Quantum Technology and Quantum Phenomena in Macroscopic Systems
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Lecture - 32
Problem Solving Session - 7

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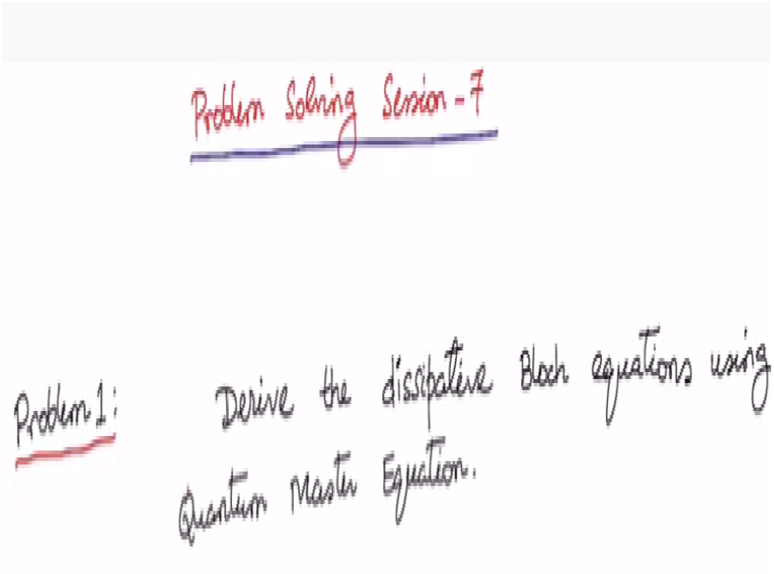

Welcome to this Problem-solving Session number 7. In this problem-solving session, we are going to solve problems related to quantum master equation, transmon qubit and Fabry-Perot cavity. As the first problem, let us begin with this one. You are asked to derive the dissipative Bloch equations using quantum master equation.

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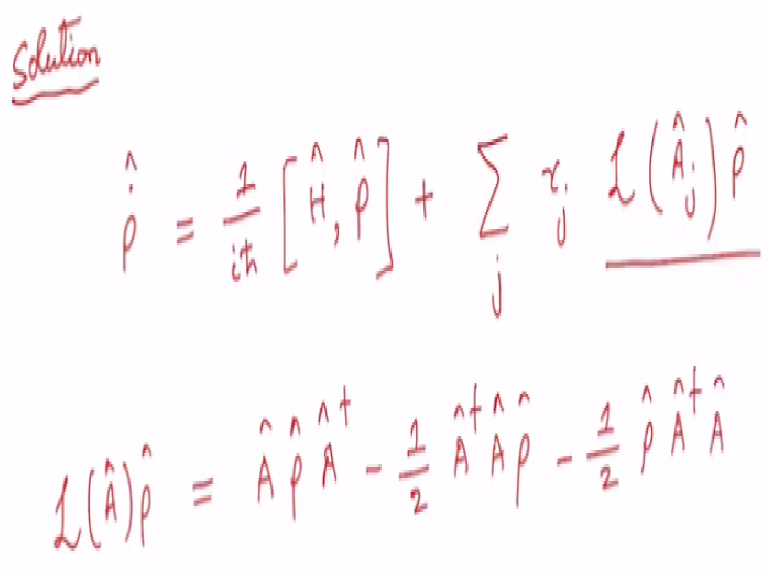

We have already learned about quantum master equation in the lecture classes. The evolution this equation quantum master equation basically gives the time evolution of the density operator or the density matrix. So, this is equal to 1 by ih cross H is the Hamiltonian H rho. This part gives the coherent evolution of the density matrix. And then, we have terms pertaining to the dissipative processes.

For, suppose the jth process is happening at the rate gamma j, there are all kinds of processes. And then, we have this Lindblad operator and A j is the operator that connects the system and the environment. So, this is the Lindblad operator. And, this Lindbladian operator is given as this. This part is equal to A rho A dagger minus half A dagger A rho minus half rho A dagger A.

So, using this master equation now, let us derive the Bloch equation in the presence of various dissipative processes. So, we have a 2-level system with energy states denoted as a ket g.

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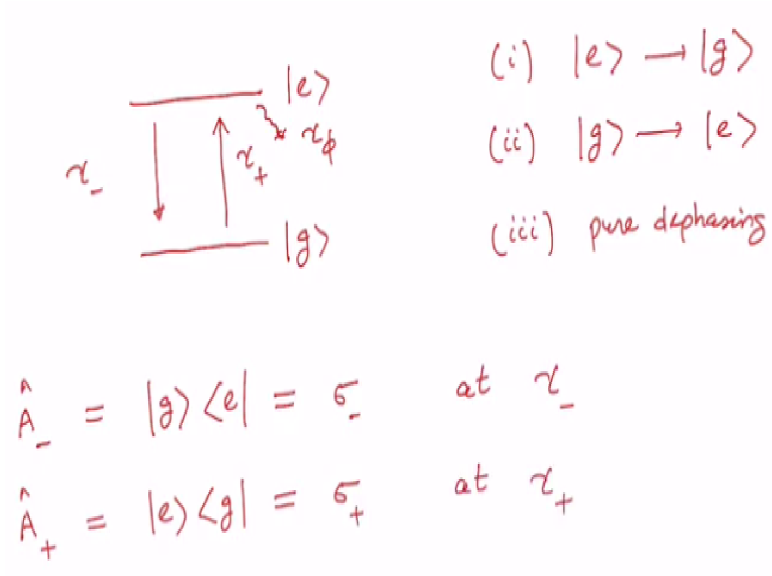

That is the ground state. And, ket e is the excited state. And, there are primarily 3 processes that occur. First of all, transition from the upper state to the lower state. Say, excited state to ground state. It happens at the rate gamma minus. So, this process refers to emissions and there are process, so, first process, we are having transition from the excited state to the ground state.

And, we have also process where transition can occur from the ground state to the excited state and that rate of occurrence is say gamma plus. So, this is also a process we have. And finally, we in addition to all these things, we have this pure dephasing and that occurs at the rate so pure dephasing also occurs and that happens at the rate say gamma phi. So, we are going to consider these 3 processes.

And, we can define relaxation operator corresponding to these 3 processes. For the emission process, we have the operator let us say this is the relaxation operator A minus is equal to we are going from the excited state to the ground state. And, this is, as you can see, this is the atomic lowering operator sigma minus. And, this process is happening at the rate gamma minus.

And then, we have this absorption process A plus where we are going from the ground state to the excited state. And, this is the atomic raising operator. And, this process is happening at a rate gamma plus.

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$$A_+ = |e\rangle\langle g|$$

$$\hat{A}_\phi = \sqrt{2}\,|e\rangle\langle e| \quad \text{at} \quad \gamma_\phi$$

$$\dot{\hat{\rho}} = \frac{1}{i\hbar}\left[\hat{H}, \hat{\rho}\right] + \gamma_-\left(\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \frac{1}{2}\hat{\sigma}_+\hat{\sigma}_-\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{\sigma}_+\hat{\sigma}_-\right)$$
$$+ \gamma_+\left(\hat{\sigma}_+\hat{\rho}\hat{\sigma}_- - \frac{1}{2}\hat{\sigma}_-\hat{\sigma}_+\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{\sigma}_-\hat{\sigma}_+\right)$$

And finally, we are having the relaxation, these pure dephasing processes and here the relaxation operator is we are going from the excited state to the excited state and there is a pre-factor of square root 2. And, it is happening at the rate gamma phi. Now, incorporating all these things, we can write down the quantum master equation. So, the master equation would be rho dot is equal to 1 by i h cross H rho. I will soon tell you, what is H the Hamiltonian?

And, let me now put up all the processes here. For gamma minus, I have, I can write sigma minus rho sigma plus minus half sigma plus sigma minus rho all these are operators. And, we have here minus half rho sigma plus sigma minus. And, for the process related to this absorption gamma plus, that would be sigma plus rho sigma minus, minus half sigma minus sigma plus rho minus half rho sigma minus sigma plus.

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$$\begin{aligned}\dot{\hat{\rho}} = \frac{1}{i\hbar}\left[\hat{H},\hat{\rho}\right] &+ \gamma_-\left(\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \frac{1}{2}\hat{\sigma}_+\hat{\sigma}_-\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{\sigma}_+\hat{\sigma}_-\right)\\ &+ \gamma_+\left(\hat{\sigma}_+\hat{\rho}\hat{\sigma}_- - \frac{1}{2}\hat{\sigma}_-\hat{\sigma}_+\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{\sigma}_-\hat{\sigma}_+\right)\\ &+ 2\gamma_\phi\left(|e\rangle\langle e|\hat{\rho}|e\rangle\langle e| - \frac{1}{2}|e\rangle\underbrace{\langle e|e\rangle}_{1}\langle e|\hat{\rho}\right.\\ &\qquad\left. - \frac{1}{2}\hat{\rho}|e\rangle\underbrace{\langle e|e\rangle}_{1}\langle e|\right)\end{aligned}$$

And finally, I also have, we have 2 gamma phi. And, that is due to the pure dephasing. We will, let me write it explicitly. We have here terms like this minus half e e e e rho. And then, I have minus half rho e e e. This would become actually simplified because we know that this part is actually scalar product. This is equal to 1 and this is equal to 1. Anyway, we will do that.

So, this is the quantum master equation where this Hamiltonian is I write the diagonalized form here.

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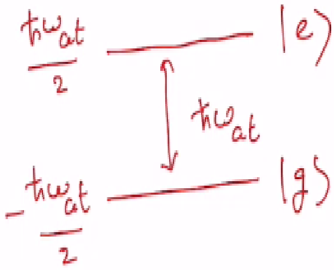
$$\hat{H} = \frac{\hbar\omega_{at}}{2}\hat{\sigma}_z$$



$$\hat{\rho}_e = \langle e|\hat{\rho}|e\rangle$$

h cross omega atom by 2 sigma z is the Hamiltonian. That means that my excited state has energy h cross omega atom by 2 and the ground state I am taking the energy as minus h cross omega atom by 2. So, therefore, the energy difference is simply h gross omega atom between the 2 energy levels. So, this is what we have. So, now, let us work out various terms. For example, how the, this element of the density matrix evolves in time?

So, rho e which is equal to e rho dot e. And, we can work it out. In fact, let me do the calculations term by term.

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$$\# \quad \langle e|[\hat{H},\hat{\rho}]|e\rangle$$
$$= \langle e|\hat{H}\hat{\rho} - \hat{\rho}\hat{H}|e\rangle$$
$$= \langle e|\hat{H}\hat{\rho}|e\rangle - \langle e|\hat{\rho}\hat{H}|e\rangle$$

$$\hat{H}|e\rangle = \frac{\hbar\omega_{at}}{2}\hat{\sigma}_z|e\rangle$$

So, first term we have here is e. Let me first calculate H rho e. And, if I open it up, it would be e H rho minus rho H e or H rho e minus e rho it is rho e. So, you know that H of e is equal to the Hamiltonian is h cross omega atom by 2 sigma z e.

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$$\hat{H}|e\rangle = \frac{\hbar\omega_{at}}{2}\hat{\sigma}_z|e\rangle$$

$$= \frac{\hbar\omega_{at}}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$= \frac{\hbar\omega_{at}}{2}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{\hbar\omega_{at}}{2}|e\rangle$$

$$|e\rangle = \begin{pmatrix}1\\ 0\end{pmatrix} \qquad |g\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}$$

$$\hat{H}|g\rangle = -\frac{\hbar\omega_{at}}{2}|g\rangle$$

Now, if I know sigma z operator is 1 0 0 minus 1 and if I take this vector ket as this column matrix 1 0. So, I am taking ket e as 1 0 and ket g as 0 1. Then, I will have, if I do the math, you will get it as h cross omega atom by 2 1 0. So, this is basically h cross omega atom by 2 ket e. Similarly, we will require it later. When it operates on the ground state, we will get h cross minus h cross omega atom by 2 ket g.

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$$= \frac{\hbar\omega_{at}}{2}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{\hbar\omega_{at}}{2}|e\rangle$$

$$\hat{H}|g\rangle = -\frac{\hbar\omega_{at}}{2}|g\rangle \quad ; \quad \hat{H}|e\rangle = \frac{\hbar\omega_{at}}{2}|e\rangle$$

$$\langle e|\hat{H} = \frac{\hbar\omega_{at}}{2}\langle e|$$

$$\langle e|\hat{H}\hat{\rho}|e\rangle = \frac{\hbar\omega_{at}}{2}\langle e|\hat{\rho}|e\rangle$$

$$\langle e|\hat{\rho}\hat{H}|e\rangle = \frac{\hbar\omega_{at}}{2}\langle e|\hat{\rho}|e\rangle$$

And therefore, we have e H rho e is equal to by the way, just note that because H e is equal to h cross omega atom by 2 ket e. So, if the Hamiltonian because this is a Hermitian Hamiltonian if it operates on the bra of that, so, you are bra e, you are going to get again h cross omega atom by 2 bra e. And therefore, from here, I can write this as h cross omega atom by 2 e rho e.

And the other one, e rho H e is also equal to h cross omega atom by 2 e rho e. And, as you can see, because both these terms are equal, so, what I will have here is this.

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$$\langle e|[\hat{H},\hat{\rho}]|e\rangle = 0$$

$$\langle e|\gamma_-\left(\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \frac{1}{2}\hat{\sigma}_+\hat{\sigma}_-\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{\sigma}_+\hat{\sigma}_-\right)|e\rangle$$

That e, this operator commutator H rho e is going to give me simply 0. Now, let us go to the other terms. Now, let us calculate this particular term e gamma minus sigma minus rho sigma plus minus half sigma plus sigma minus rho minus half rho sigma plus sigma minus e. To do this, let me first work out this term.

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$$\langle e|\gamma_-\left(\underline{\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+} - \frac{1}{2}\underline{\hat{\sigma}_+\hat{\sigma}_-\hat{\rho}} - \frac{1}{2}\hat{\rho}\hat{\sigma}_+\hat{\sigma}_-\right)|e\rangle$$

$$\# \quad \langle e|\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+|e\rangle = 0$$

$$\hat{\sigma}_+|e\rangle = 0$$

$$|e\rangle\underline{\underline{\langle g|e\rangle}} = 0$$

$$\langle e|\hat{\sigma}_+\hat{\sigma}_-\hat{\rho}|e\rangle$$

So, that is e gamma, just let me, gamma minus is anyway is a constant parameter. So, let me work it out, sigma minus rho sigma plus e. Now, this is easy to see because sigma plus is the atomic raising operator and if you are operating on the excited state, you can explicitly do the calculation also. You will immediately get it to be 0. And, to show it to you sigma plus you are basically going from the ground state to the excited state.

So, sigma plus is, you are going from the ground state to the excited state. Then, you are operating on this ket e. And, because this is orthogonal, so, this is going to give you 0. So, this term is going to give you 0. Then, we have to work out. Once this is done, what we are left? We have to work out e. Say, sigma plus sigma minus rho e, this term.

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$$\langle e|\hat{\sigma}_+\hat{\sigma}_-\hat{\rho}|e\rangle$$

$$= \underbrace{\langle e|e\rangle}_{=1}\underbrace{\langle g|g\rangle}_{=1}\langle e|\hat{\rho}|e\rangle$$

$$= \langle e|\hat{\rho}|e\rangle$$

Sly, $\langle e|\hat{\rho}\hat{\sigma}_+\hat{\sigma}_-|e\rangle = \langle e|\hat{\rho}|e\rangle\langle g|g\rangle\langle e|e\rangle$

$$= \langle e|\hat{\rho}|e\rangle$$

So, this if you, if I just put it the whole thing explicitly, then you will get e sigma plus is you are going from the ground state to the excited state. And, sigma minus is you are going from the excited state to the ground state. And then, here, you have rho e. So, this is, as you can see, this is a scalar product. It is equal to 1. This is normalized. This is also 1. So, you will be left out with simply e rho e.

Again, similarly, you can show that e rho sigma plus sigma minus e. That would be equal to, if I e if I do it explicitly, you have rho and here you will have e. This is g here g e e. So, this is also equal to e rho e. So, therefore, we have calculated all these terms.

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$$\langle e|\gamma_-\left(\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \frac{1}{2}\hat{\sigma}_+\hat{\sigma}_-\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{\sigma}_+\hat{\sigma}_-\right)|e\rangle$$

$$= \gamma_-\left(-\frac{1}{2}\underline{\langle e|\hat{\rho}|e\rangle} - \frac{1}{2}\langle e|\hat{\rho}|e\rangle\right)$$

$$= -\gamma_-\hat{\rho}_{ee}$$

So, I can therefore work out this particular term. So, this would be if I put all the terms there, I will have it as gamma minus, minus half e rho e minus half e rho e. And therefore, I will get it as minus gamma minus e rho e. I can write it as rho ee. So, this is what I have.

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$$= -\gamma_-\rho_{ee}$$

Sly, $\langle e|\gamma_+\left(\hat{\sigma}_+\hat{\rho}\hat{\sigma}_- - \frac{1}{2}\hat{\sigma}_-\hat{\sigma}_+\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{\sigma}_-\hat{\sigma}_+\right)|e\rangle$

$$= \gamma_+\hat{\rho}_{gg}$$

Similarly, I can work out this term with the absorption process. The term involving the absorption process is gamma plus sigma plus rho sigma minus minus half sigma minus sigma plus rho minus half rho sigma minus sigma plus. You will have e here. And, if you work it out, please do that. You will immediately get it as gamma plus rho gg.

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Finally:

$$\langle e|\, 2\gamma_\phi \Big(|e\rangle\langle e|\hat{\rho}|e\rangle\langle e| - \frac{1}{2}|e\rangle\langle e|\hat{\rho} - \frac{1}{2}\hat{\rho}|e\rangle\langle e| \Big) |e\rangle$$

$$= 2\gamma_\phi \left(\hat{\rho}_{ee} - \frac{1}{2}\hat{\rho}_{ee} - \frac{1}{2}\hat{\rho}_{ee} \right)$$

$$= 2\gamma_\phi \left(\hat{\rho}_{ee} - \hat{\rho}_{ee} \right)$$

$$= 0$$

So, finally, we are left out the pure, term with a pure dephasing. And, that is e 2 gamma phi. We have e e rho e e. Just let me remind you. Here, I have do this simplification. As you can see that this is equal this is 1 and similarly, this is 1. So, therefore, I can write it very simply as, the first term is fine. Then, we have minus half e e rho minus half rho cap e e. And, the whole thing is, I have to put e.

So, if you do it, you can immediately see that you are going to get twice gamma phi. And here, you will have first term will give you rho ee. Then, the second term you will have half rho ee and the third one is going to give you half rho ee. So, therefore, you have 2 gamma phi rho ee minus rho ee. So, contribution from this term would be 0.

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Thus,

$$\dot{\hat{\rho}}_e = \langle e|\dot{\hat{\rho}}|e\rangle$$

$$= -\gamma_- \hat{\rho}_{ee} + \gamma_+ \hat{\rho}_{gg}$$

Because: $\hat{\rho}_{ee} + \hat{\rho}_{gg} = 1$

Therefore, what we have is rho ee. Actually, you will have time evolution of this matrix element in the, would be in the density matrix for rho would be e rho e is equal to minus gamma minus rho ee plus gamma plus rho gg. Now, because we know that rho ee plus rho gg is equal to 1.

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$$\dot{\hat{\rho}}_{ee} = \langle e|\dot{\hat{\rho}}|e\rangle$$

$$\Rightarrow \boxed{\dot{\hat{\rho}}_{ee} = -\gamma_{-}\hat{\rho}_{ee} + \gamma_{+}\hat{\rho}_{gg}}$$

Because: $\boxed{\hat{\rho}_{ee} + \hat{\rho}_{gg} = 1}$

$$\boxed{\dot{\hat{\rho}}_{gg} = \gamma_{-}\hat{\rho}_{ee} - \gamma_{+}\hat{\rho}_{gg}}$$

So, without doing any further calculation, I can simply write the time evolution of this rho gg would be equal to, I say, actually I should put rho ee here. Time evolution of this maybe earlier also I have to make the same correction here. I just left one. This you have to put. So, time evolution of the, this particular element rho gg would be equal to, because of this relation, we will have simply it as gamma minus rho ee minus gamma plus rho gg. So, these are the 2 relations we get.

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$$\dot{\hat{\rho}}_{ge} = \langle g|\dot{\hat{\rho}}|e\rangle$$

$$\langle g|[\hat{H},\hat{\rho}]|e\rangle$$

$$= \langle g|\hat{H}\hat{\rho}|e\rangle - \langle g|\hat{\rho}\hat{H}|e\rangle$$

Now, let us calculate rho ge time evolution of the density matrix element rho ge. That would be is equal to g rho dot e. Again, let us do it term by term. First of all, let me work out g H rho e. So, this would be equal to g H rho e minus g rho H e.

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$$= \langle g | H \rho | e \rangle$$

$$= -\frac{\hbar\omega_{at}}{2}\langle g|\hat{\rho}|e\rangle - \frac{\hbar\omega_{at}}{2}\langle g|\hat{\rho}|e\rangle$$

$$\boxed{\begin{aligned}\hat{H}|e\rangle &= \frac{\hbar\omega_{at}}{2}|e\rangle \\ \hat{H}|g\rangle &= -\frac{\hbar\omega_{at}}{2}|g\rangle\end{aligned}}$$

$$= -\hbar\omega_{at}\,\hat{\rho}_{ge}$$

$$\langle g|\gamma_-\left(\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \frac{1}{2}\hat{\sigma}_+\hat{\sigma}_-\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{\sigma}_+\hat{\sigma}_-\right)|e\rangle$$

And because, already, we have written down this, let me remind again you again that H e is equal to h cross omega atom by 2 e. And, H when operates on the ground state, you will get minus h cross omega atom by 2 g because of this eigenvalue equations. I will get here the terms as here I will get it, minus h cross omega atom by 2 g rho e. And, from this term, I will get it as minus h cross omega atom by 2 g rho e.

And, combining these 2, I will get minus h cross omega atom rho ge. Now, let us work out this term, g gamma minus sigma minus rho sigma plus minus half sigma plus sigma minus rho minus half rho sigma plus sigma minus.

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$$\langle g|\, \alpha_- \left(\hat{\sigma}_- \hat{\rho}\, \hat{\sigma}_+ - \frac{1}{2} \underline{\hat{\sigma}_+ \hat{\sigma}_-} \hat{\rho} - \frac{1}{2} \hat{\rho}\, \hat{\sigma}_+ \hat{\sigma}_- \right) |e\rangle$$

$$\# \qquad \langle g|\, \hat{\sigma}_- \hat{\rho}\, \underline{\hat{\sigma}_+ |e\rangle} = 0 \qquad\qquad \hat{\sigma}_+ |e\rangle = 0$$

$$\# \qquad \langle g|\, \underline{\underline{\hat{\sigma}_+}}\, \hat{\sigma}_- \hat{\rho}\, |e\rangle \qquad\qquad \langle g|\, \hat{\sigma}_+ = 0$$

$$= \underbrace{\langle g|e\rangle}_{=0} \langle g|g\rangle \langle e|\hat{\rho}|e\rangle$$

$$= 0$$

Let us do it term by term again. So, first of all, this one, g sigma minus rho sigma plus e, these actually as you can see would be equal to 0 because sigma plus when it is operating this part if you see when it operating on the excited state, there is no more excited state to you know raise it. So, therefore, this is going to give us 0. Then, let us look at this term now. So, we have g this particular term, sigma plus sigma minus rho e.

Here, again, when sigma plus operates on the ground state, effectively it is going to give me 0 because it is operating on the bra. So, this is going to give me 0. Let me do it explicitly. You will have g here and sigma plus is you are going from the ground state to the excited state. And here, sigma minus is you are going from the excited state to the ground state. And, you will have rho e here.

And because, these are orthogonal, so, this is going to give me 0. So, therefore, this would be simply 0.

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$$\# \quad \langle g | \hat{\rho}\, \hat{\sigma}_+ \underline{\underline{\hat{\sigma}_-}} | e \rangle = \langle g | \hat{\rho}\, \underline{\underline{\hat{\sigma}_+}} | g \rangle$$

$$= \langle g | \hat{\rho} | e \rangle = \hat{\rho}_{ge}$$

Thus,

$$\langle g | \gamma_- \left(\hat{\sigma}_- \hat{\rho}\, \hat{\sigma}_+ - \frac{1}{2} \hat{\sigma}_+ \hat{\sigma}_- \hat{\rho} - \frac{1}{2} \hat{\rho}\, \hat{\sigma}_+ \hat{\sigma}_- \right) | e \rangle$$

$$= - \frac{\gamma_-}{2} \hat{\rho}_{ge}$$

Now, let us look at this term. That is g rho sigma plus sigma minus e. Now, when sigma minus the atomic lowering operator, it will lower the excited state to the ground state, so, we will have g rho sigma plus g here. And then, sigma plus, it is going to raise it. So, therefore, we will have g rho e. You can actually do the explicit calculation also. So, therefore, I can finally write the whole thing as g gamma minus sigma minus rho sigma plus minus half sigma plus sigma minus rho minus half rho sigma plus sigma minus e.

That is equal to, this is, I can write it as rho ge. And, you will have it as I think another term in the similar way you will have only this. So, because of this half term, you will have minus gamma minus by 2 rho ge. So, this is what you will get.

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Sly,

$$\langle g | \gamma_+ \left(\hat{\sigma}_+ \hat{\rho}\, \hat{\sigma}_- - \frac{1}{2} \hat{\sigma}_- \hat{\sigma}_+ \hat{\rho} - \frac{1}{2} \hat{\rho}\, \hat{\sigma}_- \hat{\sigma}_+ \right) | e \rangle$$

$$= - \frac{\gamma_+}{2} \hat{\rho}_{ge}$$

Finally,

$$\langle g | 2\gamma_\phi \Big(|e\rangle\langle e| \hat{\rho} |e\rangle\langle e| - \frac{1}{2} |e\rangle\langle e| \hat{\rho} - \frac{1}{2} \hat{\rho}\, |e\rangle\langle e| \Big) |e\rangle$$

$$= 2\gamma_\phi \left(-\frac{1}{2} \right) \hat{\rho}_{ge}$$

$$= -\gamma_\phi\, \hat{\rho}_{ge}$$

Again, proceeding exactly the same way, we will get g gamma plus. The term associated with gamma plus that would be sigma plus rho sigma minus minus half sigma minus sigma plus rho minus half rho sigma minus sigma plus e. This if you work it out, you will find that this would be minus gamma plus by 2 rho ge. Then finally, we are left out with the, this pure dephasing term. That would be, let us work it out.

That would be g here 2 gamma phi. I have here e e rho e. Then, I have minus half ket e by rho minus half rho e e. Then, I have here e. So, this is very simple. As you can see, you will have 2 gamma phi. And, rest of the terms from there you will get simply minus half so, here you will have it also I think you will have rho ge. So, you have minus gamma phi rho ge. Now, you can combine all the terms.

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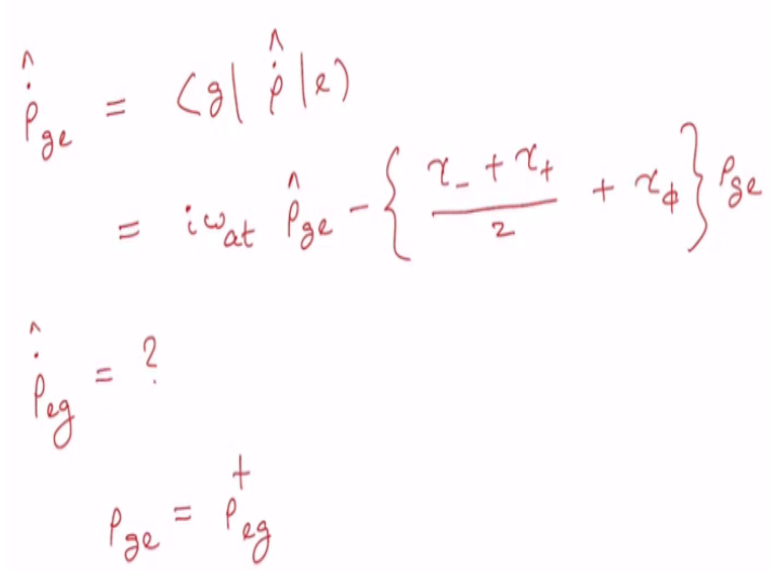

And, we will get the time evolution of this density matrix element rho ge is equal to which is g rho dot e. And, combining everything, you will get it as i omega atom rho ge minus. You will have gamma minus plus gamma plus by 2. And, you will have plus gamma phi rho ge. Now, we are left out to find out what is the time evolution of this density matrix element? And, that is easy because we know that rho ge is equal to rho dagger eg.

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$$\dot{\rho}_{eg} = ?$$

$$\rho_{ge} = \rho_{eg}^{\dagger}$$

$$\dot{\hat{\rho}}_{eg} = -i\omega_{at}\,\hat{\rho}_{eg} - \left\{\frac{\gamma_- + \gamma_+}{2} + \gamma_\phi\right\}\rho_{eg}$$

So, utilizing this, we can immediately write time evolution of the density matrix element rho eg is equal to minus i omega atom rho eg minus gamma minus plus gamma plus by 2 plus gamma phi rho eg. So, we have obtained all the evolution terms for the density matrix element.

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Problem 2: The entangled state of a two transmon Qubits is given as:

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Find the reduced density matrix corresponding to either of the tranomon Qubit.

Solution

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle}{\sqrt{2}}$$

Now, let us work out this particular problem. The entangled state of a 2 transmon qubit is given as the wave function is or the state vector is given to you. You are asked to find out the reduced density matrix corresponding to either of the transmon qubit. So, there are 2 transmon qubit and their combined state is given as psi is equal to 0 0 plus 1 1 by root 2. In fact, you know that the first 0 here correspond to the first qubit.

And, second 0 correspond to the second qubit. And, here one, this first one corresponds to the first qubit. And, one here corresponds to the second qubit.

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$$|\psi\rangle = \frac{\qquad}{\sqrt{2}} \qquad \sqrt{2}$$

$$\neq |\phi_1\rangle \otimes |\phi_2\rangle$$

$$\hat{\rho} = |\psi\rangle\langle\psi|$$

$$= \frac{1}{2}\left[\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right)\right]$$

$$= \frac{1}{2}\Big[|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \Big]$$

Now, this you cannot write it as a product of 2 states. Say, state of the first qubit is a phi 1 and the state of the second qubit as phi 2. We cannot write it. We cannot separate it in into the 2 such product states. This is not possible. And, that is the reason this particular state is in entangled state. Now, to solve this problem, let me first find out the density operator. So, rho would be equal to ket psi bra psi. Take the outer product.

And, I will have, here, let me write the terms explicitly. I have here because of 1 by root 2 1 by root 2 I will have half here. And here, I have 0 0 plus 1 1 and then, bra 0 0 plus bra 1 1. This is I have. If I now open it up, then I will have half ket 0 bra 0 you see this first qubit I will take the direct product with this qubit first qubit of this term here in the bra. Similarly, this one, I will have with this one. And similarly, this one, I will have with this one.

This, I will have with this and similarly, other term. In total, I will get 4 terms. So, let me write all the terms here. This would be, I will have 0 0 plus 0 1, direct product 0 1. And we will have 1 0 direct product 1 0 plus 1 1 direct product 1 1. So, this is what I will get. This is for the first qubit and this for the second qubit.

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$$\hat{\rho}_1 = tr_2\, \hat{\rho}$$

$$= \frac{1}{2}\Big[|0\rangle\langle 0| \overset{=1}{\langle 0|0\rangle} + |0\rangle\langle 1| \overset{=0}{\langle 0|1\rangle} + |1\rangle\langle 0| \underset{=0}{\langle 1|0\rangle} + |1\rangle\langle 1| \underset{=1}{\langle 1|1\rangle} \Big]$$

$$\Rightarrow \hat{\rho}_1 = \frac{1}{2}\Big[|0\rangle\langle 0| + |1\rangle\langle 1| \Big]$$

$$\equiv \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

And now, to find out the reduced density matrix for the first qubit first transmon qubit, let me write, I will have to take the trace I will have to trace out second transmon qubit. So, I will take trace over the second qubit transmon qubit. And, as you know, when we will take the trace operation, we have to operate it on the, I am now taking the trace operation on the second qubit here.

And if I take that operation, you know, this outer product will simply become the scalar product. And I will have it like this, here 0 0 and we will have 0 0 plus 0 1. I will have 0 1. This is all I will also have 1 0. I will have 1 0 here. And, I will have 1 1 here. Now, 0, ket 0 and ket 1 are orthogonal to each other and because of that, this is going to give me 1. This is going to give me 1. But, these are orthogonal. So, this term will vanish.

And therefore, I will have rho 1 reduced density matrix for the first transmon qubit would be half 0 0 plus 1 1. In fact, I can write it in a matrix form as well. And, that would be half 1 0 0 1.

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$$\Rightarrow \rho_1 = \frac{1}{2}\left[\ldots \right]$$

$$\equiv \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{\rho}_2 = tr_1 \hat{\rho} = \frac{1}{2}\Big[\langle 0|0\rangle\, |0\rangle\langle 0| + \langle 0|1\rangle\, |0\rangle\langle 1| + \langle 1|0\rangle\, |1\rangle\langle 0| + \langle 1|1\rangle\, |1\rangle\langle 1| \Big]$$

$$= \frac{1}{2}\Big[|0\rangle\langle 0| + |1\rangle\langle 1| \Big]$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad //$$

Exactly in the similar way, I can calculate the reduced density matrix for the second transmon and transmon qubit and for that I have to trace out first transmon qubit. And, if I do it again in the similar way, the operation scalar operation would be on the first qubit. So, here, I will get 0 0. And here, I will have the second transmon qubit part will remain as it is. And here, I will have 0 1 0 1 plus 1 0 1 0 and 1 1 1 1. And clearly, you will get.

This term would be half 0 0 plus 1 1. And, this is the same as rho 1. This would be half. In matrix form, we can write it as 1 0 0 1.

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Problem 3

Consider a planar cavity of length $d = 1$ cm. The cavity is resonant with light of $\lambda = 1\,\mu m$. The photon lifetime inside the cavity is measured to be $\tau_p = 1/\tau_p$ where $\tau_p / 2\pi = 1$ MHz.

(a) Find the free spectral range of the cavity.

(b) Compute the ratio $Q/\mathcal{F}$ for this cavity at $\lambda = 1\,\mu m$.

(c) Find the Finesse $\mathcal{F}$ of the cavity

(d) Find the wavenumbers at which the intracavity intensity is half the maximum value.

Now, let us work out this problem on Fabry-Perot cavity. I believe doing this problem will teach you quite a bit about cavity. The problem goes like this. You are asked to consider a planer cavity of length 1 centimeter. The cavity is resonant with light of wavelength 1 micrometer. The photon lifetime inside the cavity is measured to be tau P is equal to 1 by gamma P where gamma P by 2 pi is equal to 1 megahertz. The problem has 5 parts.

And, we will do it part by part one by one. In the first part of the problem, you are asked to find the free spectral range of the cavity. So, let us do it first.

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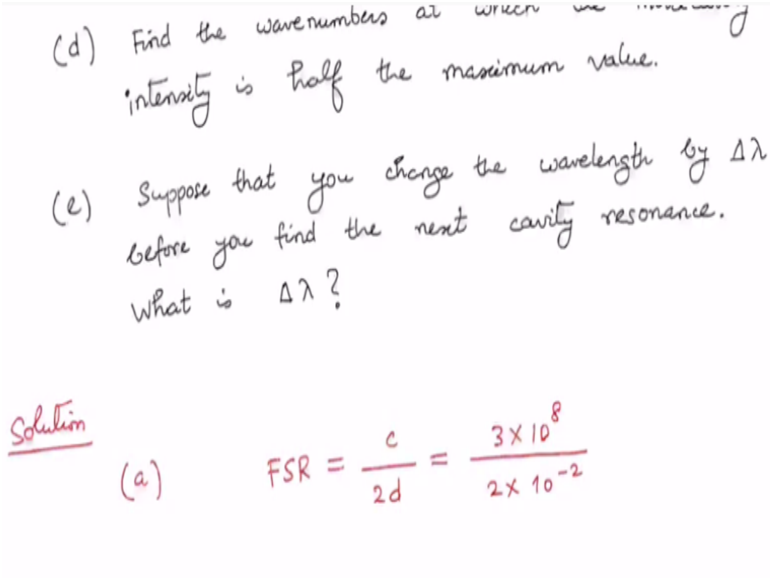

So, you have already learned about free spectral range what is it in Lecture 22. We know that the free spectral range is given by this formula. That is c divided by twice into the length of the cavity. And, c is the speed of light in free space. And, that is 3 into 10 to the power 8 meter per second and d is 1 centimeter which is 10 to the power minus 2.

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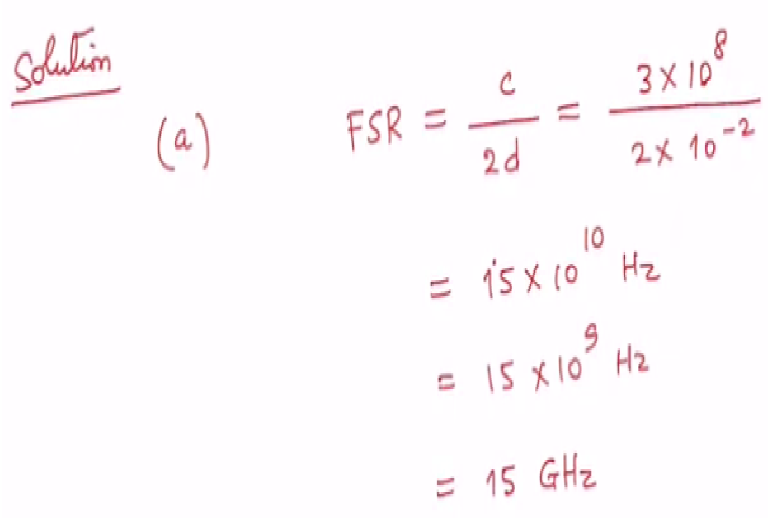

So, therefore, it is easy to see took that you will get 1.5 into 10 to the power 10 hertz which I can write as 15 into 10 to the power 9 hertz. And, we know that 10 to the power 9 hertz is 1 gigahertz. So, therefore, I can write it as 15 gigahertz. So, this is the free spectral range.

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(b) $$\frac{Q}{F} = m$$

$$m\lambda = 2d \Rightarrow m = \frac{2d}{\lambda}$$

$$\frac{Q}{F} = m = \frac{2 \times 10^{-2}}{10^{-6}} = 2 \times 10^{4}$$

Now, let me go to the 2nd part. In b, you are asked to compute the ratio of Q by F. That is the quality factor by finesse. This ratio you have to calculate for this cavity at the wavelength lambda is equal to 1 micrometer. We know from our class that this ratio of quality factor and the finesse of the cavity is an integer, say m. And, this integer gives the order of the cavity resonance.

And, we know that this order of the cavity that is m is equal to the number of wavelengths in the length of 2d. So, we know this relation that m into lambda is equal to 2d. So, from here, we can easily find out m. m is equal to 2d divided by lambda. Therefore, this ratio of quality factor and the finesse is equal to 2d. d is 1 centimeter. That is 10 to the power minus 2 meter and lambda is 1 micrometer.

So, therefore, it is 10 to the power minus 6 meter. So, if you do it, you will see that this ratio would be 2 into 10 to the power 4. Let us go to the 3rd part. Now, you are asked to find out the finesse of the cavity.

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$$(c) \quad \mathcal{F} = 2\pi \tau_p (FSR)$$

$$= 2\pi \frac{1}{\gamma_p} (FSR) \; ; \quad \gamma_p = 2\pi \times 1\,\text{MHz}$$

$$= 2\pi \frac{15 \times 10^9}{2\pi \times 10^6}$$

$$= 15 \times 10^3$$

$$= 15000$$

So, we also know the formula for the finesse. You please refer to the class lecture class. You will see that I have written down this particular formula. Finesse is equal to 2 pi into the cavity lifetime photon the lifetime there into the free spectral range. Now, this cavity photon lifetime is tau P is equal to 1 by gamma P. And, we already calculated free spectral range. And, gamma P is given to us as 2 pi into 1 megahertz.

So, if I put the parameters in the formula. I will get 2 pi into gamma P is 2 pi into 1 megahertz. This is 10 to the power 6 hertz. And, free spectral range, we have already calculated. That is 15 gigahertz, so, 15 into 10 to the power 9. So, you will see we will get it as 15 into 10 to the power 3 or simply 15,000. That is the finesse.

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$$(d) \quad I = \frac{I_{max}}{1 + \left(\frac{2\mathcal{F}}{\pi}\right)^2 \sin^2 kd}$$

$$\text{For} \quad I = \frac{I_{max}}{2}$$

we have:

$$\left(\frac{2\mathcal{F}}{\pi}\right)^2 \sin^2 k_{\pm} d = 1$$

$$\Rightarrow \quad \sin k_{\pm} d = \pm \frac{\pi}{2\mathcal{F}}$$

Now, let us go to the 4th part. In part d, you are asked to find the wave numbers at which the intracavity intensity is half the maximum value. The intracavity field intensity is related to the maximum intensity by this expression. So, I is equal to I max divided by 1 plus 2 into finesse divided by pi whole square sine square k into d. d is the length of the cavity. k is the wave number. F is the finesse. Now, for I is equal to I max by 2.

We can see that we will have 2F by pi whole square sine square k d. k can take 2 values. So, it may be, so, let me just put it appropriately here. Let me write k plus minus d is equal to 1. And from here, I have sine k plus minus d is equal to plus minus pi divided by twice into the finesse.

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$$\Rightarrow \boxed{k_{\pm} = \pm \frac{1}{d} \sin^{-1}\left(\frac{\pi}{2F}\right)}$$

we are given: $d = 1\ \text{cm} = 10^{-2}\ \text{m}$

$$F = 15 \times 10^{3}$$

$$k_{\pm} \approx \pm 0.0105\ \text{m}^{-1}$$

$$= \pm 1.05 \times 10^{-2}\ \text{m}^{-1}$$

So, we have k plus minus is equal to plus minus 1 by d sine inverse pi divided by twice that of the finesse. Now, we are given d is equal to 1 centimeter or 10 to the power minus 2 meter. And finesse, we have calculated as 15,000 or 15 into 10 to the power 3. So, if I put all these things there in this formula, then you will see. You please do that. And, you will see that you will get k plus minus is equal to plus minus 0.0105 per meter.

Or, I can write it as plus minus 1.05 into 10 to the power minus 2 per meter. So, this is the required wave number. Now, let us do the final part of the problem. Suppose that you change the wavelength by delta lambda before you find the next cavity resonance. What is delta lambda? Let us do it.

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(c) If the wavelength increases, the next resonance is when one fewer wavelength fits into cavity.

$$\Rightarrow \quad \boxed{2d = m\lambda_m = (m-1)\lambda_{m-2}}$$

$$\Delta\lambda = \lambda_{m-1} - \lambda_m$$

You know if the wavelength increases, let me write here, if the wave length increases, the next resonance that you are going to get, the next resonance is the one when we will have one fewer wavelength fits into cavity. So, what does it mean? This implies that the resonance condition we know that this is twice into the length of the cavity is m into integral multiple of the wavelength.

And, this would be where you go to the next one you will have lambda m minus 1. And here, you will have m minus 1. So, we can use this relation to work out the solution of this particular part because delta lambda here is difference between the wavelengths of the nearby resonances. This is lambda m minus 1 minus lambda m.

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$$\Rightarrow \quad \boxed{2d = m\lambda_m = (m-1)\lambda_{m-2}} \ \#$$

$$\Delta\lambda = \lambda_{m-1} - \lambda_m$$

$$= \frac{m}{m-2}\lambda_m - \lambda_m$$

$$= \lambda_m\left(\frac{2}{m-2}\right)$$

$$\approx \frac{\lambda_m}{m}$$

Now, using this relation here, I can now write lambda m minus 1 as m divided by m minus 1 lambda m minus lambda m. From here, you can easily see that I will get lambda m into 1 divided by m minus 1. m is the order. So, we know this is already a very huge number. So, I can approximately write it as lambda m divided by m.

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$$= \lambda_m \left(\frac{1}{m-1} \right)$$

$$\approx \frac{\lambda_m}{m}$$

$$= \frac{10^{-6}}{2 \times 10^{4}}$$

$$= 0.5 \times 10^{-10} \text{ m}$$

$$= 0.05 \text{ nm}$$

And, we have, we are given lambda m as 1 micrometer. That is 10 to the power minus 6 meter. And, we already calculated the order m. That is 2 into 10 to the power 4. That we have calculated in the first part of the problem. And, if we do this, we will get 0.5 into 10 to the power minus 10 meter which I can write as 0.05 nanometer. So, this is the required answer.

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Problem 4

(a) Find the radiation pressure force exerted by a steady stream of photons with power P on a perfectly reflecting mirror.

(b) Work out the ratio of the displacement of a mirror on which photons are incident with respect to its zero-point fluctuation in position.

Now, let us work out this problem. It has 2 parts. You are asked to find the radiation pressure force exerted by a steady stream of photons with power P on a perfectly reflecting mirror. Let us do it first.

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Solution

(a)

$$\Delta p = \frac{h}{\lambda} - \left(\frac{h}{\lambda}\right) = \frac{2h}{\lambda}$$

$$F = N\,\frac{\Delta p}{\Delta t}$$

$$= N\,\frac{2h}{\lambda(\Delta t)} = 2N\left(\frac{hc}{\lambda}\right)\frac{1}{c\Delta t}$$

$$F = \frac{2\,\Delta E}{c\,\Delta t} \qquad \Delta E = \frac{Nhc}{\lambda}$$

$$= \frac{2P}{c}$$

As discussed in Lecture 22, we know that the momentum imparted by photon of wavelength lambda on a perfectly reflecting mirror is the change in momentum delta P of the photon and which is given by h by lambda minus h by lambda. So, this is equal to 2h by lambda. So, that is the change in momentum. The radiation pressure force imparted by a steady stream of photons.

Suppose the stream has n number of photons, then the force would be equal to N into rate of change of momentum by per photon. So, this is the total force. Now, we have N delta P. We already know. That is 2h by lambda into delta t. This I can write as N into let me write 2 here. 2N into hc by lambda and because I have multiplied c up all up, so, I, we have to put here also c delta t. This guy hc by lambda is the energy of a single photon.

So, the total energy would be N into hc by lambda for the whole bunch of photons. So, if I write delta E is equal to N into hc by lambda, then I can write force as 2 into delta E divided by c into delta t. Now, energy per unit time is the power. So, therefore, I will have 2 into P divided by c. So, this is the required answer. Now, let us do the 2nd part. Work out the ratio of the displacement of a mirror on which photons are incident with respect to its zero point fluctuation in position. Let us do it.

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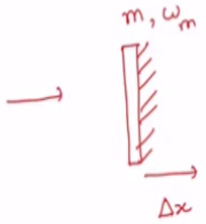
(b)



$$\boxed{\frac{1}{2} m\omega_m^2 (\Delta x)^2 = \frac{(\Delta p)^2}{2m}} \qquad \Delta p = \frac{2h}{\lambda}$$

$$\Rightarrow \quad \Delta x = \frac{\Delta p}{m\omega_m} = \frac{2h}{\lambda\, m\omega_m}$$

Say, due to the photon, the mirror having mass m and resonance frequency omega m is getting displaced by an amount delta x when a photon is getting incident on it. The kinetic energy gained by the mirror due to the momentum imparted by the photon gets converted to potential energy of the mirror. So, as per the conservation of energy principle, so, we will have a half m omega m square delta x square. That is the potential energy of the mirror.

And, the kinetic energy would be delta P square by 2m where delta P we know that this is equal to 2h by lambda. From this equation, we can find out what is delta x? Delta x would be equal to delta P divided by m omega m which I can now write. Putting the value of delta P, I can write it as 2h divided by lambda m omega m.

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$$\Rightarrow \quad \Delta x = \frac{\Delta p}{m\omega_m} = \frac{2h}{\lambda\, m\omega_m}$$

$$x_{ZPF} = \sqrt{\frac{\hbar}{2m\omega_m}} \qquad \hbar = \frac{h}{2\pi}$$

$$\Rightarrow \quad x_{ZPF}^2 = \frac{\hbar}{2m\omega_m} = \frac{h}{4\pi\, m\omega_m}$$

$$\Rightarrow \quad \frac{h}{m\omega_m} = 4\pi\, x_{ZPF}^2$$

Also, we know that the zero-point fluctuation is given by h cross divided by twice m omega m where h cross is the reduced Planck's constant. It is h by 2 pi. And, from this expression, I have square of this zero-point fluctuation term would be equal to h cross by twice m omega m or I can write it as h divided by 4 pi m omega m. And, from here, let me work out this particular expression.

So, I have h divided by m omega m. That would be equal to 4 pi x square zero-point fluctuation.

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$$\Rightarrow x_{ZPF}^2 = \frac{\hbar}{2m\omega_m} = \frac{h}{4\pi m\omega_m}$$

$$\Rightarrow \frac{h}{m\omega_m} = 4\pi x_{ZPF}^2$$

$$\Delta x = \frac{2}{\lambda}\frac{h}{m\omega_m}$$

$$= \frac{2}{\lambda} 4\pi x_{ZPF}^2$$

$$\Rightarrow \boxed{\frac{\Delta x}{x_{ZPF}} = \frac{8\pi}{\lambda} x_{ZPF}}$$

Now, we have delta x is equal to 2 by lambda into h divided by m into omega m. So, clearly, we can now write it as 2 by lambda into 4 pi x square zero-point fluctuation. And, from here, I can write the, it as delta x divided by x zero-point fluctuation is equal to 8 pi by lambda into x zero point fluctuation. So, this is the required answer to the problem.