Quantum Technology and Quantum Phenomena in Macroscopic Systems
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Module-03
Lecture-39
Problem Solving Session-9.

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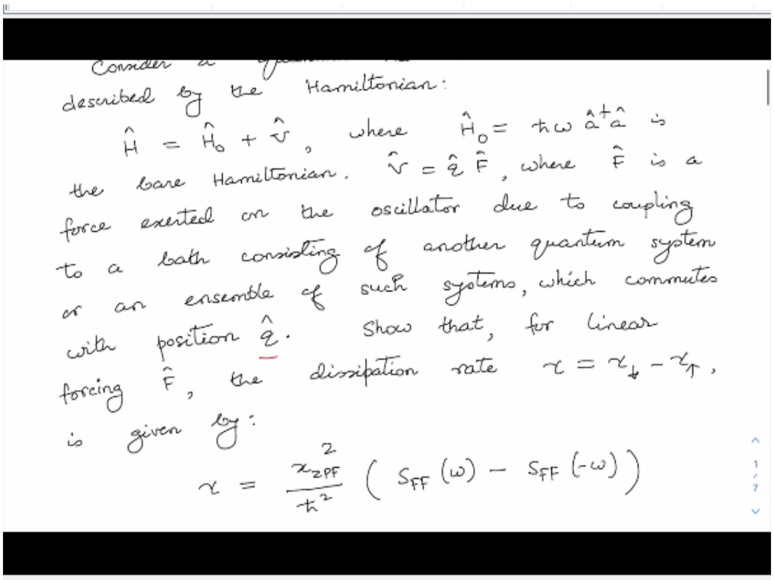

Welcome to this problem solving session number 9. In this problem solving session we will solve problems related to quantum harmonic oscillator interacting with a bath or environment. Now the first problem, consider a quantum harmonic oscillator described by the Hamiltonian H = H 0 + V, where H 0 is h cross omega a dagger a, is the bare Hamiltonian. So, I have not written + half here, so because that is constant.

And V = q F, where F is a force exerted on the oscillator due to coupling to a bath consisting of another quantum system or an ensemble of such systems, which commutes with the position q cap. Show that, for the linear forcing F, the dissipation rate is given by this expression.

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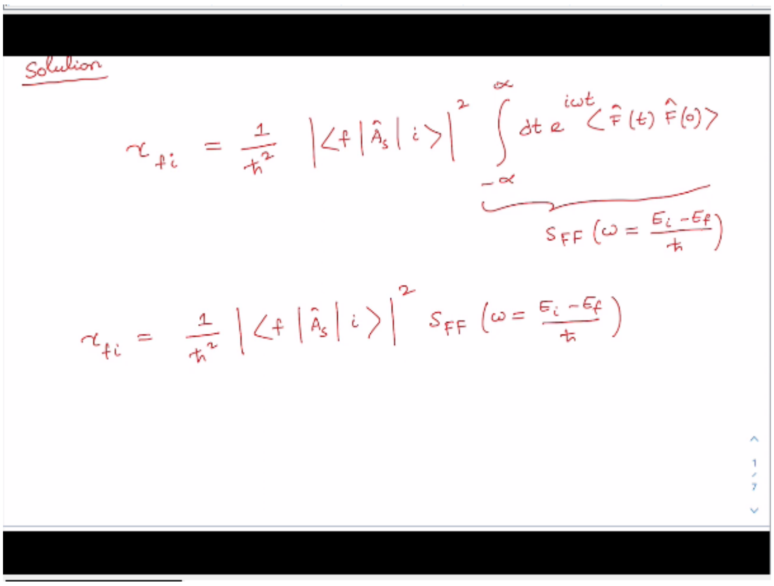

So, to do this problem you have to recall the Fermi golden rule which we have discussed in details in the supplementary lecture, also I have even if you have missed the supplementary lecture or you have not gone through that but you know the Fermi golden rule that we wrote in the lecture class. So, the Fermi golden rule says that the transition rate for a system going from the state i to the state f is given by this expression that is 1 by h cross square. You are going from the state i to the state f and A s is the system operator.

So, this mod square and the Fourier transform of the correlator that is F of t, these are the bath operators, the system is interacting with the bath operator by via this operator F and this is the Fourier transformation. So, this we know. In fact, we know that this is nothing but the spectral noise density or simply S FF where omega = E i - E f divided by h cross or let me write it here as gamma f i = 1 by h cross square modulus of this matrix element.

That is i to f you are going via A s and S FF evaluated at omega = E i - e f by h cross. So, here while I write this formula omega is considered to be positive because this i if you recall the lecture we are going from the higher energy state to the lower energy state. So, this is the transition rate.

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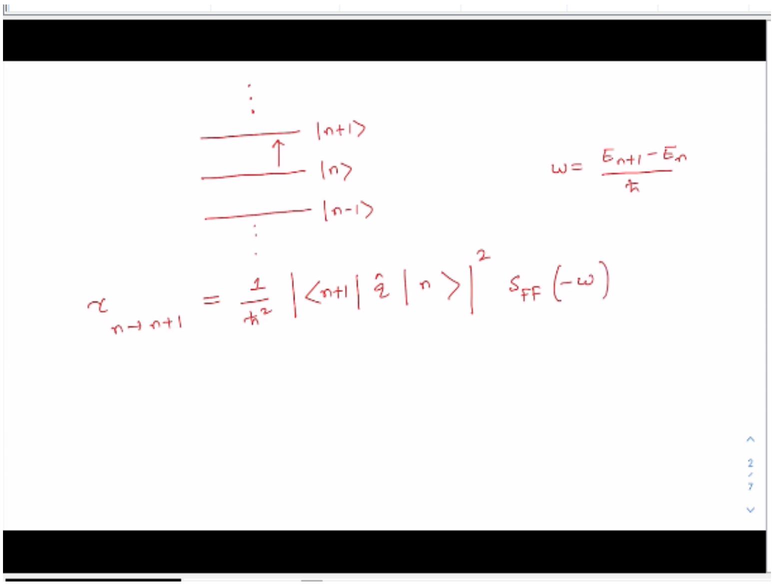

Now let us consider our quantum harmonic oscillator and let us say we have energy levels like that; all of them are equally spaced. So, this is the state n and this one is n + 1 then you have a higher energy levels and here it is say n – 1. Now the rate of transition for the system when it is going from the state say n to n + 1 that is the system is going in the upward direction then the formula if I apply I will have 1 by h cross square modulus you are going from n to n + 1 and the system operator here is q cap that is the position operator and mod square.

And S FF as per our expression we will have omega = E n because we are going from E n to n + 1. So, this is what I have. So, actually because I am taking positive omega, so omega as per my definition I will take it positive so En +1, here we are going from the upward direction. So, going from the lower energy state to the higher energy state so and omega we are taking to be positive. So, clearly I will write it simply as – omega. So, this would be – omega, I hope you get the idea here.

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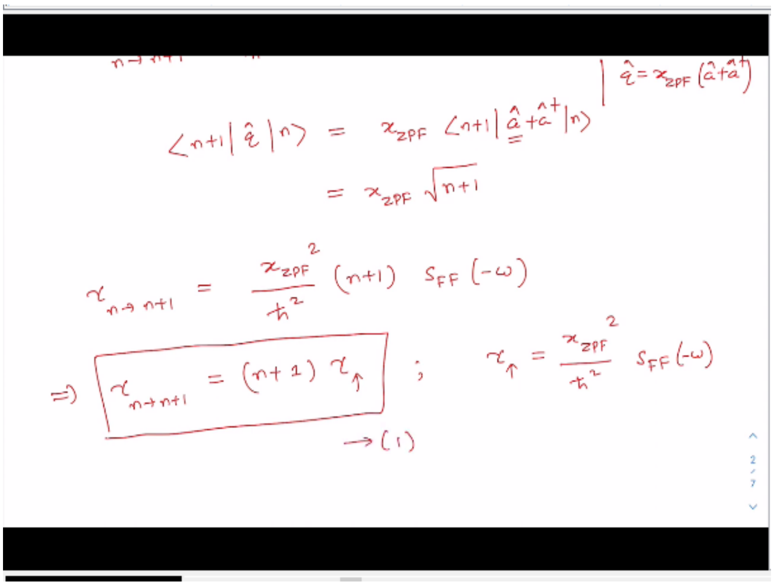

Now let me work out this quantity n + 1 q cap n and q cap is you know it is zero point fluctuation, q cap is the zero point fluctuation into a + a dagger. So, therefore I have here it would be n + 1 a + a dagger n. So, if you evaluate it you can see that you will get x, you will have zero point fluctuation and this will give you square root of n + 1; that is what you will get because then you will have n + 1, n + 1 that will give you 1.

But if you apply a on n then you will get n - 1 because of the orthogonality the other term will vanish. So, you will simply you will end up with this expression. Now you can write gamma n to n + 1 = x zero point fluctuation square divided by h square n + 1 spectral noise density evaluated at - omega or I can write it as n + 1 gamma up, where gamma up is the upward transition rate for a single proton. So, that is I will just write gamma up is equal to x zero point fluctuation square S FF evaluated at – omega. So, this is one expression we have. Let me say this is my equation number 1.

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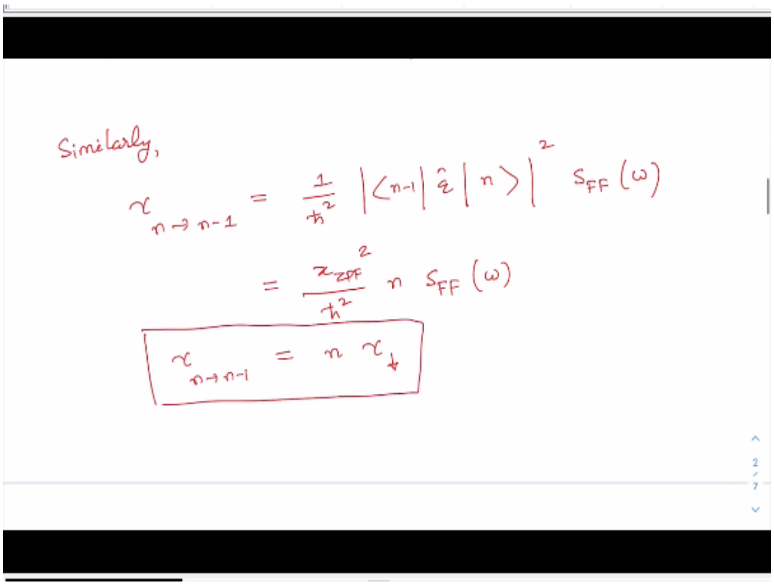

Similarly, you can work out the transition rate when you are going in the downward direction n to n - 1 that would be equal to 1 by h cross square. You have to evaluate this quantity you are going from n to n - 1 in the downward direction you are going, q is the system operator mod square, this time you will evaluate S FF + omega, because you see that here omega = E n - E n - 1 by h cross and harmonic oscillators are equally spaced.

So, this is a positive frequency, so therefore we have to write here + omega and similarly if you evaluate it you will get it as x zero point fluctuation square divided by h square n S FF at + omega or I can write it as n into gamma. Now we are going in the downward direction gamma. So, therefore we have another formula n to n – 1.

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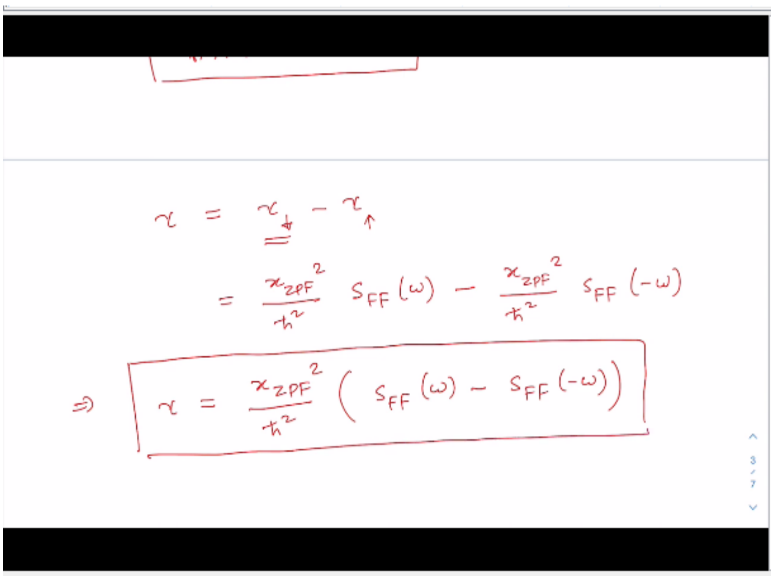

Now as per the definition the dissipation rate is the downward transition minus the upward transition for dissipation to happen this quantity the downward transition rate has to be higher than the upward transition rate. So, gamma down this quantity from here as you can see this is gamma up and gamma down also you can read out from here very easily. So, therefore if I put it I will have here x zero point fluctuation square by h square S FF at omega - gamma up is x zero point fluctuation square by h square S FF evaluated at – omega.

So, this implies that we get gamma is equal to x zero point fluctuation square by h square S FF at omega - S FF evaluated at – omega. So, this is the dissipation rate and it is an important formula worth remembering.

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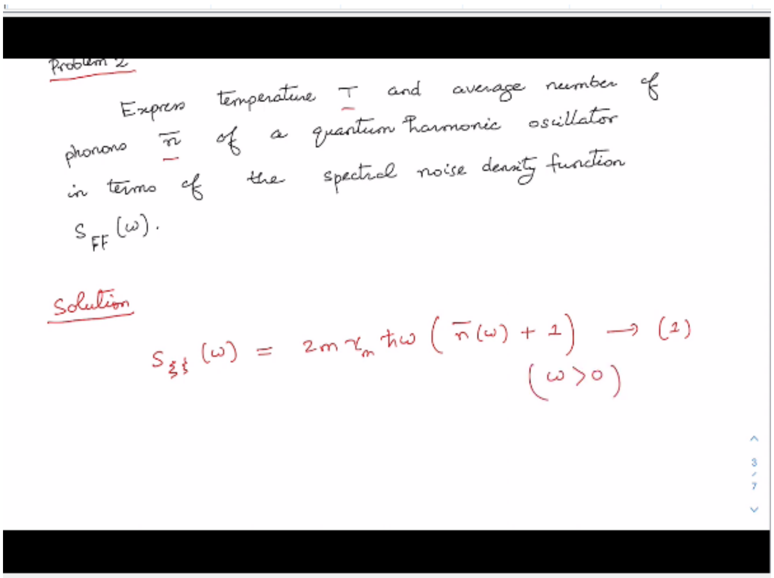

Now let us work out this problem. Express temperature T and the average number of phonons n of a quantum harmonic oscillator in terms of the spectral noise density function S FF omega. Let us do it. Now please refer to lecture 28, the lecture class where we obtain these expressions for spectral noise density S Xi Xi of omega where Xi is the Langevin noise; this was equal to twice m gamma m h cross omega.

Gamma m is the dissipation rate of the mechanical oscillator into n of omega that is the average number of phonons or quanta of the harmonic oscillator. This was for when omega is greater than 0.

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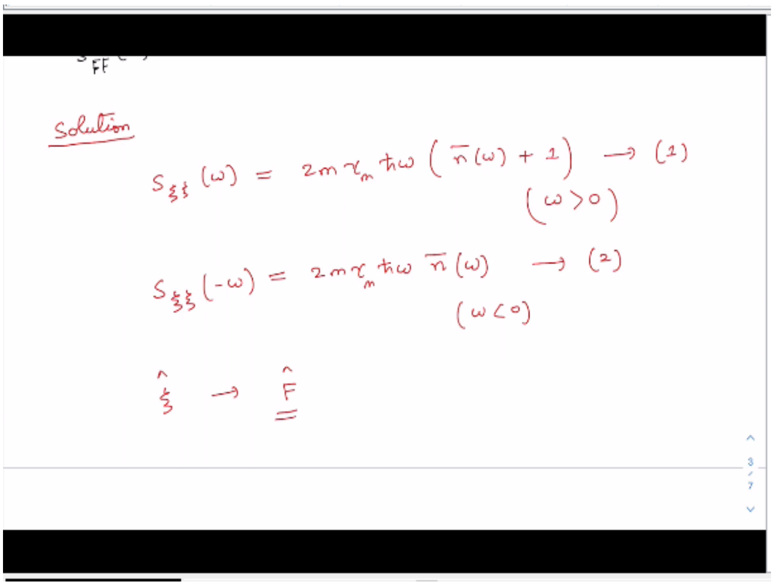

And then we also got S Xi Xi at - omega the negative frequency that is equal to twice m gamma m h cross omega n of omega and this is for omega less than 0, let us say this is formula number 2. Now here the Langevin noise operator Xi takes the role of the bath operator F as explained in the previous problem. We have explained about this bath operator in the previous problem.

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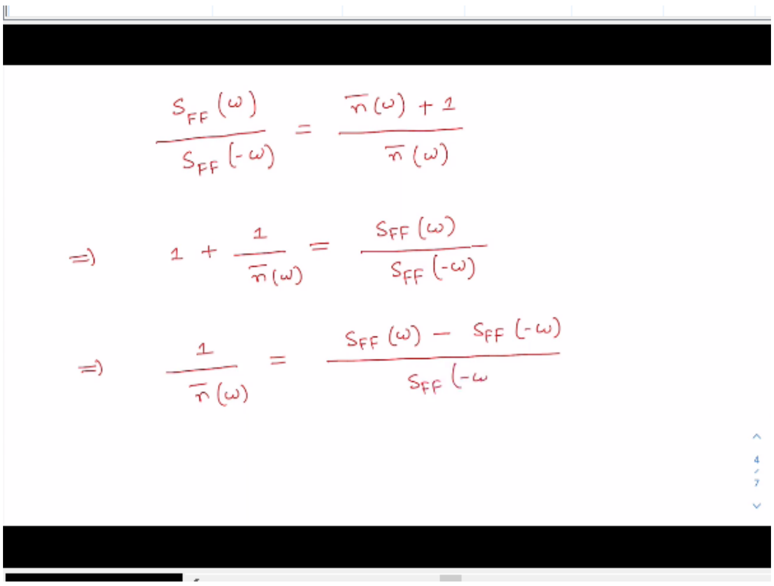

Now from equation 1 and 2 we can write if I take the ratio of these 2 expressions then you will see that instead of Xi now I will write F. S FF of omega divided by S FF at - omega that would be equal to n of omega + 1 divided by n bar of omega. From here you can see I can write this as 1 + 1 by n bar of omega that is equal to S FF omega divided by evaluated at – omega. So, from here I can write 1 by n of omega = S FF omega – S FF - omega divided by S FF at – omega.

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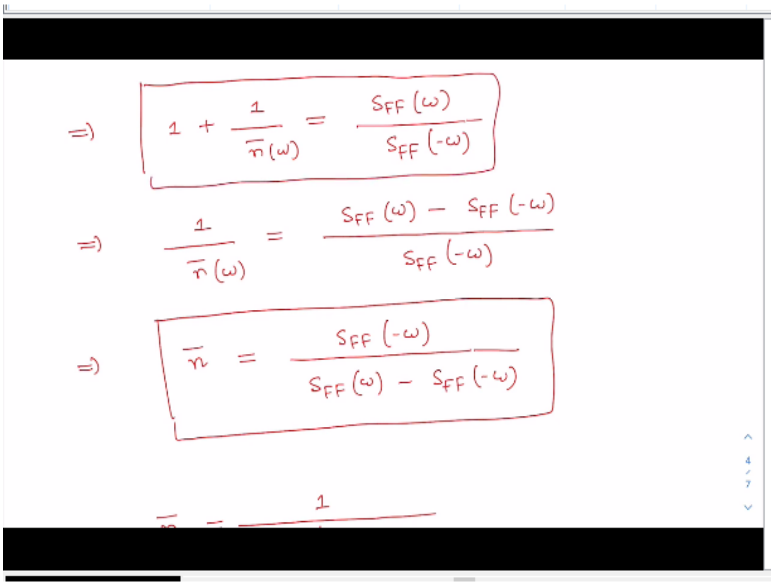

So, from here I can work out the average number of phonons in terms of the spectral noise density function as S FF evaluated at - omega divided by S FF at omega – S FF at – omega. So, this is what I obtain. Now regarding the temperature, you see we know that average number of phonon is given by this expression 1 divided by e to the power h cross omega by K B T – 1. So, from here I can write e to the power h cross omega by K B T = 1 + 1 by n bar and which is actually we have worked out this expression I already worked out.

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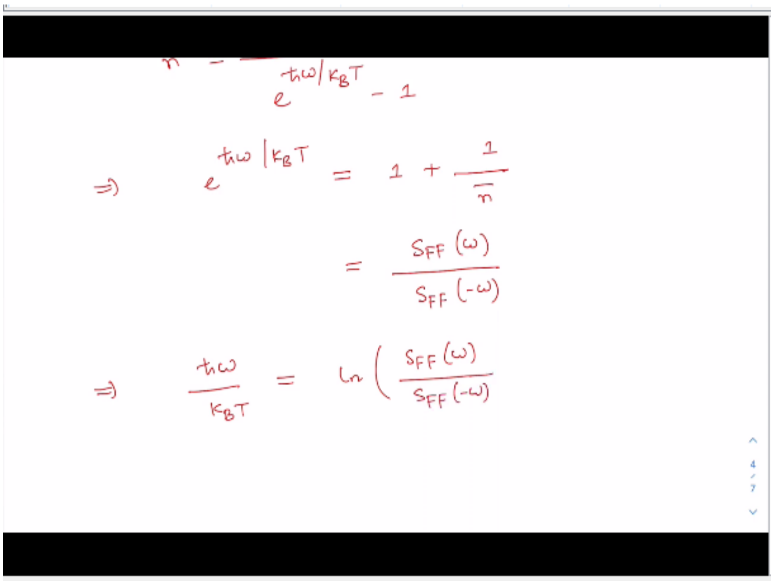

So, if I put it here then I will get it as S FF omega divided by S FF at - omega and you can see now if I take the logarithm on both sides then I will get h cross omega by K B T = logarithm of S FF omega divided by S FF at – omega.

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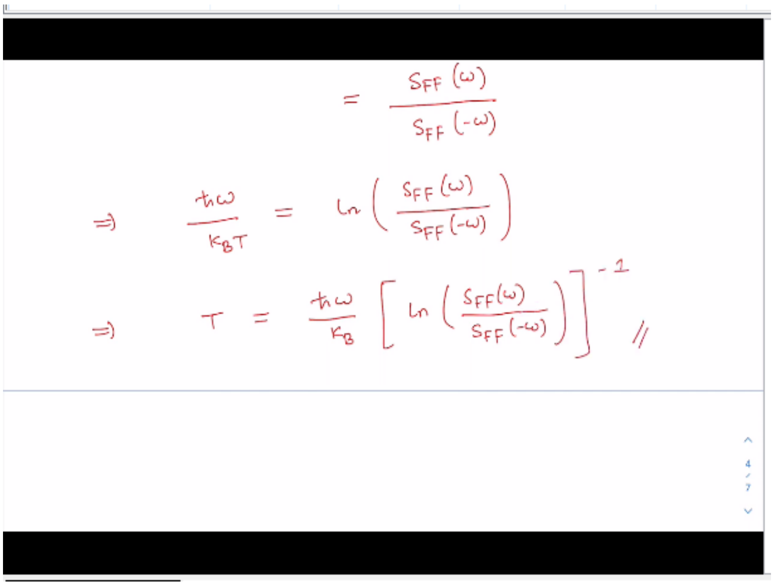

And from here I can write down the expression for the temperature T very easily that would be T = h cross omega by K B, K B is the Boltzmann's constant, logarithm of S FF omega divided by S FF at - omega to the power – 1. Now let us work out this problem.

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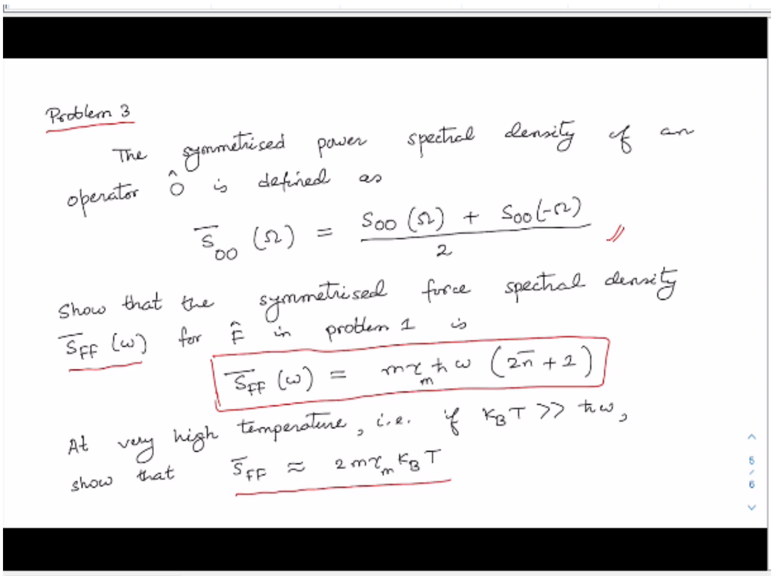

The symmetrized power spectral density of an operator O cap is defined as this, this is the definition for symmetrized power spectral density. You are asked to show that the symmetrized force spectral density S FF omega for F cap which is the fluctuating operator that we have discussed in problem number 1 is this. So, this is what we are asked to find out and also at very high temperature that is when K B T is much, much greater than h cross omega we have to show that this expression for the symmetrized force spectral density reduces to this expression. Let us do it.

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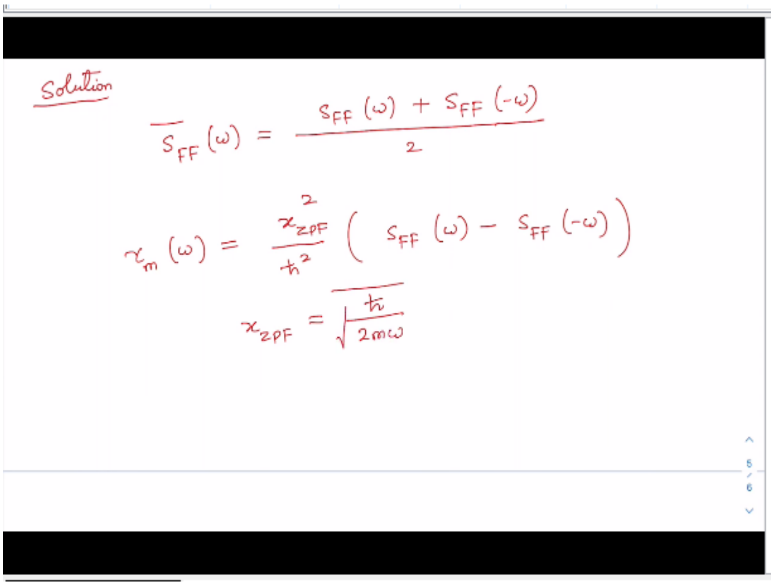

So, as per definition of the symmetrized power spectral density of an operator for the fluctuating operator F this quantity is now defined as S FF at omega + S FF at - omega divided by 2. As you can see that this quantity S FF bar or the symmetrized force spectral density it quantifies the fluctuation introduced to the quantum harmonic oscillator from environment and this fluctuation arises from the sum of power spectral density at positive frequency omega and at the negative frequency – omega.

And also, you know that in the quantum regime these 2 quantities which we have actually shown in the class, S FF omega is not equal to S FF at - omega in the quantum regime they are not equal but in the classical regime they are equal. So, these things we know and also, I told you that F plays the role of quantum Langevin noise here, these are already we know.

Now let us work out the expression we know that the dissipation rate gamma for the mechanical oscillator that is gamma m evaluated at frequency omega this expression already we have worked out in the previous problem that is x square zero point fluctuation divided by h cross square S FF evaluated at omega - S FF at – omega. This is known to us where this x zero point fluctuation for the harmonic oscillator is h cross divided by twice m omega square root.

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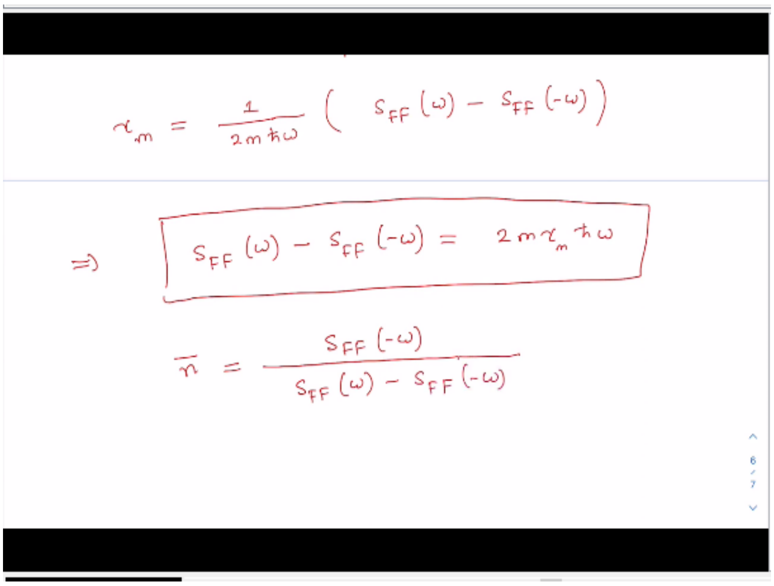

And therefore, I can write gamma m = 1 by twice m h cross omega into S FF at omega – S FF evaluated at – omega. So, from here I get S FF at omega – S FF at - omega = twice m gamma m h cross omega. So, this is one expression we have. Also, from the previous problem we have worked out the average number of quanta in the harmonic oscillator in terms of the spectral noise density that is S FF at - omega divided by S FF at omega – S FF at – omega.

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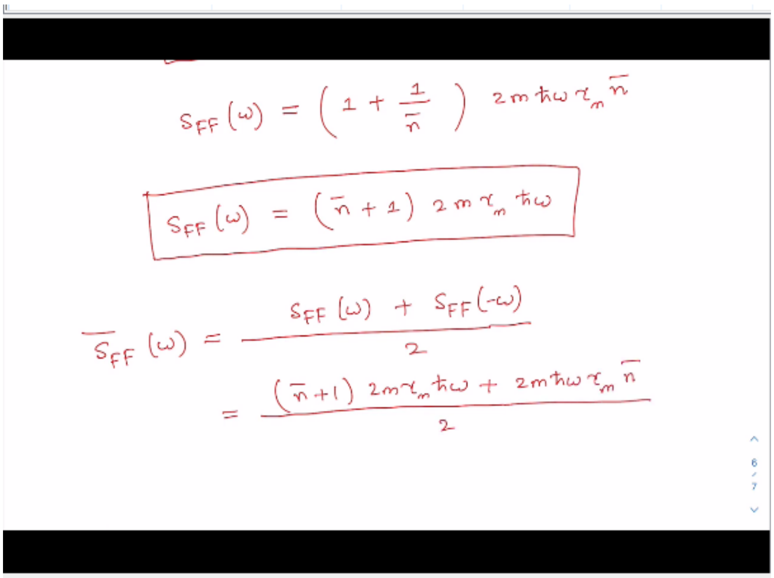

From here I have this expression S FF at omega minus this quantity evaluated at – omega. This is equal to 1 by n bar S FF evaluated – omega. Now I have these 2 expressions at my disposal this is say equation 1 and this is equation 2. I can play with these 2 and from here one thing I can immediately write is from these 2 expressions one quantity that I can work out is S FF, because I know what is this expression from here.

So, using this I have S FF at - omega = twice m h cross omega gamma m n bar. So, I know this quantity, so I have to find out the other quantity that is S FF at + omega that would be equal to 1 + 1 by n bar and S FF at - omega is known to me. So, let me put the expression for that twice m h cross omega gamma m n bar, from here I have S FF at omega is equal to I will have n + 1 into twice m gamma m h cross omega.

So, I now know the noise spectral density at + omega as well as – omega. So, it is easy for me to work out this symmetrized function and frequency omega that would be S FF omega + S FF – omega by 2. If I put down the expressions, you can see, let me put it down, first one I have n bar + 1 twice m gamma m h cross omega, the other one is twice m h cross omega gamma m n bar divided by 2.

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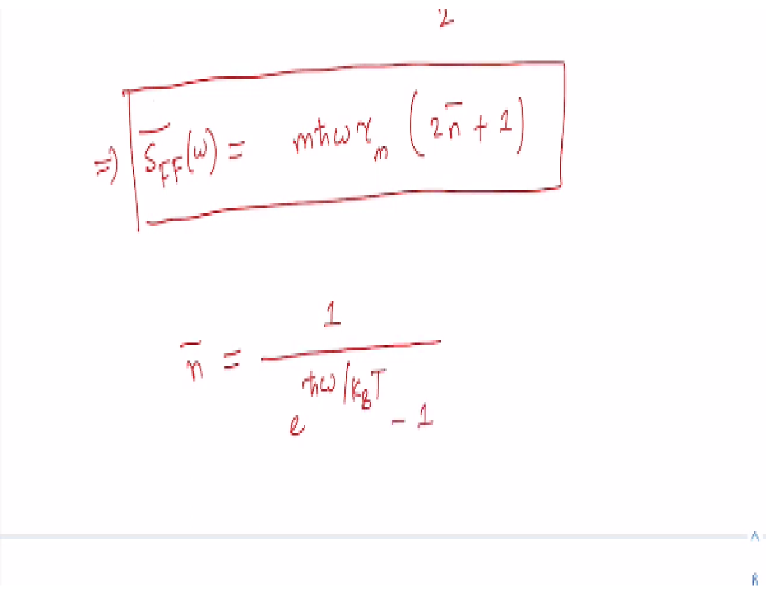

And from here I get S FF bar at omega = m h cross omega gamma m 2 n + 1. So, this is the required expression. Now the other part is what happens at very high temperature? For that to analyze that we know that this average number of quanta n bar = 1 by e to the power h cross omega by K B T – 1.

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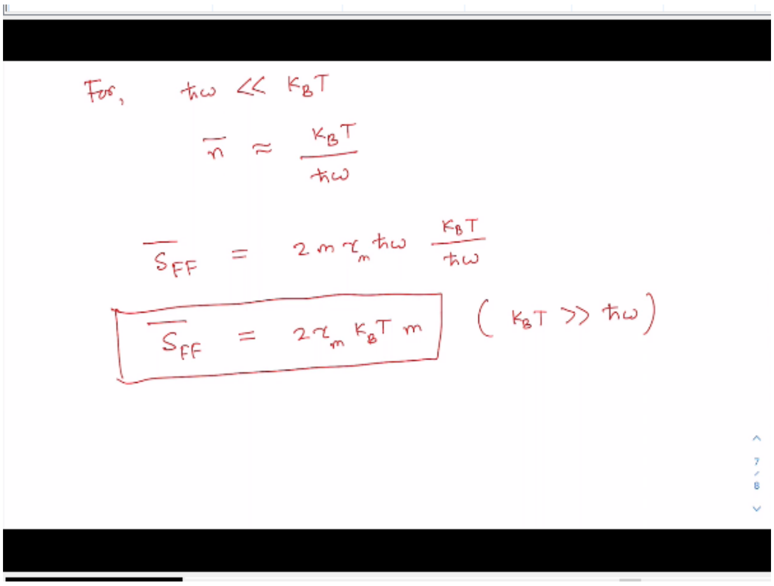

Now at very high temperature that means when h cross omega is much, much less than KB T, for this we have this n bar is nearly equal to K B T divided by h cross omega and therefore I can now write S FF this average or this symmetrized function would be equal to twice m gamma m h cross omega and then K B T by h cross omega. So, what I will have here is twice gamma m K B T into m is also there. So, this is what I have at very high temperature. That means when K B T is much, much greater than h cross omega.

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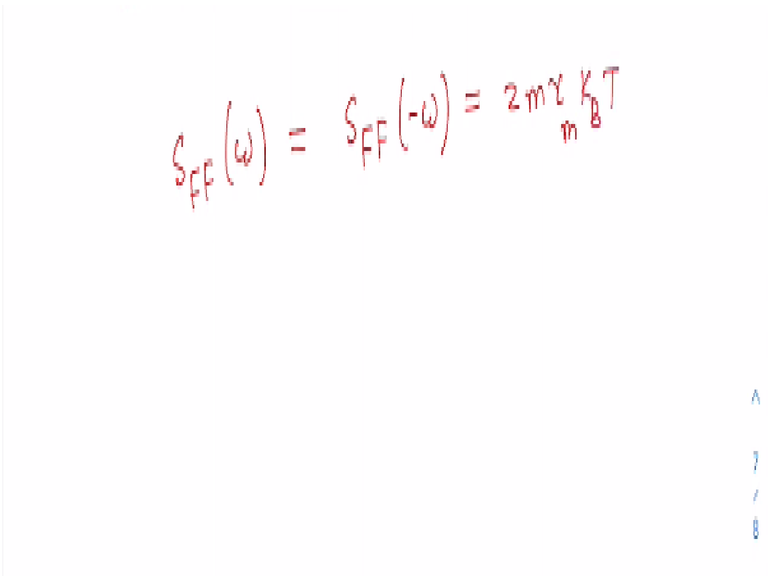

As you can see that this particular function is now independent of frequency, so we have that this spectral noise density is obviously frequency independent. So, they are symmetric in frequency and this is the case for when we are in the classical regime that we also discussed in the class.

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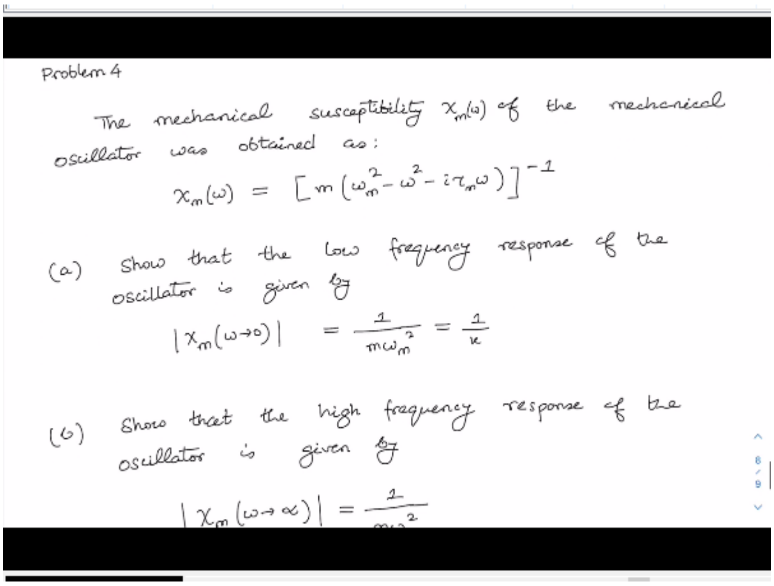

Let us now work out this problem. This is a simple problem where our goal is to explore various limits of the mechanical susceptibility. So, as we have learned in our lecture classes that the mechanical susceptibility of the mechanical harmonic oscillator was obtained as this, where Chi m is the mechanical susceptibility, omega m is the resonance frequency of the oscillator, m is the mass and gamma m is the dissipation rate.

You are asked to show that the low frequency response of the oscillator is given by, low frequency means when the omega tends to 0 that modulus of the susceptibility would be given by this where K is the spring constant, K = m omega, m square and you are asked to show that at high frequency the response of the oscillator is given by this expression.

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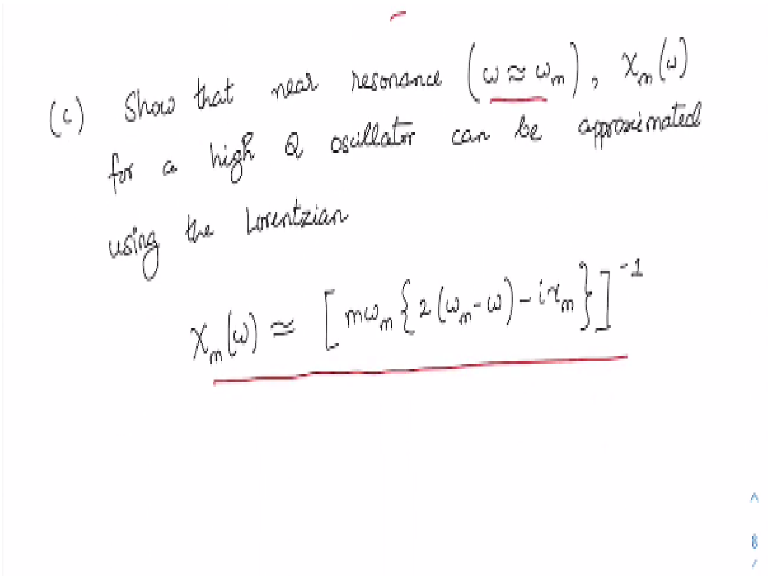

And finally, you are asked to show that near resonance the susceptibility for a high Q oscillator can be approximated by using this expression which is a Lorentzian. So, let us do it, it is a simple problem. Let me first of all write down the expression for Chi m the mechanical susceptibility that is equal to 1 divided by m into omega m square - omega square - i gamma m omega. So, this is what we have.

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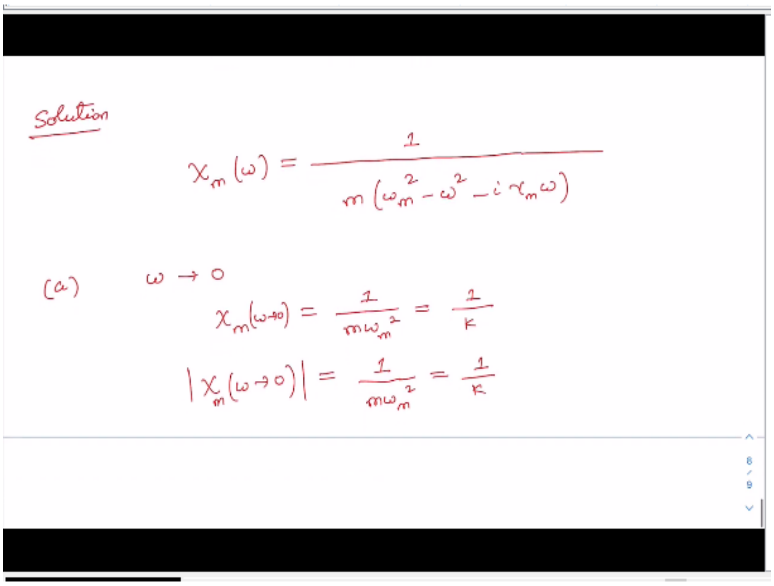

The first part of the problem is very easy because as you can see as omega tends to 0 I can write Chi m of omega = 1 divided by m omega m square. These are the characteristic parameter of the oscillator and 1 by m omega square is nothing but the spring constant K or if I talk about the modulus and this actually I have in the limit omega tends to 0, so you have to be careful here. This I am writing it in the limit omega tends to 0. So, modulus of the susceptibility is simply 1 by m into omega m square and that is the spring constant.

In fact physically speaking this result quantifies the response of a constant force and therefore it is independent of friction as you can see as well as the mass because K is the characteristic parameter of the system.

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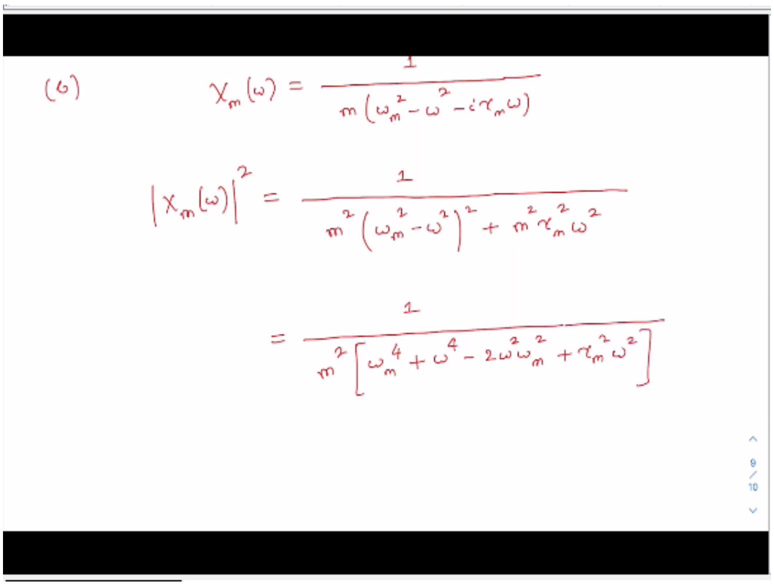

Now going over to the second problem in the high frequency limit what happens? To do this first of all let me simplify this expression because I have to work out the modulus, let me do it, Chi m of omega = 1 divided by I have m into omega m square - omega square - i gamma m omega. So, if I want to find out the modulus, so first of all let me find out Chi m of omega mod square that would be equal to it is easy to see you have to take the multiplication of the complex conjugate then you will get m square omega m square - omega square whole square.

And then you will have + m square gamma m square omega square. So, this is the expression you are going to get. Let me open it up, if I open it up then I can have 1 divided by let me take m square common then I have omega m to the power 4 + omega to the power 4 - twice omega square omega m square + gamma m square omega square.

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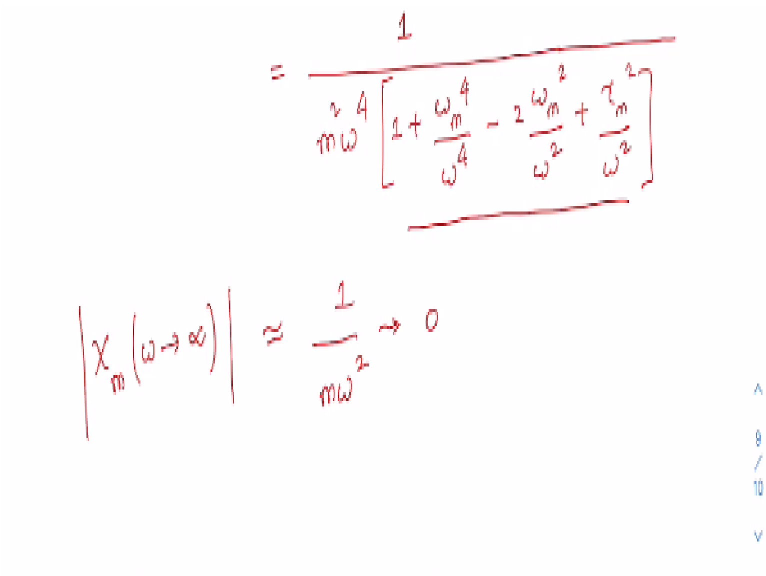

Now if I take omega to the power 4, let me take it outside, 1 divided by m square omega to the power 4. That would be 1 + omega m to the power 4 by omega to the power 4 - twice omega m square by omega square + gamma m square by omega square. So, the modulus of the mechanical susceptibility at very high frequency that is omega tends to infinity that would be equal to 1 by m omega square as these terms will vanish.

In fact, this is nearly equal to 0. What it mean is physically that the oscillator essentially behaves like a free particle and the response of the oscillator is inertial and is independent of stiffness and damping. Now let us do the last part of the problem.

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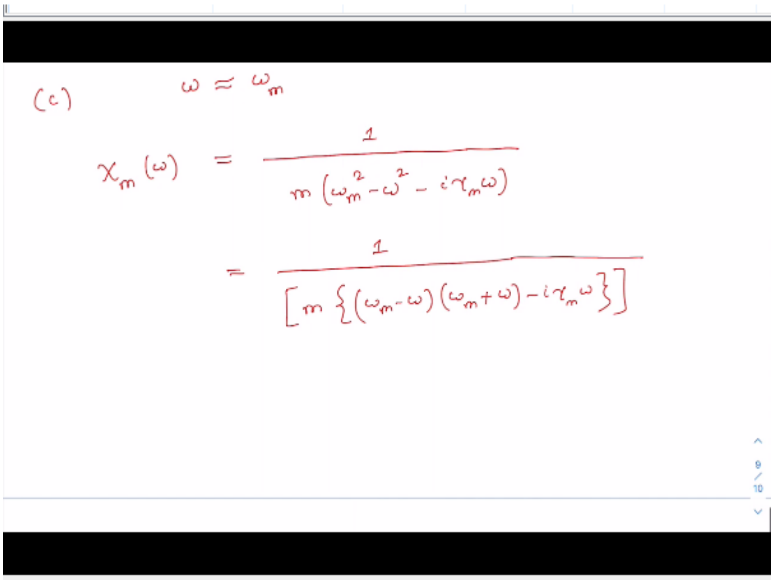

In the last part we are asked how the response function behaves at the resonance that is omega is nearly equal to say omega m. Then this susceptibility expression Chi m of omega let me write it once again, this is 1 divided by m into omega m square - omega square - i gamma m omega. This I can write as 1 divided by m into omega m - omega into omega m + omega - i gamma m omega. So, this is what I have.

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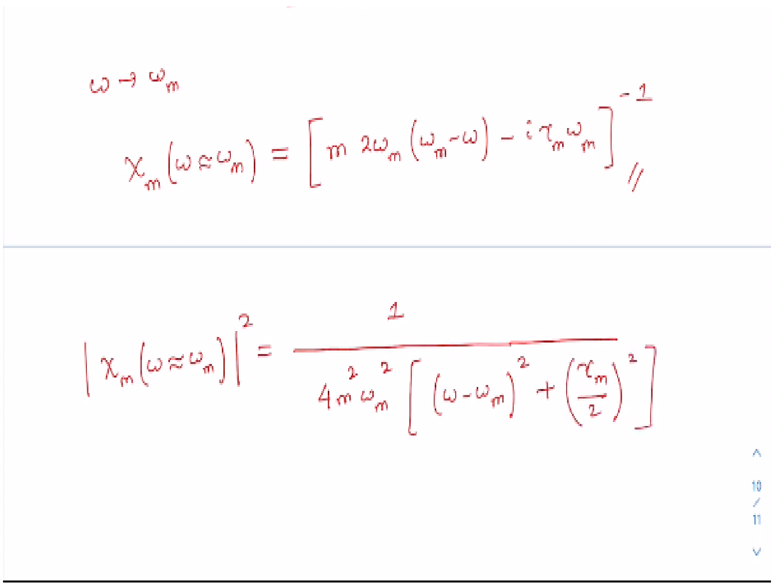

Now at resonance as omega tends to omega m at resonance I can write this expression chi m of omega at omega m, I can write it as m into this I can nearly write it as omega is nearly equal to omega m. So, let me write it as twice omega m and it is not exactly equal to omega m. So, let me retain this term it is not 0 and then I have here i gamma m omega m m to the power – 1.

So, this is what I have. So, from here I can find out the modulus of Chi m, actually this is as you can say this is a Lorentzian, so you can see it more clearly if I take the modulus square of this function that would be equal to 1 divided by 4 m square omega m square, you can do it yourself, it is easy to do omega - omega m square + gamma m by 2 whole square.

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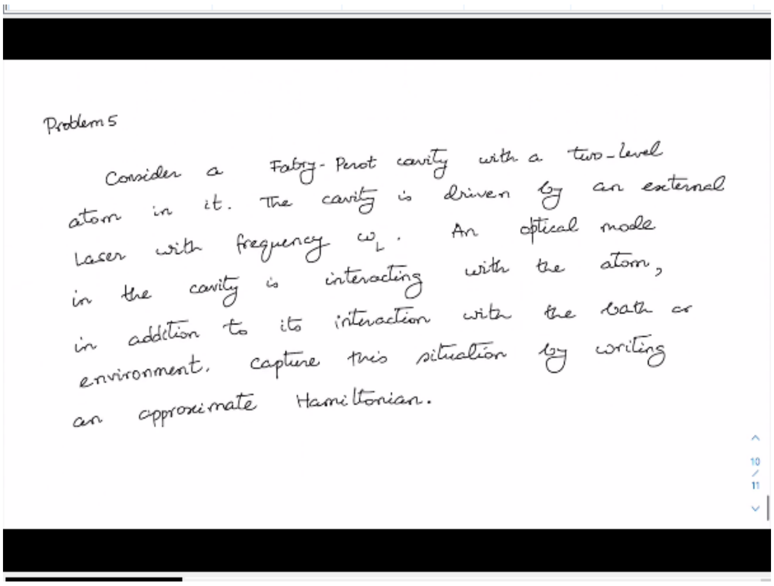

Finally let us work out this problem. Consider a Fabry-Perot cavity with a two-level atom in it the cavity is driven by an external laser with frequency omega L. An optical mode in the cavity is interacting with the atom in addition to its interaction with the bath or environment. Capture the situation by writing an approximate Hamiltonian. Let us do it. This problem is taken to the Jaynes-Cummings Hamiltonian.

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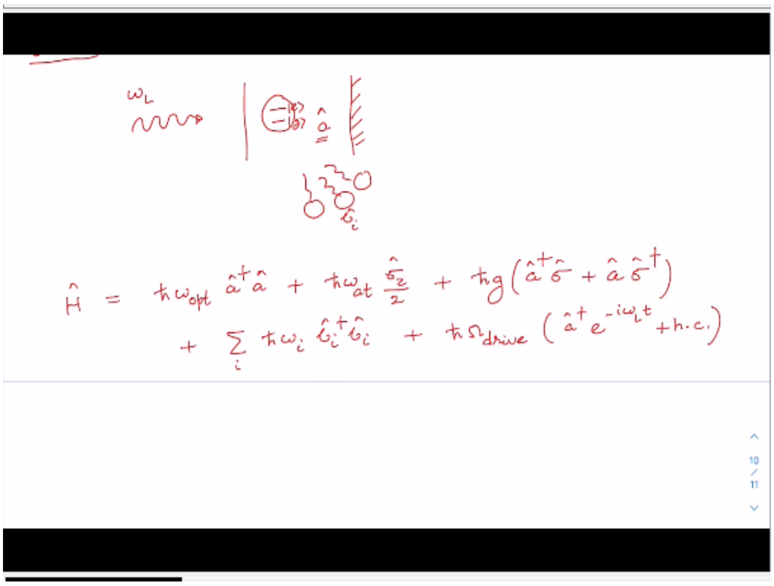

So, we have this situation, we have this Fabry-Perot cavity and this cavity is driven by an external laser with frequency omega L and we have a two-level atom inside the cavity. So, this one it has energy ket e and ket g and we have an optical mode inside it, let me represent it by a cap. So, in addition to this, this mode is interacting with the bath and bath we can as you know that we can consider or model the bath as a collection of external harmonic oscillator.

Each bath can be represented by an operator b i cap ith oscillator, bath oscillator can be represented by this b i cap. Now let me write down the Hamiltonian term by term. First of all, let me consider this optical mode. This optical mode is a harmonic oscillator, so it has a frequency omega optical. So, this is the energy of this whole mode would be h cross omega optical a dagger a.

So, this is the first part of the Hamiltonian referring to the energy of the optical mode. Then the atom we can model it as a two-level atom as we have already seen. So, this is h cross omega atom we can write it as sigma z by 2. So, this is the atom part of the Hamiltonian and this atomic field interaction I can write down as, so this also you have already learned in the context of Jaynes-Cummings models.

So, that would be h cross g that is the interaction, strength between the atom and the mode. So, that would be a dagger sigma, sigma is the atomic lowering operator, a sigma dagger that is the atomic raising operator. So, this is what we have for the atom field interaction. Then this bath is also a harmonic oscillator, it is a collection of harmonic oscillators that I can write as h cross omega i b i dagger b i.

And K B T is driven by this external laser, so that we can capture it by writing it as h cross omega drive that is the driving amplitude and because of this drive a photon is created inside this cavity, so we have a dagger e to the power - i omega L t. So, it has to be Hermitian, so we have this Hermitian conjugate is also there. Now we are left the last thing is that there is a interaction between the bath and the mode.

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$$+\sum_i \hbar\left(\Omega_i^* \hat{a} \hat{b}_i^\dagger + \Omega_i \hat{a}^\dagger \hat{b}_i\right)$$

And that interaction we can capture by this particular term. So, we have h cross suppose this interaction between the bath and the optical mode is omega i that strength and it is a complex quantity. So, let us write it say omega i star, this is a is the optical mode and this interaction. Because of this interaction an optical mode may be annihilated and a bath mode may be generated inside the cavity. So, all these things are captured by this particular Hamiltonian.

So, a i, it has to be Hermitian. So, this part is the Hermitian conjugate part. So, this is what we have. So, this Hamiltonian captures the whole situation that is described in the problem, only condition here is that this particular quantity b i, b j dagger has to be equal to delta i j, thank you.