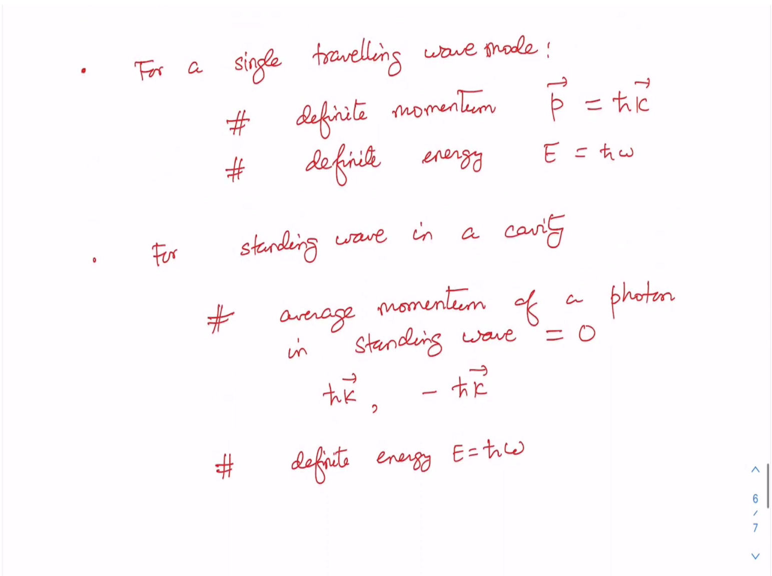

Now let us express A and E as a function of alpha and alpha star and if we do that similar the way we have done earlier. So, you will get 1 by omega h cross omega epsilon 0 V under root alpha + alpha star sine k z and this electric field E of z t would be eta i h cross omega epsilon 0 V under root alpha - alpha star sine k z. So, that is what we are going to get. Now quantization is straight forward because already we established that q and p are canonically conjugate variables.

So, they would be replaced by the operators q p is equal to i h cross and alpha and alpha star. So, because I have already utilized the symbol A for propagating waves. Now here let me use the symbol B. So, alpha is represented by this annihilation operator B and alpha star would be represented by creation operator B dagger such that B and B dagger you can verify it that would be equal to this commutation relation would be equal to 1.

So, for standing wave I have this operator A of z. I can write it as eta 1 by omega h cross omega epsilon 0 V, b + b dagger sine k z and for the electric field I have i eta h cross omega epsilon 0 V under root b - b dagger sine k z. So, now you can compare the electric field expression that we obtained for traveling EM wave in the last class. For traveling EM wave the electric field operator that we obtained in the last class is like this, it was h cross twice epsilon 0 V. V is the quantization volume and we had A of t e to the power i k dot r - a dagger e to the power - i k dot r.

Now while this is for this is for propagating wave or traveling wave and these are for standing waves. Now while both the travailing wave mode and the standing wave mode behaves like harmonic oscillators when quantized there is a subtle difference with regard to the photon concept associated with standing wave in the travailing wave. For example, for a single propagating wave for traveling wave for a single traveling wave the photon has a traveling wave mode the photon has a definite momentum.

It has a definite momentum say p is equal to h cross k. If you make a measurement you will always get the momentum to be like that and it would have a definite energy definite energy E is equal to h cross omega. On the other hand, for a single this standing wave for standing wave standing wave in a cavity for because you know that the standing wave is a superposition of 2 plane waves with opposite wave vectors the average momentum of the standing wave.

Average momentum of a photon in standing wave standing wave is equal to 0 and because of that if a measurement is made then either one will get h cross k as the momentum or one will get -h cross k as the momentum. However, the photon in a single standing wave will have energy, it would have a definite energy. So, that would be h cross omega. This is the fundamental difference between standing wave mode and propagating wave mod for a single mode I am talking about.

So, we have established that standing electromagnetic wave when quantized behave like a harmonic oscillator. Now let us discuss some useful quantum states of harmonic oscillator. This discussion is going to be valid for both quanta of electromagnetic radiation that is called photons and also the quanta of mechanical oscillators called phonons.

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$$\hat{D} = \exp\left[\frac{\hat{p}}{i\hbar}\Delta x\right]$$

$$\hat{D}(\Delta x) = \exp\left[\frac{\Delta x}{2q_0}\left(\hat{a}^{+} - \hat{a}\right)\right]$$

$$= \exp\left[\underline{\underline{Re(\alpha)}}\left(\hat{a}^{+} - \hat{a}\right)\right] \qquad \Delta x = 2Re(\alpha)q_0$$

Let us now discuss about various quantum states of harmonic oscillators. Earlier when we discussed quantum mechanical oscillators we discussed about the so-called number states and these number states were represented by ket n, where n refers to the number of quanta in the state and these quanta were known precisely and they may be photons or they may be phonons.

Now in that context we also this we talked about ground state of harmonic oscillator and ground states are also kind of number states. Only thing is that there is no quanta in the state. Now I am going to talk about one very important class of states that is called coherent states and these states are particularly useful for optical radiations and they are represented by this ket alpha and we will see that these coherent states are nothing but displaced ground states.

Now here the number of quanta number of quanta in coherent states are generally are infinite and because of that these coherent states have a definite phase. On the other hand, in the case of number states the phase is random because you know the number of quanta precisely. So, there is an uncertainty in phase and uncertainty in the number of quanta is 0 in the case of number states and this is also the reason why generally it is very difficult to generate number states experimentally.

Now before I proceed further let me first talk about this displaced ground state. What I mean by that? That means I can generate a coherent state by just using this ground state if I apply this displacement operator. So, let me first talk about briefly about this displacement operator displacement operator let me define that first let us say we have a wave function psi of x and it has a structure of this form say it looks like this.

And this wave function is now displaced by some distance say delta x and after displacement it say takes this form. So, it is displaced by delta x and this is psi of x and this displaced form of the wave function is, say, psi tilde of x. Now you can clearly you can see that psi tilde of x is actually the same as this wave function psi of x at distance x - del x.

Now if this displacement delta x is very small, if this is very small then I can expand it into a Taylor series and Taylor series expansion of this is going to give me psi of x - delta x delta psi of delta x and you will get all other terms. I can also write it 1 - delta x delta x + the terms and psi of x or I can write it in an exponential form that would be e to the power - delta x delta x psi of x.

So, you see this is the so-called displacement operator I am talking about. This is what the displacement operator is. Let me make it more formal because you know that quantum mechanically the momentum operator is defined as -i h cross delta of x. Here, I am talking about displacement along only one direction x direction generally if it is 3 dimensional this thing then I will have i h cross this delta.

So, what I can write, so therefore I can write e to the power - delta x delta of x as in terms of this momentum operator, I can write it as exponential p by i h cross where p is the operator here delta x. So, again you remember that earlier we defined this operator p in terms of creation in any relation operator like this i q 0 m omega a dagger – a. Here, I am talking about a mechanical harmonic oscillator but it is equally valid for photons as well where you just have to put m is equal to 1.

And for mechanical harmonic oscillator q 0 is the zero point fluctuation and it was defined as h cross divided by twice m omega. So, in terms of this new annihilation and creation operator I can write down this displacement operator D here. This displacement operator I can represent it by this D cap here then I have exponential p by i h cross delta x which if I put this expression from here then I would be able to write it as exponential. Doing all these things you can see that you can write it as exponential twice q 0 delta x by 2 k 0 a dagger – a.

This is what you will get and this is only for one particular direction that is along the x direction we have work it out or you can actually you know you can just write it as later on I will show that you will be able to write it as exponential real part of alpha and you will have a dagger – a. I think I will come to this particular point what I am saying as real this thing later on you will see that delta x is equal to twice real part of alpha into this q 0, that is what you can just take it like this that delta x by 2 q 0 is nothing but the real part of this parameter alpha.

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$$\boxed{e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]}}$$

if $\underline{[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0}$

$$\boxed{D(\alpha) = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha \hat{a}^{\dagger}} e^{-\alpha^{*} \hat{a}}}$$

Now as you know that x and p are on equal footing, x and p are on equal footing. So, the displacement could occur in the momentum direction or any arbitrary direction in the phase space. So, we define the displacement operator general definition of displacement operator or the expression would be D alpha is equal to exponential alpha a dagger - alpha star a. So, this is what is the definition of the so-called displacement operator is where alpha is a here alpha is an arbitrary complex number. It is an arbitrary complex number, okay it is an arbitrary complex number.

So, you can actually write it, simplify it further because we know from quantum mechanics that if we have 2 operators A and B then e to the power A + e to the power B this operator this relation is there e to the power A e to the power B and then you have e to the power - half A B the commutation relation between A B. If the operators satisfy these relations A A B right and B A B is equal to 0 then this identity this expression that I have written here this is valid provided subject to the fulfillment of these conditions and you will see that in our case here both a and a dagger they satisfy this.

So, you can just take it as an operator say this is A this is B then you can put it there in the expression then you should be able to show that D alpha is equal to exponential or just let me write it as e to the power - half mod alpha square and you will have e to the power alpha a dagger and you will have e to the power - alpha star a. This is also another form of the displacement operator you can use.

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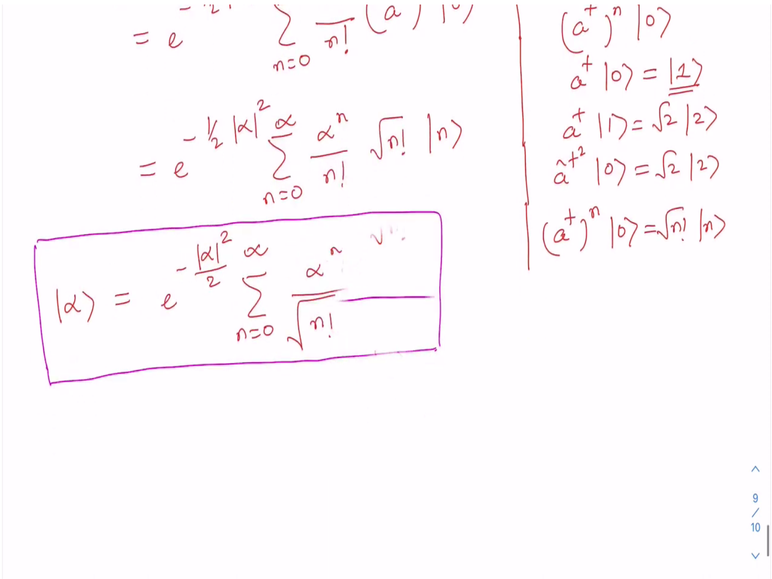

Now going back to the fact that alpha this coherent state is nothing but the displaced ground state. So, this when you operate on the ground state 0 ket by this displacement operator one will get the coherent state. So, let us see what we will get by our expression that I just got it here. So, if I put it there let me just quickly show you e to the power - half mod alpha square that is a constant quantity.

So, let me take now let me put the operator alpha a dagger which I can write in exponential form alpha a dagger to the power n divided by n factorial where n is equal to 0 to infinity you may remember that e to the power x I can write it as summation x to the power n by n factorial where n goes from 0 to infinity the same thing here I am doing here and then for the another expression I have m is equal to 0 to infinity this one I can write it as - alpha star of a to the power m divided by m factorial then you operate on 0.

So, let us do it, it is very simple. So, let me just write here. It implies that this ket alpha is equal to e to the power - half mod alpha square and now just look at this expression here if you break it up you will see that this summation the first term for m is equal to let me just open it up only this one if you open it up you will see for m is equal to 0 you will have this ket 0 and for the another then the second term for m is equal to 1 you will have it as you will have it as - alpha star a 0.

Now because you know that annihilation operator when it operates on the ground state it is going to give you 0 and similarly for all other terms. So, ultimately you will be left out with only ket 0 only. So, therefore I have this summation is going to give me simply ket 0. So, therefore I write it as n is equal to 0 to infinity here alpha a dagger to the power n divided by n factorial then ket 0.

And now it is the second step let me do this - half mod alpha square you could have it write it as n is equal to 0 to infinity alpha to the power n by n factorial and then you have a dagger to the power n operated on this ket 0. Now again you know that a dagger n operated over ket 0, first of all you know a dagger 0 is equal to simply 1, ket 1 and a dagger applied on ket 1 is going to give you root 2 ket 2 and in this way you again a dagger square on 0 therefore you see a dagger 1 is there.

So, if you just put in. So, this is nothing but a dagger 0 would be equal to root 2 2. So, therefore I think you are getting the idea where I can just extrapolate it to a dagger n 0 would be root over n factorial and ket n and so from here I can write it as e to the power - half mod alpha square summation n is equal to 0 to infinity alpha to the power n by n factorial root over n factorial and n ket n.

And thus, I can write the expression for the coherent state as e to the power - mod alpha square by 2 n is equal to 0 to infinity alpha to the power n by n root n factorial ket n. So, this is actually the one of the most popular form of the coherent state

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$$\langle \hat{x} \rangle = \langle \alpha | \hat{x} | \alpha \rangle$$
$$= q_0 \langle \alpha | \hat{a} + \hat{a}^{\dagger} | \alpha \rangle$$
$$= q_0 (\alpha + \alpha^*)$$
$$\boxed{\langle \hat{x} \rangle = q_0 \, 2 \, \mathrm{Re}(\alpha)}$$

$$\hat{x} = q_0 (a + a^{\dagger})$$
$$q_0 = \sqrt{\frac{\hbar}{2m\omega}}$$
$$\hat{a} | \alpha \rangle = \alpha | \alpha \rangle$$
$$\langle \alpha | \hat{a}^{\dagger} = \langle \alpha | \alpha^*$$

$$\langle \hat{p} \rangle = \langle \alpha | \hat{p} | \alpha \rangle$$
$$= 2m\omega q_0 \, \mathrm{Im}(\alpha)$$

$$\hat{p} = i m\omega q_0 (\hat{a}^{\dagger} - \hat{a})$$

Please note that that this coherent state is normalized and it is very easy to prove it just by utilizing this relation. Now, one can get more insight by evaluating the expectation value of the oscillator position and momentum in the coherent state. For example, the expectation value of the position operator in the coherent state. Let us evaluate it. So, this is the expectation value and we know that x operator we can write it as q 0 a + a dagger and where q 0 is the zero-point fluctuation and that is h cross divided by twice m omega under root.

So, if I use this then I can write it as q 0 or the zero-point fluctuation alpha a + a dagger alpha and if you work it out it is very trivial first term a a dagger a. So, a ket alpha is going to give you simply alpha alpha on the other hand this a dagger operates on the bra alpha. So, you will get it as bra alpha and alpha star. So, if you utilize it you can immediately see that it would be q 0 into alpha + alpha star and which is nothing but q 0 twice of the real part of alpha because alpha is a complex number.

And in fact, this is the reason you see earlier I used this definition here we just defined it say I promise to come back, yes here, you see I have taken this delta x as twice real part of q 0 this is the reason because then and here you see that this expectation value of position operator is related to the real part of alpha. Similarly, one can find out the expectation value of the momentum operator and that would be simply alpha p alpha again you know that this momentum operator is simply i m omega q 0 a dagger - a.

So, if you utilize it then you can show that this would be simply twice m omega q 0 imaginary part of alpha. This is trivial.

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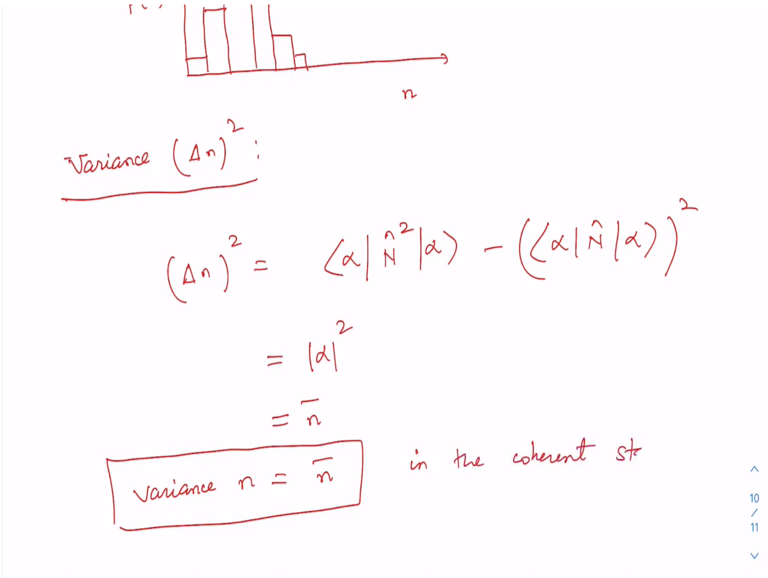

Now, what about the probability of finding n photons in the coherent state? So, let us work it out. Probability of finding n photons in the state alpha. This is of course valid for any n number of quanta in the coherent state alpha. So, this is given by you just have to take the scalar product of this quantity mod square and because you know that mod alpha is equal to e to the power - mod alpha square by 2 and you have n is equal to 0 to infinity alpha to the power n by n factorial under root then this ket n.

So, immediately you should be able to get this expression it would be e to the power - mod alpha square mod alpha to the power 2 n and divided by n factorial okay. Now also note that the mean number of photons in the coherent state or mean number of quanta in the coherent state is given by this expression you just have to take the expectation value of this number operator and this number operator here is a dagger a and you can immediately see that this is nothing but mod alpha square.

So, in terms of this mean number you can express this probability expression like this e to the power - n bar and it would be n bar to the power n divided by n factorial. So, you may recognize that this is nothing but a familiar Poissonian distribution. So, clearly the probability distribution of a coherent state is Poissonian and one can just to give an example how the distribution look like let me plot in here this is the kind of distribution you can expect in a Poissoning distribution.

So, this is what you are going to have okay. Another important quantity is the so-called variance. So, variance of coherent state variance delta n square in the coherent state that is given by delta n squared. So, you just have to take the n bar alpha - alpha n alpha whole square. So, if you work it out it is very straightforward to work out. So, if you work it out you should be able to get let me leave it for you to do it what you will get is that you will get this expression mod alpha square which is nothing but n bar.

So, variance is equal to the mean number. So, variance is exactly equal to mean number for coherent state in the coherent state. So, this is an important result.

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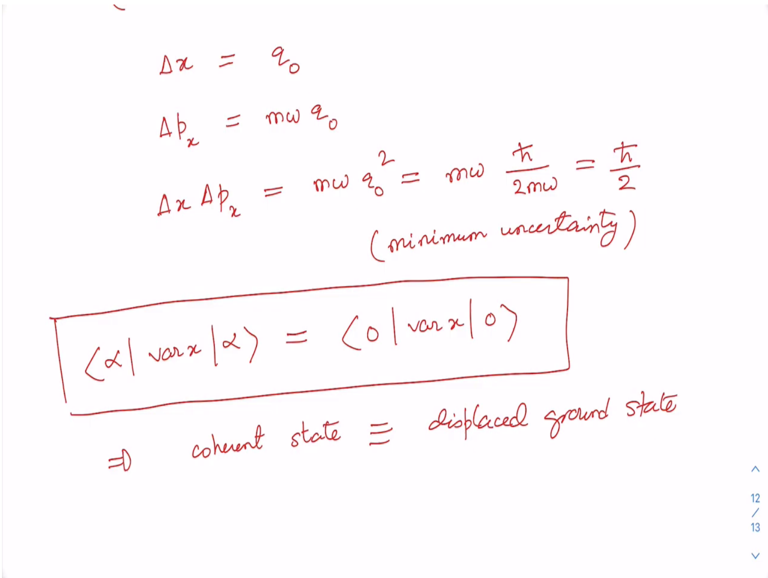

Finally let us look at some properties of the displacement operator D alpha which is going to be very useful later on in the course properties of D alpha let me just list them out. Already you know that displacement operator D alpha one of the expression is of this step e to the power alpha a dagger - alpha star a and immediately from here you should be able to see that if you take D dagger alpha then you will get D alpha star a - alpha a dagger and which is nothing but D - alpha

And so, this is actually one important relation to remember and secondly, you can work it out say d dagger alpha a d alpha you will find that this would be simply equal to a + alpha, let me prove it to. To show it you just have to use this relation which you know already from quantum mechanics hopefully e to the power A - A. It is called Baker Hausdorff formula I think. So, you just have to work it out, just have to utilize this relation there.

Let me just write it A A B. Let us find it out. D dagger alpha a D alpha and D dagger alpha is e to the power alpha star a - alpha a dagger a e to the power alpha a dagger - alpha star a. Now let me take it as a here capital A and then apply the formula. So, this is my B. So, I have here a + you have the commutation relation between capital A capital B. So, that would be alpha star a - alpha a dagger you will have here a and then.

So, from here you will get a + if you work it out you will simply get alpha then let me just work out the next term also you will have here half a is your alpha star a - alpha a dagger then commutation relation between A B already we have worked out and found that this is simply alpha and which is a constant. So, obviously this term is going to give you 0 and similarly all higher order terms will go to 0.

And you will be left out with a + alpha. So, hence it is very straightforward to proof. Similarly, you can prove that D dagger alpha a dagger D alpha would be a dagger + alpha star. Also, please note that D dagger alpha D alpha is equal to identity operator this also you can prove easily. Let me now quickly show you some usefulness of these relations that we wrote.

So, for example you can immediately find out what is D dagger alpha x D alpha because you know that x is equal to q 0 D dagger alpha here it is a + a dagger the alpha q 0 is the zero-point fluctuation. So, if you work it out you should be able to get immediately, please show it, it is trivial it would be x twice q 0 and real part of alpha. Similarly, you can work out the expression for say x square as well as p square expectation value of x square p square and so on.

And very quickly you can work out the expectation value. For example, let me just quickly show you this this one. So, in the coherent state, I just have to utilize the fact that coherent states are displaced ground state. So, I can just write it as D alpha here and this would be D dagger alpha and then in between I have x square here. Now because of these properties I have here x I can sandwich identity operator here that would be D alpha.

Then let me put here D dagger alpha x D alpha 0, this ket 0. I think you are getting it this is the identity operator from here and because I know this already I know this relation just we know that this is x cap + twice q 0 real part of alpha and the other one is also x cap twice q 0 real part of alpha. So, it is very straightforward to work it out if you work it out you will get simply q 0 square + 4 q 0 square real part of alpha square.

And therefore, immediately if I ask you to find out the standard deviation or for say the variance of the position operator that would be x square x whole square here this is it is in the coherent state. And in fact, using this relationship you should be able to show that delta x this standard deviation would turn out to be simply q 0 which is basically the zero-point fluctuation.

And in fact, similarly you will be able to show that the corresponding standard deviation in the momentum operator also we can work out and there to turn out to be m omega for a mechanical harmonic oscillator this is what you are going to get it. In fact, you can verify that delta x into delta p x for coherent state it would be m omega q 0 square and that is q 0 square is h cross divided by twice m omega.

So, therefore it is simply h cross by 2 and this is the minimum uncertainty product. So, for coherent state the product of this position uncertainty and momentum and uncertainty is minimum and that is the reason why these coherent states are also known as Gaussian states. In fact, you can show that the variance of x in the coherent state is equal to variance of the position operator in the ground state and this is also you can establish.

And this is the reason why coherent state is defined as or said to be nothing but the displaced ground state. Let me stop for today in this class we have learned how to quantize electromagnetic radiation and also, we learned about coherent states of harmonic oscillator. In the next class we will continue our discussion on quantum states of harmonic oscillator we will discuss squeeze states. It is a very important class of quantum states and also we are going to conclude module 1 of this course in the next lecture, thank you..