Quantum Technology and Quantum Phenomena in Macroscopic Systems
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Lecture - 40
Input-Output Relation

Welcome to lecture 8 of module 3. This is lecture number 29 of the course. In this lecture, we will see how quantum noise, quantum Langevin noise in particular, affects the optical mode of a Fabry-Perot cavity. And this will lead us to the very famous input-output relation. And this relation is going to be extremely useful for our discussion on quantum cavity optomechanics. So let us begin.

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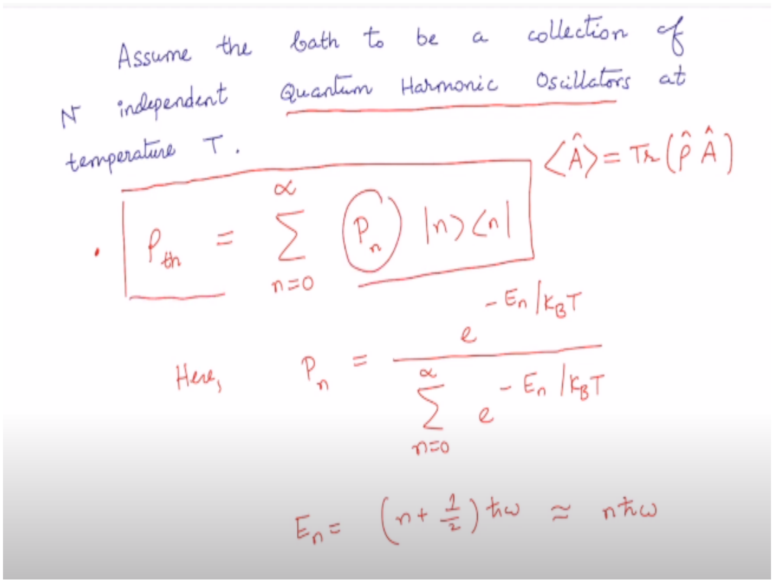

In the last class, we started discussing the quantum counterpart of the classical Langevin noise. In this regard, we have assumed the bath to be a collection of N independent quantum harmonic oscillator at some finite temperature T. As we have to calculate the expectation value of various operators, we needed to know the appropriate density operator as we know that the expectation value of any operator A is given by trace of rho into the operator.

So we first wrote down the density operator in the so called number state basis where this guy P n is the probability.

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• Average phonon number:

$$\langle \hat{b}^\dagger \hat{b} \rangle = n(\omega_m) = \frac{1}{e^{\hbar\omega_m/k_BT} - 1}$$

And we calculated the average phonon number.

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• Position – position correlation for an ensemble of N- harmonic oscillators

$$\boxed{\langle \hat{q}_i \hat{q}_j \rangle}$$

$$\rho_{th} = \left(\sum_{n_1=0}^{\infty} P_{n_1} |n_1\rangle\langle n_1|\right)\left(\sum_{n_2=0}^{\infty} P_{n_2} |n_2\rangle\langle n_2|\right) \cdots \left(\sum_{n_N=0}^{\infty} P_{n_N} |n_N\rangle\langle n_N|\right)$$

$$\rho_{th} \equiv \sum \prod P_{n_k} \left|\{n_k\}\right\rangle\left\langle\{n_k\}\right|$$

After that we worked out the position-position correlation function for an ensemble of N-harmonic oscillator. And first we started by calculating the expectation value of product of two position operators for i-th and the j-th oscillator.

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$\{n_K\}$ K

$$\hat{q}_i = q_{i0}\left(\hat{b}_i + \hat{b}_i^{+}\right), \quad q_{i0} = \sqrt{\frac{\hbar}{2m_i\omega_i}}$$

$$\hat{q}_j = q_{j0}\left(\hat{b}_j + \hat{b}_j^{+}\right)$$

$$q_{j0} = \sqrt{\frac{\hbar}{2m_j\omega_j}}$$

We have expressed the position coordinate in terms of the quantum annihilation and creation operators.

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$$\langle \hat{q}_i \hat{q}_j \rangle = \sum_{\{n_K\}} \prod_{K} P_{n_K} q_{i0} q_{j0} \left[\sqrt{n_i n_j} + \sqrt{(n_i+1)(n_j+1)}\right] \delta_{ij}$$

$$= \sum_{n_i} P_{n_i} q_{i0} q_{j0} \left[\sqrt{n_i n_j} + \sqrt{(n_i+1)(n_j+1)}\right] \delta_{ij}$$

Finally, summing

$$\sum_{i,j=1}^{N} \langle q_i q_j \rangle = \sum_{i,j}^{N} \sum_{n_i} q_{i0} q_{j0} P_{n_i} \left[\sqrt{n_i n_j} + \sqrt{(n_i+1)(n_j+1)}\right] \delta_{ij}$$

$$= \sum \sum q_{i0}^2 P_{n_i} (2n_i + 1)$$

And thereby, we have arrived at the expression for the expectation value of q i and q j. And finally summing it over the all oscillators, we got the expression.

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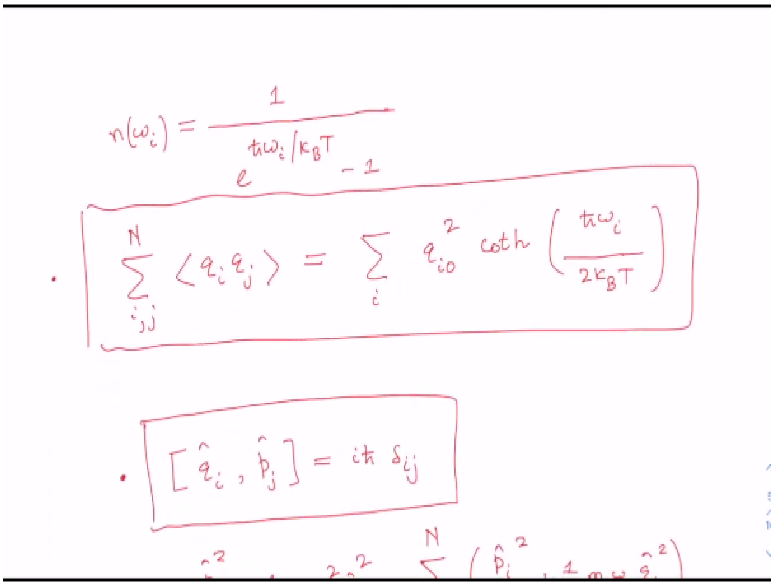

And we expressed it in the in this particular form here.

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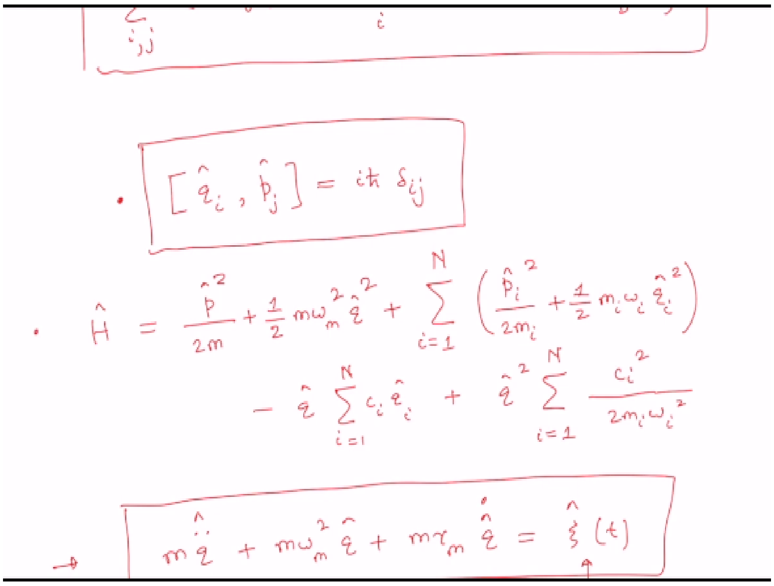

Then we wrote down the quantum mechanical Hamiltonian for the bath oscillator system. Here the Hamiltonian has exactly the same form as that of the classical one. Only thing is that this variable position and the momentum variable are the operators and they has to satisfy the commutation relation as defined in this equation.

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$$- \hat{q} \sum_{i=1}^{N} c_i \hat{q}_i + \hat{q}^2 \sum_{i=1}^{N} \frac{c_i^2}{2 m_i \omega_i^2}$$

$$\boxed{m \ddot{\hat{q}} + m \omega_m^2 \hat{q} + m \gamma_m \dot{\hat{q}} = \hat{\xi}(t)}$$

Quantum Langevin noise

$$\boxed{\hat{\xi}(t) = \sum_{i=1}^{N} c_i \left[\hat{q}_i(0) \cos \omega_i t + \frac{\hat{p}_i(0)}{m_i \omega_i} \sin \omega_i t \right]}$$

So using Heisenberg equation of motion, we can get an equation analogous to the classical Langevin equation where this classical Langevin noise is now represented by this operator. And it has also exactly the same form in the limit when the bath oscillator coupling is assumed to be weak.

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$$\hat{\xi}(t) = \hat{\xi}^{\dagger}(t)$$

$$\hat{\xi}(t)\, \hat{\xi}(0) \neq \hat{\xi}(0)\, \hat{\xi}(t)$$

$$\langle \hat{\xi}(t)\, \hat{\xi}(t') \rangle$$

$$= \sum_{ij} c_i c_j \Big[\langle \hat{q}_i(0) \hat{q}_j(0) \rangle \cos \omega_i t \cos \omega_j t' + \frac{\langle \hat{q}_i(0) \hat{p}_j(0) \rangle}{m_j \omega_j} \cos \omega_i t \sin \omega_j t' + \langle \hat{p}_i(0) \hat{q}_j(0) \rangle \sin \omega_i t \cos \omega_j t'$$

We find that this is very easy to see that this Langevin noise opera quantum Langevin noise operator is Hermitian and because of the fact that these are quantum operators, so xi of t, xi of 0 is not equal to xi of 0 xi of t. That means the order, time order also this depends. We calculated the expectation value of this product of this quantum Langevin noise which is the auto correlator.

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$$\bullet\ \langle \hat{\xi}(t)\hat{\xi}(t') \rangle = \sum_{i=1}^{N} \frac{\hbar c_i^2}{2m_i\omega_i}\left[\coth\left(\frac{\hbar\omega_i}{2k_BT}\right)\cos\omega_i(t-t') - i\sin\omega_i(t-t')\right]$$

$$\bullet\ \boxed{J(\omega) = \pi\sum_{i=2}^{N}\frac{c_i^2}{2m_i\omega_i}\delta(\omega-\omega_i)}$$

$$\bullet\ \langle \hat{\xi}(t)\hat{\xi}(t') \rangle = \frac{\hbar}{\pi}\int_0^{\infty} d\omega\, J(\omega)\left[\coth\left(\frac{\hbar\omega}{2k_BT}\right)\cos\omega(t-t') - i\sin\omega(t-t')\right]$$

And calculating it we got this particular expression which further can be expressed in a little bit simpler form defining the as usual the bath spectral density. Using bath spectral density we have written down the autocorrelation function auto correlator here.

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$$J(\omega) = m\gamma_m\omega$$

$$\boxed{\langle \hat{\xi}(t)\xi(t') \rangle = \frac{m\gamma_m}{\pi}\lim_{\omega_c\to\infty}\int_0^{\omega_c} d\omega\,\hbar\omega\left[\coth\left(\frac{\hbar\omega}{2k_BT}\right)\cos\omega(t-t') - i\sin\omega(t-t')\right]}$$

• In the classical limit: $\hbar \to 0$

$$\cot\left(\frac{\hbar\omega}{2k_BT}\right) \approx \frac{2k_BT}{\hbar\omega}$$

$\omega_c \to \infty$

$$\boxed{\langle \xi(t)\xi(t') \rangle = 2m\gamma_m k_BT\,\delta(t-t')}$$

And in considering the Ohmic damping, we got the expression for the autocorrelation for the Langevin noise which is also known as the second moment of the Langevin noise and as usual in the classical limit, it gives it should give the classical expression which we obtained. We see that this autocorrelator depends only on the time difference.

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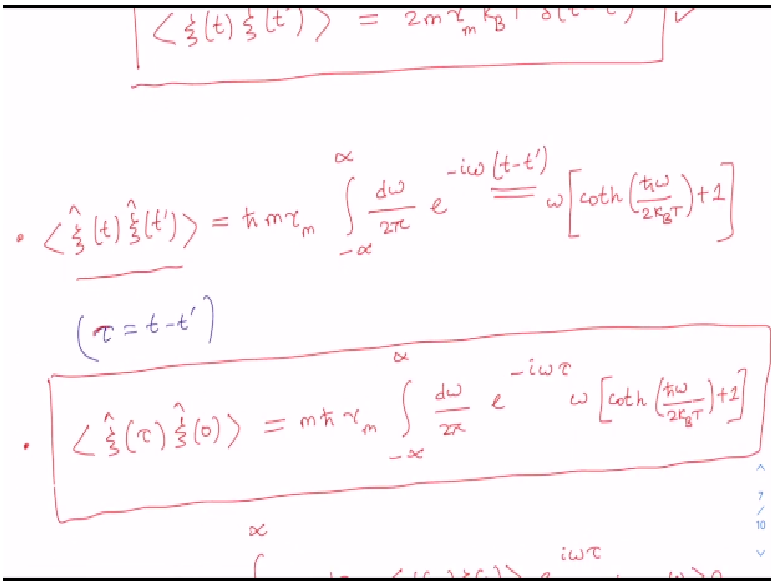

So defining a parameter tau which is the time difference of this which basically denotes the time difference. So using this parameter tau, we write down the autocorrelation function in this particular form. After that, we calculated the quantum spectral noise density. And to do that, we just have to work out the Fourier transform of the correlator.

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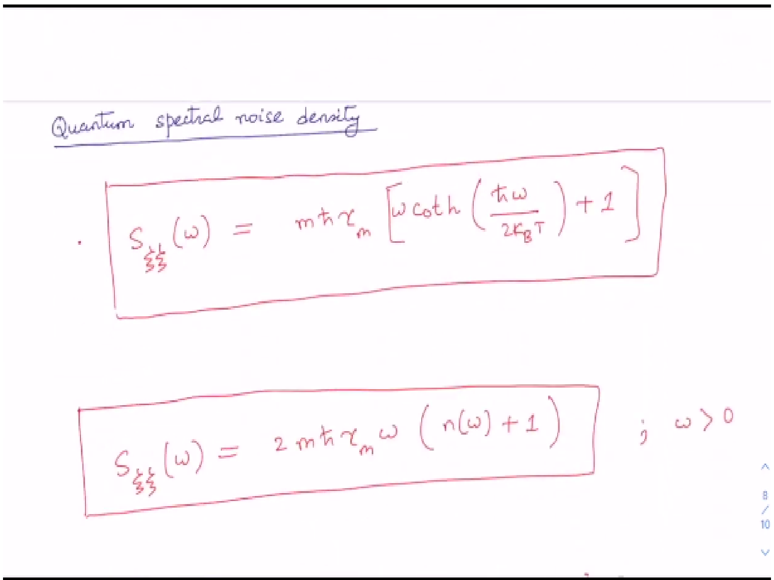

And doing that we get the expression for the spectral, quantum spectral noise density for omega greater than zero and we have worked it out for omega less than zero also.

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$$\boxed{S_{\xi\xi}(\omega) = 2m\hbar\gamma_m\omega\left(n(\omega)+1\right)} \quad ; \ \omega > 0$$

$$S_{\xi\xi}(-\omega) = \int_{-\infty}^{\infty} d\tau \, \langle \xi(\tau)\,\xi(0) \rangle \, e^{-i\omega\tau} \quad ; \ \omega < 0$$

$$\boxed{S_{\xi\xi}(-\omega) = 2m\hbar\gamma_m\omega\, n(\omega)}$$

So we worked out the quantum spectral noise density at plus omega and at minus omega.

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$$\boxed{S_{\xi\xi}(-\omega) = 2m\hbar\gamma_m\omega\, n(\omega)}$$

$$\boxed{S_{\xi\xi}(\omega) \neq S_{\xi\xi}(-\omega)} \qquad \underline{\text{Not symmetric}}$$

And it turns out that this function is not symmetric which is unlike the classical case. In the classical case, we saw that this noise, spectral noise density is a symmetric function.

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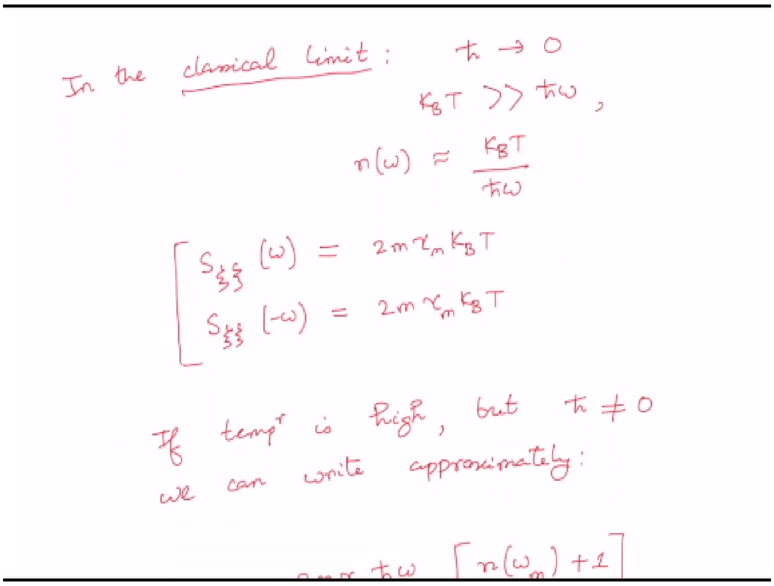

In the classical limit obviously, again we obtain the classical expression for the spectral noise density.

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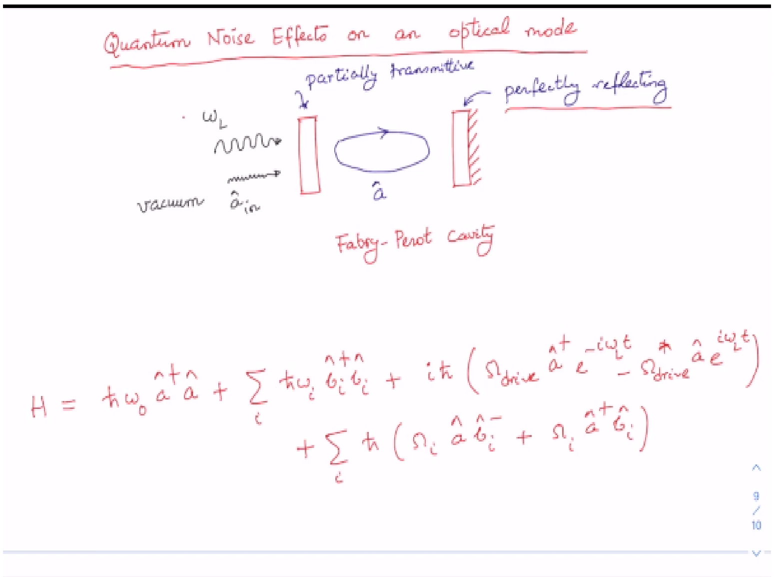

Finally, we started discussing the quantum noise effect on an optical mode in the context of a Fabry-Perot cavity, because the Fabry-Perot cavity is at the backbone of any cavity optomechanical system. Here we considered one of the mirror in the Fabry-Perot cavity to be perfectly reflecting and the other mirror to be partially transmittive. And vacuum fluctuation enters into the cavity from one side.

And if it is a cavity optomechanical system then it is usually always drive by a single mode laser having frequency omega L.

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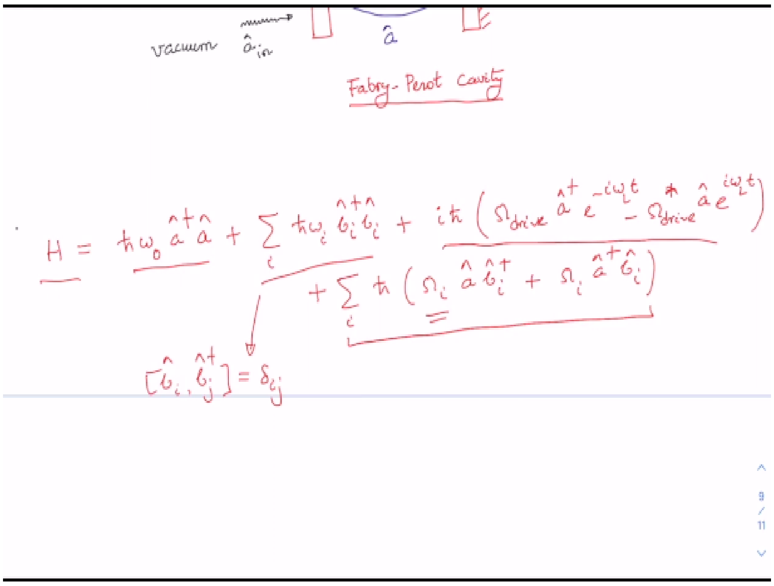

We wrote down the quantum mechanical Hamiltonian for a system N bath. Here the first term refers to the energy of the cavity mode. The second term describes the energy of the bath oscillator modelled as a collection of independent electromagnetic oscillators with the constraint that this commutation relation has to be obeyed.

That is the commutation between say b i, b j dagger should be equal to delta ij. And the third term describe the laser driving the cavity externally. The fourth and the final term, this is the final term. It refers to the system bath coupling. The strength of the coupling between the cavity mode and the bath operator is given by the parameter omega i. This Hamiltonian is the basis of our analysis for the rest of this lecture.

Now let us go over to the continuum limit because ultimately this bath oscillators are infinite in numbers.

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$$\sum_i \longrightarrow \frac{1}{\Delta\omega}\int d\omega$$

$$\Delta\omega \to 0$$

$$b_i \longrightarrow (\Delta\omega)^{1/2}\, b(\omega)$$

$$b_i^{\dagger} \longrightarrow (\Delta\omega)^{1/2}\, b^{\dagger}(\omega)$$

$$\int \delta(\omega-\omega')\, d\omega = 1$$

So when we go over to the continuum limit, this summation sign is going to be replaced by integral. In fact, this summation is over the discrete index i and it turns into an integration over the bath oscillator frequency omega. And it has to be diamensionless. So it is divided by say delta omega here. Delta omega is the mode spacing and we take it in the limit say delta omega tends to 0.

And the bath operators undergoes this kind of transformation. So I had explained it. So we have say b i the discrete variable. Now in the continuum limit, it will become delta omega half, delta omega to the power half b of omega and b i dagger in the discrete space it is going to be delta omega to the power half b dagger of omega in the continuum.

Also please note that in continuum we have this relation delta omega minus omega there is d omega that is equal to 1.

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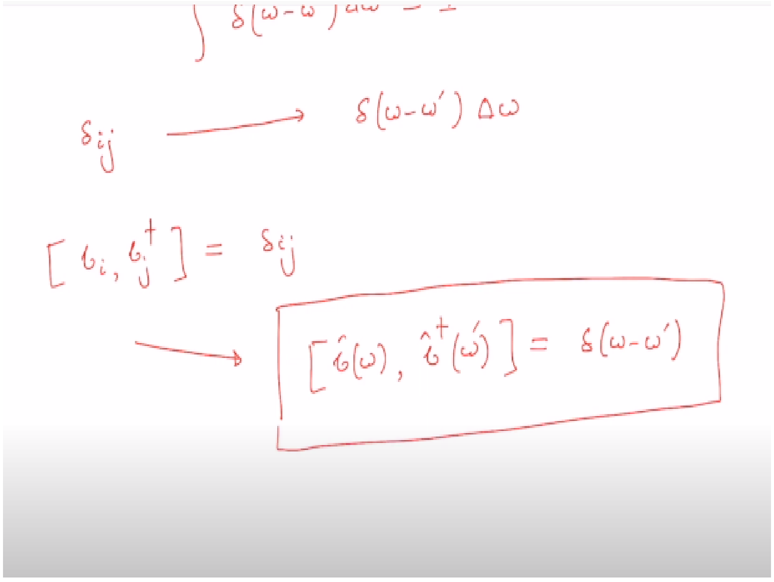

On the other hand you know that in the discrete space we have this Kronecker delta. Now in the continuum it will be replaced by delta omega minus omega dash delta omega. So therefore, this commutation relation in the discrete space b i b j dagger is equal to delta ij. In the continuum limit it would be replaced by b of omega b dagger of omega dash is equal to delta omega minus omega dash.

So these are very important relations. So utilizing all these now we can rewrite this Hamiltonian. This Hamiltonian is in the discrete space. Now in the continuum space, this Hamiltonian can be written as follows.

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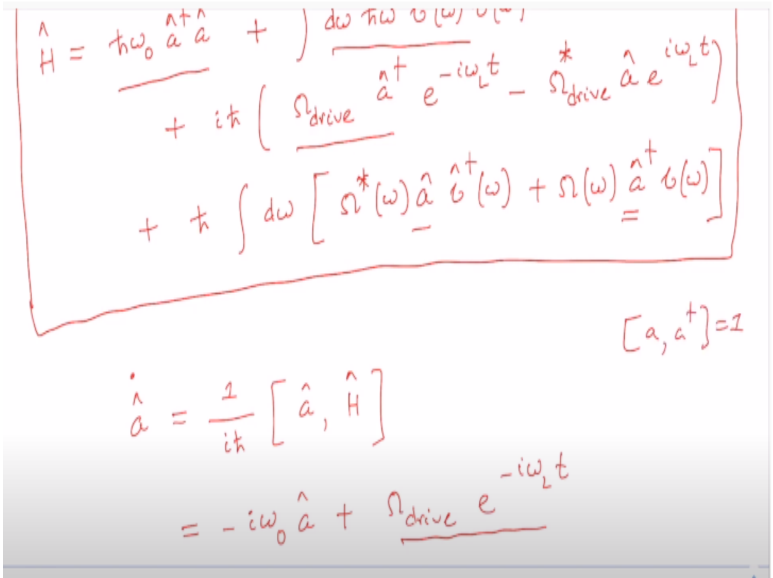

So we will write this Hamiltonian as h cross omega 0 a dagger a. So bath oscillators are now going from the discrete to the continuum. So we have now integration d omega h cross omega. Let me just show you. So this summation is now replaced by integral. So d omega h cross omega b dagger omega and b of omega. And this external laser drive would remain same because it just involve the optical mode only.

It does not involve the bath oscillator, so it will remain as it is. So you will have a dagger e to the power minus i omega L t minus omega drive the complex conjugate a e to the power i omega L t. And finally the bath and the mode coupling that would be again this bath oscillators are involved.

So it would be replaced by integral h cross integration d omega, omega star omega a b dagger omega plus omega, this capital omega omega and we have a dagger b of omega. So this is going to be the key Hamiltonian now. And we can immediately write couple of things from here. First of all, we can calculate the Heisenberg equation or motion for the mode operator.

So that would be a dot is equal to time derivative of the mode operator would be 1 by i h cross the commutation between a and H. So you can already we have done this kind of stuff too many times in the course. So immediately you can write, it will be minus i omega 0 a plus omega drive e to the power minus i omega L t. You can verify it because commutation with a and a dagger a will give you simply a.

And then this bath oscillators are independent of the mode cavity mode. So therefore, this term is not going to contribute. And then you have this particular term. From this term you are having this term and what else you will be left is the last term because a and a dagger is there. So a, a dagger is equal to 1. So we have to exploit a, a dagger is equal to 1. So exploiting that you will have this particular term now.

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$$\dot{\hat{a}} = \frac{1}{i\hbar}\left[\hat{a}, H\right]$$

$$\Rightarrow \boxed{\dot{\hat{a}} = -i\omega_0 \hat{a} + \Omega_{drive}\, e^{-i\omega_L t} - i\int d\omega\, \Omega(\omega)\, b(\omega)}$$

$$\dot{\hat{b}} = \frac{1}{i\hbar}\left[\hat{b}, \hat{H}\right]$$

That would be minus i integration d omega capital omega of omega b of omega. So this is for the optical mode, time derivative of the optical mode operator. Similarly, for the bath mode we can calculate. That would be time derivative of b. That is 1 by ih cross b of H.

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$$\dot{\hat{b}} = \frac{1}{i\hbar}\left[\hat{b}, \hat{H}\right]$$

$$= \frac{1}{i\hbar}\left[\int d\omega'\, \hbar\omega' \left[b(\omega), \hat{b}^{\dagger}(\omega')\, b(\omega')\right] + \hbar\int d\omega'\, \Omega^{*}(\omega')\, a \left[\hat{b}(\omega), \hat{b}^{\dagger}(\omega')\right]\right]$$

$$= -i\left[\int d\omega'\, \omega'\, \delta(\omega-\omega')\, b(\omega')\right.$$

So here let me show you the calculation. This is also easy. 1 by i h cross. Now I have integration because a dagger a, this part is not going to contribute because these are independent as I said. So we will have, from the next term we will have say this is d omega dash h cross omega dash b of omega and here I have b dagger of omega dash b of omega dash, okay. Let me show it here.

So I am now talking about this particular term, okay. And then we have h cross d omega dash omega star of omega dash a. And we have b of omega b dagger of omega dash. So this is what we will have now. And you can, okay let us evaluate it. You will have minus i say integration d omega dash omega dash. And we now use the commutation relation between this relation we are going to use, this one we are going to use.

If we use it then you will have here delta of omega minus omega dash b of omega dash.

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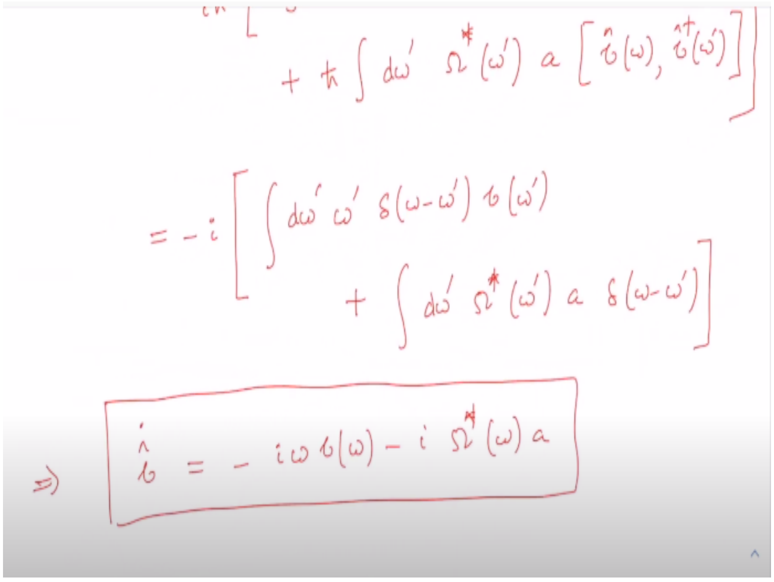

And then next term would be d omega dash omega star of omega dash a delta omega minus omega dash. Then we can utilize the property of the Dirac delta function and this will give us the equation of motion for the bath or the mode and that would be b dagger b dot is equal to minus i omega b of omega minus i omega star of omega into a. So this is the equation we get. We can write a formal solution to this part equation.

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Take, $b = \tilde{b}\, e^{-i\omega t}$

$$\dot{b} = \dot{\tilde{b}}\, e^{-i\omega t} - i\omega\, \tilde{b}\, e^{-i\omega t}$$

$$\dot{\tilde{b}}\, e^{-i\omega t} \cancel{- i\omega\, \tilde{b}\, e^{-i\omega t}} = \cancel{-i\omega\, \tilde{b}\, e^{-i\omega t}} - i\,\Omega^*(\omega)\, a$$

$$\Rightarrow \boxed{\dot{\tilde{b}} = -\, i\,\Omega^*(\omega)\, a\, e^{i\omega t}}$$

To do that, let us make a change of variables. Let me take b is equal to b tilde e to the power minus i omega t. Then I will have b dot is equal to b tilde dot. I am taking the time derivative e to the power minus i omega t minus i omega b tilde e to the power minus i omega t. Then if I put it here in this equation, then I will get it as b tilde dot e to the power minus i omega t minus i omega b tilde e to the power minus i omega t.

And on the right hand side I have minus i omega b tilde e to the power minus i omega t minus i omega star omega a. Now as you can see from this equation that this particular term and this term get cancelled out and we will have from here we will have b tilde dot is equal to minus i omega star of omega a e to the power i omega t, okay. So now integrating both sides from some initial time t 0.

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$$\int_{t_0} \dot{\tilde{b}}\, dt' = -\, i\,\Omega^* \int_{t_0} a(t')\, e \quad dt'$$

$$\Rightarrow \tilde{b}(\omega, t) - \tilde{b}(\omega, t_0) = -i\,\Omega^* \int_{t_0}^{t} a(t')\, e^{i\omega t'}\, dt'$$

$$\Rightarrow \boxed{b(\omega,t) = b(\omega, t_0)\, e^{-i\omega(t-t_0)} - i\,\Omega^*(\omega) \int_{t_0}^{t} a(t')\, e^{-i\omega(t-t')}\, dt'}$$

So let me integrate it on both sides from some say initial time t 0 to some time t and accordingly here also I have minus omega star t 0 to t a of t dash e to the power i omega t dash dt dash. So if I do the integration, this is going to give me b tilde of omega t minus b tilde of omega t 0. That would be equal to minus i omega star. This will remain the same.

It would be t 0 to t a of t dash e to the power i omega t dash dt dash. And from here I can now rewrite actually this in the variable b of omega t. So b of omega t is equal to b of omega t 0 e to the power minus i omega t minus t 0 minus i omega star omega integration t 0 to t a of t dash e to the power minus i omega t minus t dash dt dash okay. So this is what we get as our formal solution.

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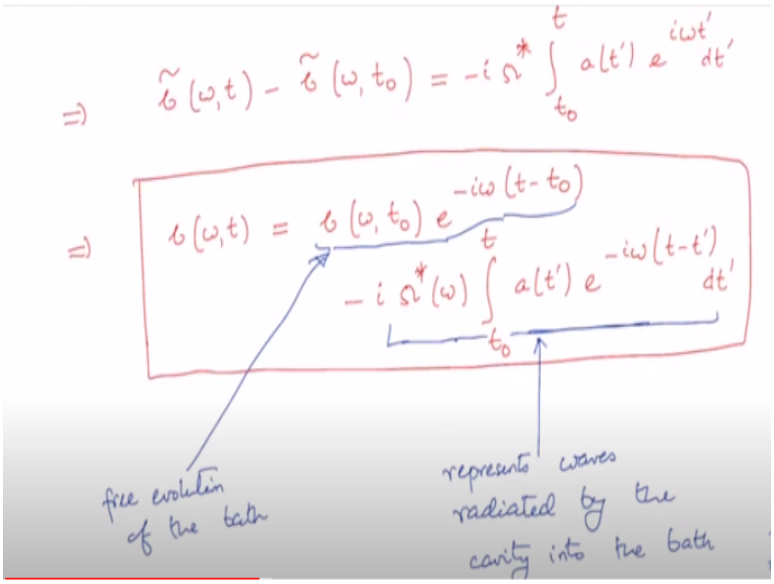

In fact, you see here the first term on the right hand side this particular term, this term corresponds to the free evolution of the bath while the second term here it represents waves radiated by the cavity into bath. So this particular term, this second term represents, it represents waves, waves radiated by the cavity into the bath.

And on the other hand this particular term represents as I said it is free evolution of the bath, okay. Now we can substitute this particular solution, this bath solution into the equation for the optical mode here. So we can put b of omega into this equation. So then what we will obtain this. Let me write that here. Okay let me first bring the solution to the address or let me first write it then I will put it.

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$$\dot{a} = -i\omega_0 a + \Omega_{drive} e^{-i\omega_L t}$$
$$-i\int d\omega\, \Omega(\omega)\, b(\omega, t_0)\, e^{-i\omega(t-t_0)}$$
$$-\int d\omega\, \underline{|\Omega(\omega)|}^2 \int_{t_0}^{t} a(t')\, e^{-i\omega(t-t')}\, dt'$$

So I have to put my bath solution in this equation a dot is equal to minus i omega not a plus omega drive e to the power minus i omega L t. And I have here minus i integration d omega, omega here. And then this whole thing I have to put. So because I have two terms I will get two terms. So let me just write it one by one. The first term I will have is this.

I will have b of omega t 0 e to the power minus i omega t minus t 0. And the second one is going to give me d omega mod of capital omega of omega whole square integration t 0 to t a of t dash e to the power minus i omega t minus t dash dt dash. This is coming because you see that this complex conjugate is there here and okay. So because of that and in the, here we have this omega, okay.

And because in the second term we have that complex conjugate that is why this mod of capital omega, omega square term is coming.

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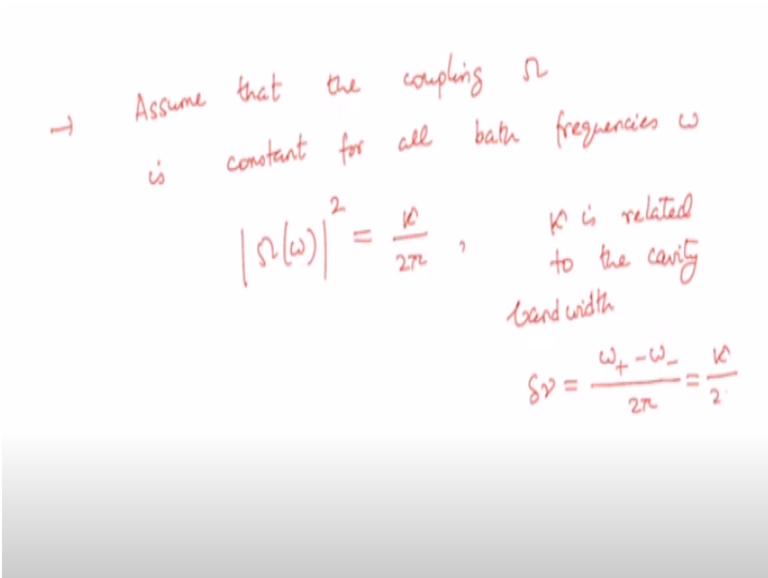

Now assume, let us assume that which we will justify it later for this particular identification that we are going to make now. Assume that the coupling, the coupling omega, omega is the coupling between the bath and the optical mode. The coupling omega is constant for all frequencies, for all bath frequencies omega.

And let us write it as this omega capital omega, omega mod square is equal to kappa by 2 pi where kappa is related to the cavity bandwidth. Kappa is related to the cavity bandwidth. So I mean to say the bandwidth would be something like this. Delta nu is equal to say omega plus minus omega minus divided by 2 pi and omega plus minus omega minus.

That is the width is say kappa divided by 2 pi. So this is what we have. Now with this, we can write the equation for the optical mode as follows.

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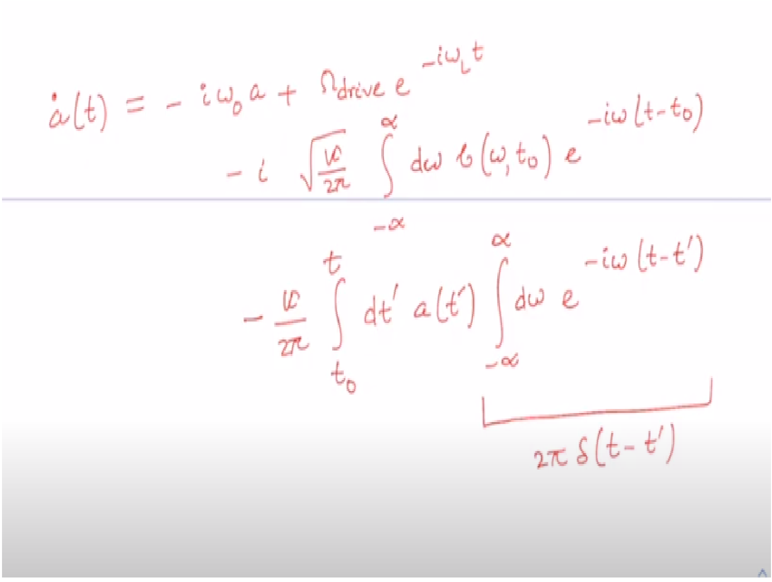

So we will have a dot t is equal to minus i omega 0 a plus omega drive e to the power minus i omega L t minus i square root of kappa by 2 pi integration minus infinity to plus infinity d omega b of omega, okay. So b of omega at t 0 e to the power minus i omega t minus t 0 minus kappa by 2 pi. I am basically replacing omega by square root of kappa by 2 pi and mod omega square capital omega square is equal to kappa by 2 pi.

So I am rewriting it only, nothing new I am doing here, t 0 to t dt dash a of t dash integration minus infinity to plus infinity d omega e to the power minus i omega t minus t dash. Now you may recognize that this particular term is nothing but the Dirac delta function. So it is 2 pi into, this is the delta function, delta t minus t dash. So let us look at this particular term this last term, this last term let us look at specifically.

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$$\frac{\kappa}{2\pi}\int_{t_0}^{t} dt'\, a(t')\; 2\pi\,\delta(t-t')$$

set $\quad t_0 \to 0 \quad$ and $\quad t \to \infty$

$$\kappa\int_{0}^{\infty} dt'\, a(t')\,\delta(t-t')$$

We have kappa by 2 pi integration t 0 to t dt dash a of t dash, and here I have 2 pi delta t minus t dash. Then setting t 0, this setting t 0 at 0 and this upper limit t at infinity, we will have it as kappa 0 to infinity dt dash a of t dash delta t minus t dash.

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$$\frac{\kappa}{2}\int_{-\infty} dt'\, a(t')\,\delta(t-t') = \frac{\kappa}{2}a$$

$$a_{in} = \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, b(\omega, t_0)\, e^{-i\omega(t-t_0)}$$

$$\boxed{\dot{a}(t) = -i\omega_0 a + \Omega_{drive}\, e^{-i\omega_L t} - \frac{\kappa}{2}a - \sqrt{\kappa}\, a_{in}}$$

But without loss of generality, rather than setting t 0 at 0, we can set it as minus infinity and then we will have it as, because we are setting at minus infinity we have to divide it by half. And then we will have dt dash a of t dash delta t minus t dash. This will give me, now applying the property of the Dirac delta function I will have it as kappa by 2 a. So in the again the last term as you see it is get simplified significantly.

We are having only this kappa by 2 a from the last term. On the other hand in this third term, if we define a parameter say defined as this say a in is equal to i divided by square root of 2 pi integration minus infinity to plus infinity d omega b of omega t 0 e to the power minus i omega t minus t 0, okay. Then we can rewrite this equation for, time evolution equation for the optical mode in a very simplified form.

And that would be a dot t is equal to minus i omega 0 a plus omega drive e to the power minus i omega L t. Now the last term the fourth term here that is kappa by 2 let me put it first here, minus kappa by 2 a. And the third term we will have here root over kappa a in. So this is a very important equation that now we have obtained.

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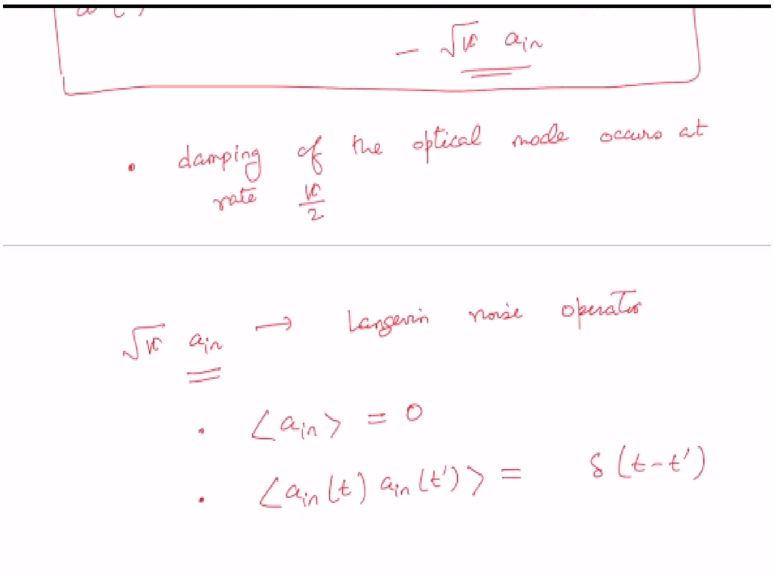

Clearly from this equation as you can see that the damping of the optical mode damping of the optical mode occurs at the rate kappa by 2 or in other words the corresponding energy loss occurs at the rate kappa and which is expected behavior of cavity oscillator and this is one of the reason why we have identified this particular term as this, alright.

And this particular term the last term now here, this term is very important. And you can recognize that this is nothing but the Langevin noise operator. This is Langevin noise operator. This is Langevin noise operator. Since a in has a vanishing mean value, we can show that it has a vanishing mean value and the autocorrelation is a delta function. So autocorrelation a in of t and in of t dash would turn out to be delta function.

So and you know that this is these are the properties of quantum Langevin noise also. And then hence we can identify this last term as the nothing but the Langevin noise. So let us actually prove it.

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$\sqrt{\kappa}\, a_{in} \longrightarrow$ Langevin noise operator

- $\langle a_{in} \rangle = 0$
- $\langle a_{in}^{\dagger}(t)\, a_{in}(t') \rangle = \bar{n}(\omega_0)\, \delta(t-t')$

$$\rightarrow \quad \langle a_{in} \rangle = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \underbrace{\langle b(\omega, t_0) \rangle}_{=\,0}\, e^{-i\omega(t-t_0)}$$

$$= 0$$

If we assume that the bath is a thermal state, then we can write the expectation value of a in is equal to i by square root of 2 pi integration minus infinity to plus infinity d omega the expectation value of this annihilation operator with e to the power minus i omega t minus t 0. Now it is very clear and it is actually obvious that since the annihilation and creation operator have no diagonal elements, so this is going to give us simply 0. So this expectation value of this Langevin noise is 0 here.

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$$= 0$$

$$\langle a_{in}^{\dagger}(t)\, a_{in}(t') \rangle$$

$$= \Big\langle \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, b^{\dagger}(\omega, t_0)\, e^{i\omega(t-t_0)} \int_{-\infty}^{\infty} d\omega'\, b(\omega', t_0)\, e^{-i\omega'(t-t_0)} \Big\rangle$$

We can now calculate the autocorrelation. To do that let us work out. a in dagger t a in at some different times say t dash. Let us first calculate it. Let us put the expression of a in from here. So if we put it here, I have 1 by 2 pi. So there are two a in. So square root of 2 pi is there from one term and for another square root of 2 pi. And in one case we are taking a in dagger.

So it will be minus i. So minus i into i will give us plus 1. So therefore, it will be 1 by 2 pi. And we will have integration minus infinity to plus infinity d of d omega b dagger omega t 0 e to the power i omega t minus t 0. And from the other a in I have to take the expectation value.

So from the other one I have minus infinity to plus infinity d omega dash b omega dash t 0 e to the power minus i omega dash t minus t 0. So let me close the bracket.

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$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty} d\omega' \underbrace{\langle b^{\dagger}(\omega,t_0)\, b(\omega',t_0)\rangle}_{\bar{n}(\omega')\,\delta(\omega-\omega')} e^{i\omega(t-t_0)}\, e^{-i\omega'(t-t_0)}$$

$$= \frac{\bar{n}(\omega_0)}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')}$$

And then I have 1 by 2 pi integration minus infinity to plus infinity d omega integration minus infinity to plus infinity d omega dash expectation value of b dagger omega t zero b of omega dash t 0 and I have e to the power i omega t minus t 0. And here I have e to the power minus i omega dash t minus t 0. Now you see this particular term this is the expectation value of the number operator for phonons.

So this is going to give us, it would become phonon number n of omega, let us say n of omega dash. And then this would be delta function, delta omega minus omega dash. So using this we can immediately write here as n of average number of phonons assuming that the bath occupation number peaks at this cavity frequency omega 0.

So then I can take this out and I have n bar omega zero divided by 2 pi integration minus infinity to plus infinity B omega e to the power i omega t minus t dash. And you know this is nothing but the delta function so along with 1 by 2 pi.

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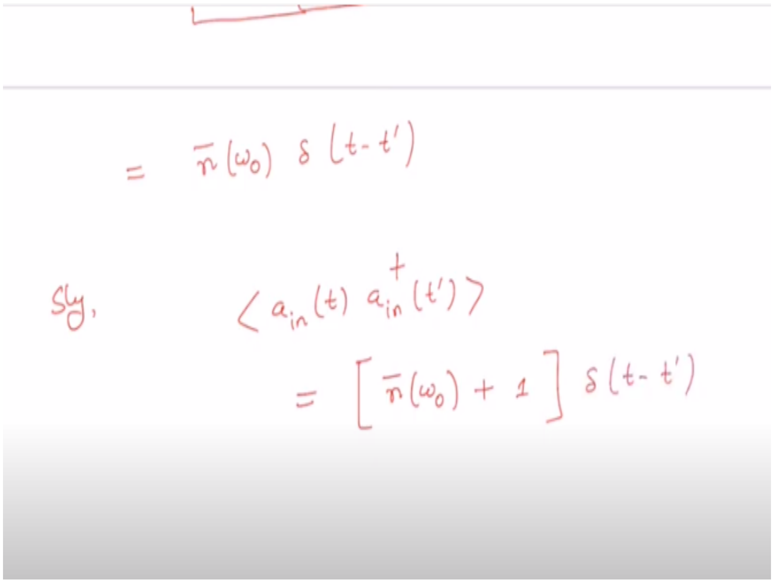

So this would be n bar of omega 0 delta t minus t dash. So in fact, what I should have written here earlier as I said we are now going to prove it. So let me write here it is a dagger here and here let me put n bar of omega 0. So this is the correct one. So as you see the autocorrelation function is a delta function. Similarly, you can show that, now here we have worked out a dagger a in.

We can show the other one also that is a in a in dagger at some different time t dash. This autocorrelation function in the similar way you can work out and you can show that this would be n bar of omega 0 assuming that again that the bath occupation number peaks at this cavity frequency. Then you will have this particular expression delta t minus t dash, okay.

Now one thing has to be kept in mind that this is a thermal oscillation, thermal oscillator. So generally these are in microwave frequency and so on.

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$$= \left[\bar{n}(\omega_0) + 1\right] \delta(t-t')$$

At optical frequencies: $\omega_0 \sim 10^{15}$ Hz, $T \sim 300$ K

$$\frac{\hbar\omega_0}{K_B T} \ll 1 \Rightarrow \bar{n}(\omega_0) \sim 0$$

$$\langle a_{in}^{\dagger}(t)\, a_{in}(t') \rangle = 0$$

$$\langle a_{in}(t)\, a_{in}^{\dagger}(t') \rangle = \delta(t-t')$$

But at optical frequency if I talk about at, optical frequencies where omega 0 is on the order of 10 to the power 15 hertz, and if we will consider room temperature that is around 300 Kelvin, in that case this h cross omega 0 by K B T is much less than 1. That means K B T is much higher than h cross omega 0. And this implies that this phonon number or optical photons, that would be, average number would be nearly 0.

And in that case at optical frequencies we will have a in dagger t a in t dash this autocorrelation will give us 0. On the other hand, the other one a in t a in dagger t dash that would be this delta function, delta t minus t dash, okay. This we have done for the, in the time domain.

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$$\langle a_{in}(\omega)\, a_{in}^{\dagger}(\omega') \rangle$$

$$= \left\langle \int_{-\infty}^{\infty} dt\, a_{in}(t)\, e^{i\omega t} \int_{-\infty}^{\infty} dt'\, a_{in}^{\dagger}(t')\, e^{i\omega' t'} \right\rangle$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt\, dt'$$

So we can work out in the frequency domain as well this correlation in the frequency domain and that is very straightforward to calculate. So I mean to say let us calculate a in of omega and a in dagger of say omega dash. So this is in the frequency domain. So you can calculate. First let me do it, let me write the Fourier transform of it that is minus infinity to plus infinity dt a in of t e to the power i omega t.

And here it is minus infinity to plus infinity for the second term, for this term. I have here say dt dash a in dagger d dash e to the power i omega dash t dash. Let me close the bracket. Then I have minus infinity to plus infinity, minus infinity to plus infinity dt dt dash. The expectation value of a in t a in dagger t dash. And I have here e to the power i omega dash t dash plus omega t.

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$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt\, dt' \, \underline{\langle a_{in}(t)\, a_{in}^{\dagger}(t')\rangle} \, e^{i(\omega' t' + \omega t)}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt\, dt' \, \delta(\underbrace{t-t'}_{\tau}) \, e^{i[\omega' t' + \omega(\tau + t')]}$$

$$= \int_{-\infty}^{\infty} dt' \, e^{i(\omega'+\omega)t'} \underbrace{\int_{-\infty}^{\infty} d\tau \, \delta(\tau)\, e^{i\omega\tau}}_{1}$$

$$\Rightarrow \boxed{\langle a_{in}(\omega)\, a_{in}^{\dagger}(\omega')\rangle = 2\pi\,\delta(\omega+\omega')}$$

Now I know the result for this one and utilizing that minus infinity to plus infinity, minus infinity to plus infinity dt dt dash. And this guy gives me delta t minus t dash this one and I have e to the power i. Actually I can write it, all right let me do it i omega, t minus t dash is I can consider it as tau. Then I will have here e to the power i omega dash t dash and t minus t dash if I replace t by tau plus t dash then I will have a term omega tau plus t dash, okay.

So this is what I will have. And using this one I can then next I can write minus infinity to plus infinity dt dash e to the power i omega dash plus omega t dash integration minus infinity to plus infinity d tau delta tau e to the power i omega tau, okay. And then this is the delta function, so apply the property of the delta function.

So we will get from here, very simply we will have a in omega a in dagger of omega dash.

That would be equal to 2 pi. And this is going to give us 1, okay. So I will have this is the Dirac delta function again. That would be 2 pi delta into delta of omega plus omega dash.

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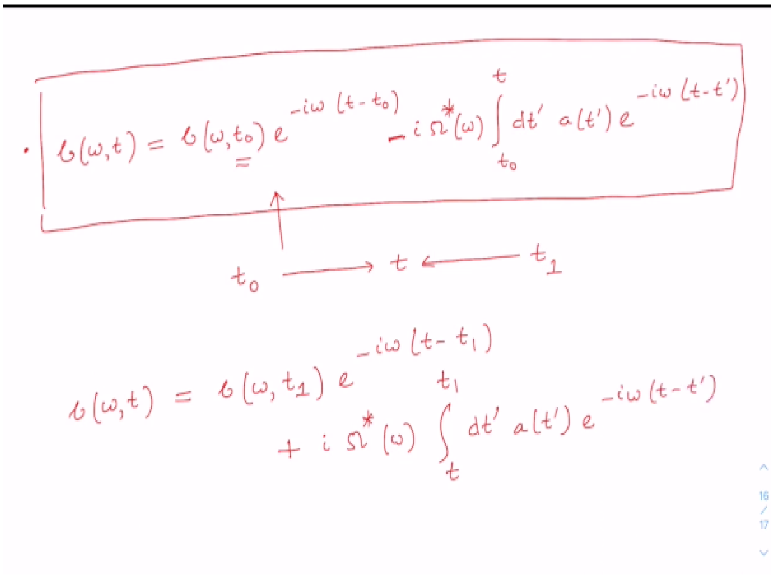

Now in contrast to the bath mode solution, we can write another solution in terms of a final time t 1, rather than the initial time t 0. While we have written this particular solution, we went from the initial time t 0 to some given time t, some instant of time t. We can have another solution where we can go from the say final time t 1 to this time t. That means we are now we can go in the backward direction in time.

If we do that, then we will get a solution of this type. That would be b of omega t. That would be equal to b of omega t 1, e to the power minus i omega t minus t 1. Here instead of this minus sign, and that is going to matter a lot, we will get a plus i omega star of omega integration from t to t 1 dt dash a of t dash e to the power minus i omega t minus t dash. This can be worked out very easily. Let me just quickly show you how to do that.

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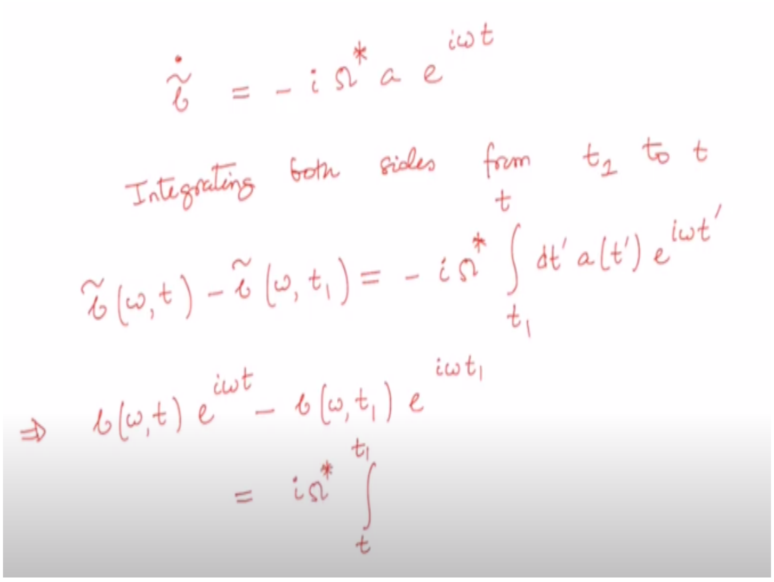

We can again begin from the change of variable for the bath and if we take the change of variable that is we introduced this quantity b tilde. So b tilde dot is equal to minus i omega star a e to the power i omega t. Let me quickly take you back to the way we have done it earlier. So while we have done it, as you see. Yes, this is where our original bath mode equation, time evolution equation for the bath mode.

Then going over to this new variable b tilde, we got rid of this particular term and we have then this particular equation. So here also in the similar way, I am starting with this particular equation. So integrating both sides, integrating both sides from this final time t 1 to some time t we can immediately write b tilde omega t minus b tilde omega t 1. That will be equal to minus i omega star integration t 1 to t dt dash a of t dash e to the power i omega t dash.

This I can now write as going back to the original variable that is b of omega t e to the power i omega t minus b of omega t 1 e to the power plus i omega t 1 and this is equal to, now let me just reverse the integration. So here I now go from t to t 1. So I will have a plus i omega star dt dash a of t dash e to the upper i omega t dash, okay.

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$$\Rightarrow b(\omega,t) = b(\omega,t_1)\, e^{-i\omega(t-t_1)} + i\,\Omega^*(\omega)\int_{t}^{t_1} dt'\, a(t')\, e^{-i\omega(t-t')}$$

$$\dot{a} = -i\omega_0 a + \Omega_{drive}\, e^{-i\omega_L t} - i\int d\omega\, \Omega(\omega)\, b(\omega,t)$$

So from here I get b of omega t is equal to b of omega t 1 e to the power minus i omega t minus t 1 plus i omega star of omega integration t to t 1 dt dash a of t dash e to the power minus i omega t minus t dash. Now we can put this solution into the equation for the cavity mode, the equation that we obtained.

That is a dot is equal to minus i omega 0 a plus omega drive e to the power minus i omega L t minus i integration d omega capital omega of omega b omega t. So let me put it here, this particular solution if I put it here.

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$$\boxed{\dot{a} = -i\omega_0 a + \Omega_{drive}\, e^{-i\omega_L t} + \frac{\kappa}{2} a - \sqrt{\kappa}\, a_{out}}$$

$$a_{out} = \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, b(\omega,t_1)\, e^{-i\omega(t-t_1)}$$

represents the waves travelling out from the cavity into the bath

And in the similar fashion, we will be able to get the equation, time evolution equation for the optical mode in this form and when we go into the backward direction in time now I have a dot is equal to minus i omega 0 a plus omega drive e to the power minus i omega L t plus now here we have plus kappa by 2 a and minus root over kappa. Here I will define a new variable a out.

Earlier we had a in. So here I am defining a variable a out which is defined as a out is equal to, this is also Langevin noise. It is i by square root of 2 pi integration minus infinity to plus infinity d omega b of omega t 1 e to the power minus i omega t minus t 1. This one actually represents, this represents waves traveling out from the cavity, traveling out from the cavity into the bath.

And intuitively you can see that this is this makes really sense because now we are going in the final time to the some instant of time t.

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$$\dot{a} = -i\omega_0 a + \Omega_{drive} e^{-i\omega_L t} + \frac{\kappa}{2} a - \sqrt{\kappa}\, a_{out} \rightarrow (ii)$$

$$\downarrow$$

$$-\kappa a - \sqrt{\kappa}\, a_{in} + \sqrt{\kappa}\, a_{out} = 0$$

$$\Rightarrow \boxed{a_{out} = a_{in} + \sqrt{\kappa}\, a}$$

So therefore, we get two equations for the bath optical mode, when we go in the forward direction in time and that equation that we got is this. It is a dot is equal to minus i omega 0 a plus omega drive e to the power minus i omega L t minus kappa by 2 a minus square root of kappa a in and let me say this is my equation 1. And another equation I got when I go in the backward in the time direction.

That is minus i omega 0 a plus omega drive e to the power minus i omega L t. And I will have here plus kappa by 2 a minus square root of kappa a out. So let me term it as equation number 2. Now if we subtract equation 2 from equation 1 then we will obtain this minus kappa a is minus square root of kappa, you can easily see this, a in plus square root of kappa a out, that is equal to 0.

And from here I get an equation for the output mode of the cavity mode, output optical field. That is a out is equal to a in plus square root of kappa a. This relation is known as the input-output relation. And it is a very useful relation because, if we can solve for the dynamics of the cavity mode a, then we can predict the observables in the cavity output. In fact, we can represent it in by diagram here.

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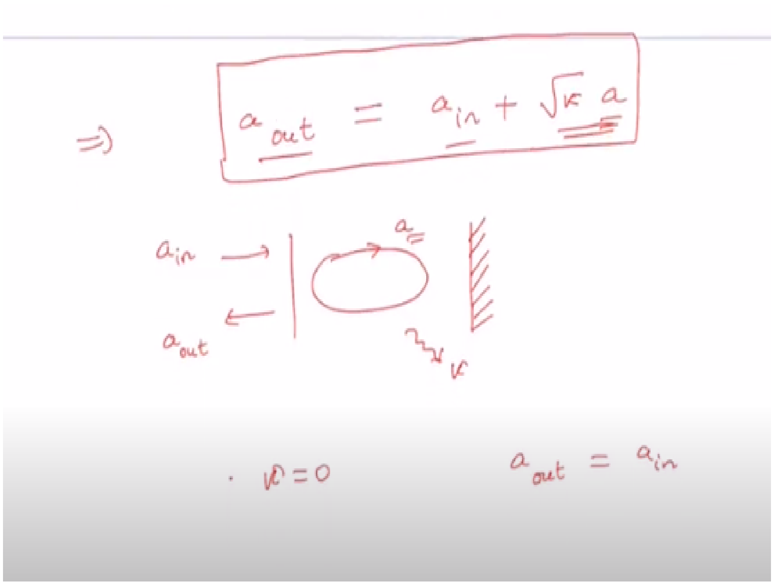

So we have this Fabry-Perot cavity where one of the mirror is perfectly reflecting and its other mirror is partially transmittive. So input is incident here and then output is the reflected part inside the cavity. And inside the cavity there is a cavity mode is there. Then this is circulating mode is there and it decays at the rate kappa. Now if kappa is equal to 0 that means cavity decay rate is 0.

Then you immediately see that whatever is getting incident that is going to get reflected. On the other hand, if kappa is not equal to 0, then the as you can see that the output is the sum of the incident field reflected at the cavity entrance and a contribution emanating from the cavity mode is given by this part. Please note that here this cavity mode, the A is a function of a in as is evident from this equation here.

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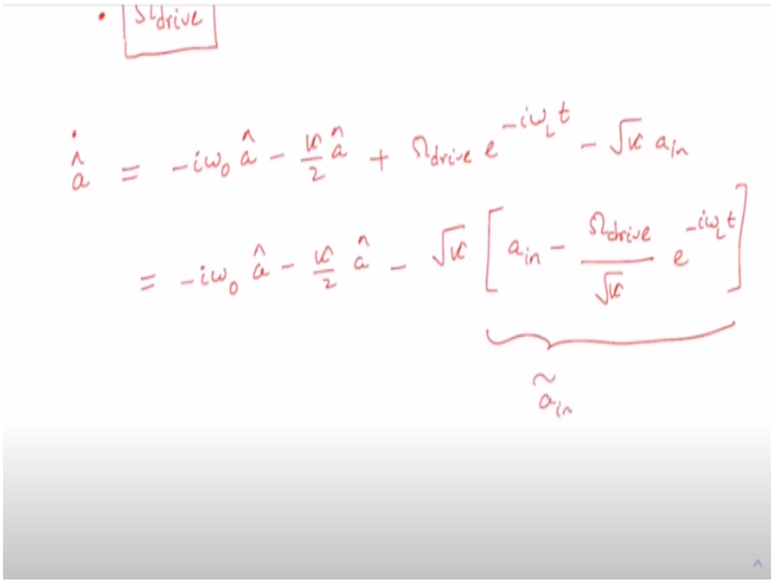

Finally, let us derive an expression for the drive coupling parameter omega drive which is the amplitude for the laser drive. When we are driving the Fabry-Perot cavity externally, we can use input output relation to work out an expression for omega drive. To do that, let us begin with this equation for the optical mode, cavity mode.

That is a dot is equal to minus i omega 0 a minus kappa by 2 a plus omega drive e to the power minus i omega L t minus square root of kappa a in. This equation we can write in a little bit different form, let me write it. Minus i omega 0 a minus kappa by 2 a. And let me write square root of kappa a in minus omega drive divided by square root of kappa e to the power minus i omega L t.

Let me define this parameter as the new input noise operator a in tilde and it includes the classical laser drive.

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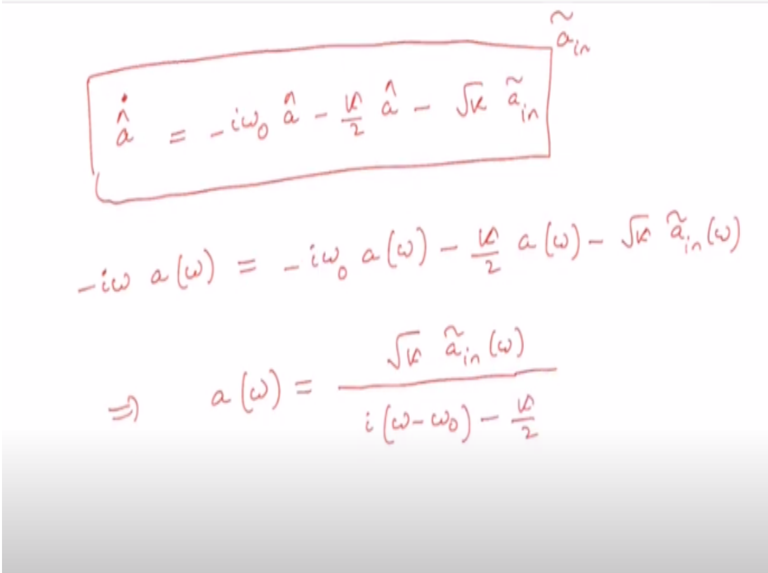

And with this equation, so let me rewrite again. We have a dot is equal to minus i omega 0 a minus kappa by 2 a minus square root of kappa a tilde in. Now if I go over to the frequency domain, that means if I take the Fourier transformation, immediately I can get this equation, minus i omega a of omega. These are all operators. Let me do not write the hat term all the time.

So you understand that these are anyway quantum operators. We have minus i omega 0 a of omega minus kappa by 2 a of omega minus square root of kappa a tilde in of omega. From here you can immediately get the expression for a omega. That would be square root of kappa a tilde in of omega i omega minus omega 0 minus kappa by 2.

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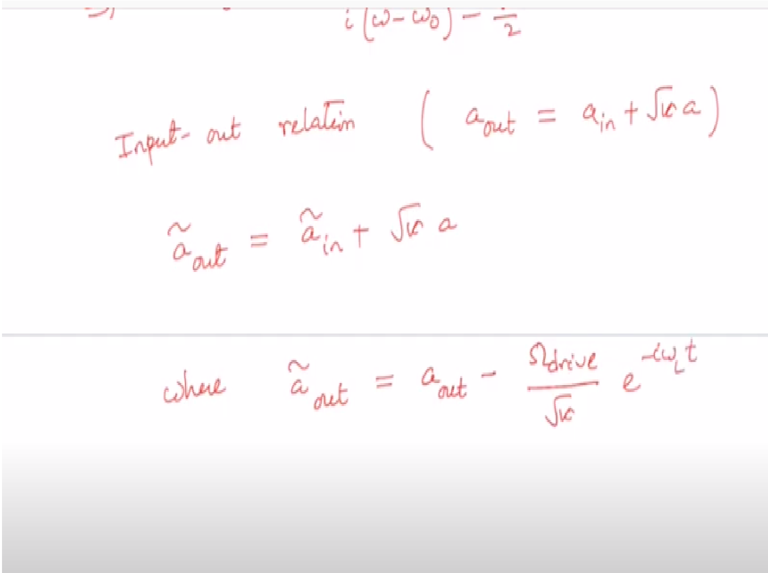

So if we now rewrite this input-output relation, let me rewrite input output relation for the new variable. We have this input output relation a out is equal to a in plus square root of kappa a. This we can write for our new variable says a tilde out is equal to a tilde in plus square root of kappa a. And where this a out is defined in a similar way that of a tilde in and a tilde out is equal to a out minus omega drive divided by square root of kappa e to the power minus i omega L t.

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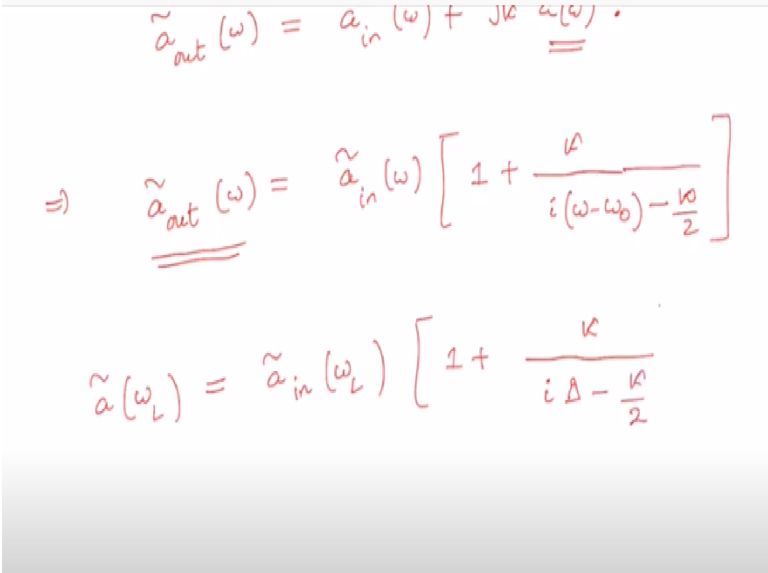

If I take the Fourier transform of this relation, so I will get it in the frequency domain is a tilde out omega is equal to a tilde in omega plus square root of kappa a of omega. Now we know the expression for a of omega from here and if I put it in this expression, so I will be able to write a tilde out of omega is equal to a tilde in of omega into 1 plus kappa divided by i omega minus omega 0 minus kappa by 2.

Let me evaluate this a tilde out at the laser frequency omega L and a tilde at omega L would be equal to a tilde in evaluated at omega L 1 plus kappa divided by i. Omega L minus omega 0 let me define it as the detuning parameter i delta minus kappa by 2, where I have defined the detuning parameter as omega L minus omega 0.

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$$\tilde{a}_{out}(\omega_L) = \tilde{a}_{in}(\omega_L)\left[1 + \frac{\kappa}{i\Delta - \frac{\kappa}{2}}\right]$$

$$\Delta = \omega_L - \omega_0$$

$$\boxed{\tilde{a}_{out}(\omega_L) = \frac{a(\omega_L)}{\sqrt{\kappa}}\left(i\Delta + \frac{\kappa}{2}\right)}$$

And again what I can do, we can write this a tilde in omega L in terms of a of omega, because we have this expression. From here I can write it in terms of a of omega and if I put it there, so I will get, so this is a tilde out, I will get an expression for a tilde out. It is very simple to work it out, just a few step and if you do it, you will get it as a of omega L divided by square root of kappa i delta plus kappa by 2, okay.

So this expression now we are going to utilize because this allows us to express the conservation of energy by equating the input power to the outgoing power.

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$$P_{in} = \hbar\omega_L \langle \hat{a}^{\dagger}_{out} \hat{a}_{out}(\omega_L)\rangle$$

$$= \hbar\omega_L \left(\Delta^2 + \frac{\kappa^2}{4}\right) \langle \hat{a}^{\dagger}(\omega_L)\hat{a}(\omega_L)\rangle$$

$$\dot{a} = -i\omega_0 a + \Omega_{drive} e^{-i\omega_L t} - \frac{\kappa}{2} a - \sqrt{\kappa}\, a_{in}$$

So input power P in has to be equal to the outgoing power that is equal to h cross omega L a dagger out plus a out and evaluated of course at the frequency of the laser that is omega L. Both these quantities a dagger as well as a, that is evaluated at omega

L. And because we have this expression for a tilde out, so from here I have h cross omega L. You can see that I will get del square plus kappa square by 4.

And we will have a dagger evaluated at omega L a of omega L. This is what I get. Now to go further, let me first do one thing. Let me take the equation for the cavity mode once again. So I have a dot is equal to minus i omega 0 a plus omega drive e to the power minus i omega L t minus kappa by 2 a minus square root of kappa a in.

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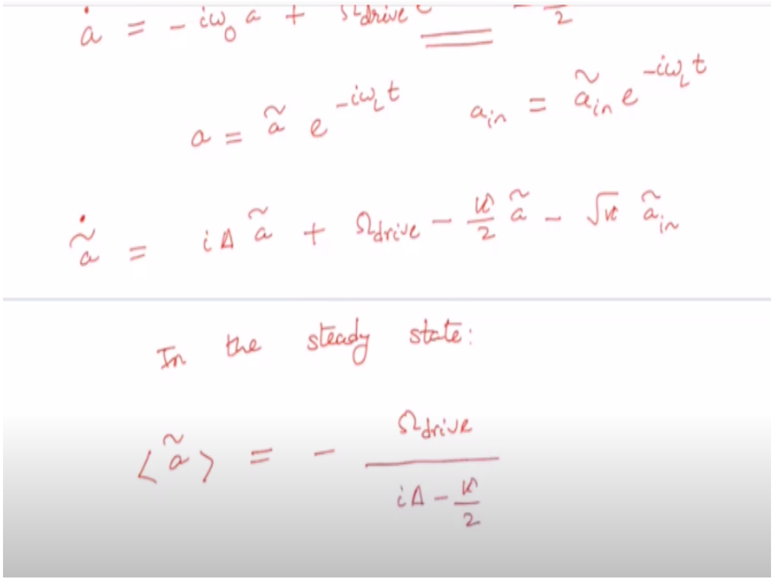

To get rid of this parameter explicit time dependence, so I apply the usual trick. I go to the change of variable. a I take it as I take a is equal to A tilde e to the power minus i omega L t and a in I take it as a in tilde e to the power minus i omega L t. So if I do that I will be able to get an equation for a in terms of a tilde.

So a tilde dot is equal to i delta a tilde plus omega drive minus kappa by 2 a tilde minus square root of kappa a tilde in. And from here in the steady state I can get the steady state value of a tilde. In the steady state I will have a tilde, average value of a tilde would be equal to minus omega drive divided by i delta minus kappa by 2.

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$$\left|\langle \tilde{a} \rangle\right|^2 = \left|\langle a \rangle\right|^2 = \frac{\left|\Omega_{drive}\right|}{\Delta^2 + \frac{\kappa^2}{4}}$$

$$\langle \hat{a}^{\dagger}(\omega_L)\, \hat{a}(\omega_L) \rangle = \left|\langle a \rangle\right|^2 = \frac{\left|\Omega_{drive}\right|^2}{\Delta^2 + \frac{\kappa^2}{4}}$$

$$P_{in} = \frac{\hbar\omega_L}{\kappa} \left|\Omega_{drive}\right|^2$$

And therefore, as you can see if I take the mod square of a tilde square that is exactly equal to mod of average of a square and that is equal to omega drive mod square divided by delta square plus kappa square by 4. So therefore, what we have here is this that a dagger of omega L a of omega L is equal to average of this quantity and this is equal to simply the one that let me again right here. It is this.

So this is what I have. So therefore, we will obtain P in is equal to h cross omega L. Okay, let me show you the expression here once again. So we have this. We now got this expression. So it will be h cross omega L divided by kappa omega drive mod square because this particular term is getting cancelled because of this term and we have this expression for input power.

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$$\langle \hat{a}^{\dagger}(\omega_L)\, \hat{a}(\omega_L) \rangle = \left|\langle a \rangle\right|^2 = \frac{\left|\Omega_{drive}\right|^2}{\Delta^2 + \frac{\kappa^2}{4}}$$

$$\boxed{P_{in} = \frac{\hbar\omega_L}{\kappa} \left|\Omega_{drive}\right|^2}$$

$$\Rightarrow \boxed{\left|\Omega_{drive}\right| = \sqrt{\frac{\kappa P_{in}}{\hbar\omega_L}}}$$

And from here we can write an expression for the drive amplitude, laser drive amplitude. That would be mod of omega drive is equal to square root of kappa P in divided by h cross omega L. This is an expression which is worth remembering and it will be useful for our discussion on cavity optomechanics in the next class. Let me stop here for today. In this lecture, we discussed how quantum Langevin noise affects the optical mode of a Fabry-Perot cavity.

This led us to the discussion of input-output relation. We applied this input output relation to derive an approximate expression for the laser drive amplitude. So we are now well equipped with all the tools to discuss quantum cavity optomechanics in the next class. So see you in the next class. Thank you.