# Quantum Technology and Quantum Phenomena in Macroscopic Systems

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## Lecture - 41
## Cavity Quantum Optomechanics

Hello, welcome to lecture 9 of module 3. This is lecture number 30 of the course. After getting equipped with all the necessary tools in the previous class, now we are ready to discuss the quantum mechanical Hamiltonian for a cavity optomechanical system. And finally in this lecture, we will obtain the quantum Langevin equation for a cavity optomechanical system. So let us begin.

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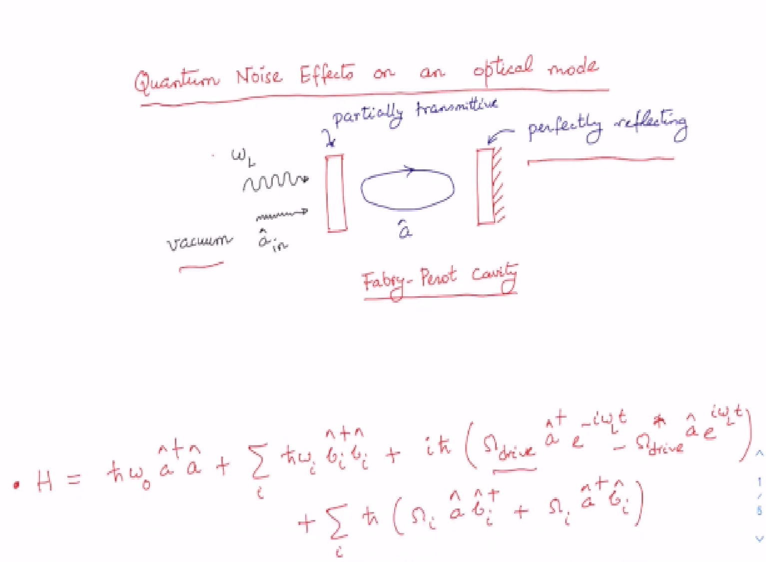


In the previous lecture, we studied the effects of environment on an optical mode in a Fabry-Perot cavity. We considered a single sided cavity where one mirror is perfectly reflecting while the other mirror is weakly transmissive. Electromagnetic fluctuations from the vacuum outside the cavity inject quantum noise into the cavity.

The cavity is driven by a single mode laser with drive amplitude omega drive and frequency omega L. We have written down the Hamiltonian and then went on to write this Hamiltonian in the continuum domain because the vacuum could be modeled as an infinite collection of independent bath oscillators.

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$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \int \ldots$$
$$+ i\hbar\left(\Omega_{drive}\hat{a}^\dagger e^{-i\omega_L t} - \Omega^*_{drive}\hat{a} e^{i\omega_L t}\right)$$
$$+ \hbar\int d\omega\left[\Omega^*(\omega)\hat{a}\hat{b}^\dagger(\omega) + \Omega(\omega)\hat{a}^\dagger \hat{b}(\omega)\right]$$

$$\dot{\hat{a}} = -i\omega_0\hat{a} + \Omega_{drive} e^{-i\omega_L t} - i\int d\omega\, \Omega(\omega)\, b(\omega)$$

$$\dot{\hat{b}} = -i\omega\, b(\omega) - i\,\Omega^*(\omega)\, a$$

So using this Hamiltonian, we have first worked out the Heisenberg equation of motion for the optical mode and the bath mode.

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Assume that the coupling $\Omega$ is constant for all bath frequencies $\omega$

$$\boxed{|\Omega(\omega)|^2 = \frac{\kappa}{2\pi}},$$

($\kappa$ is related to the cavity band width $\delta\nu = \frac{\omega_+ - \omega_-}{2\pi} = \frac{\kappa}{2\pi}$)

$$\boxed{\dot{a}(t) = -i\omega_0 a + \Omega_{drive} e^{-i\omega_L t} - \frac{\kappa}{2} a - \sqrt{\kappa}\, a_{in}} \rightarrow (i)$$

$$a_{in} = \frac{i}{\ldots}\int^{\infty} d\omega\, b(\omega, t_0)\, e^{-i\omega(t-t_0)}$$

And then we solved the bath mode equation and we got a solution for going from some initial time t 0 to some instant of time say t and putting this bath solution into the Heisenberg equation for the mode, optical mode we got this equation first. And also we have taken the coupling between the bath oscillator and the optical mode to be constant for all bath frequencies. And we have identified or taken this coupling parameter to be like this, where kappa is related to the cavity bandwidth.

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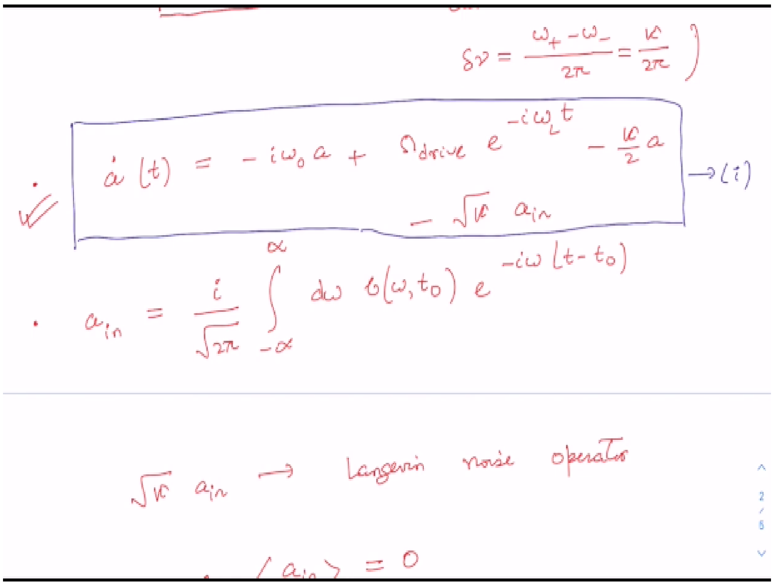


We got an equation for the, Heisenberg equation for the optical mode where we have this a in refers to the input quantum noise that is injected into the cavity.

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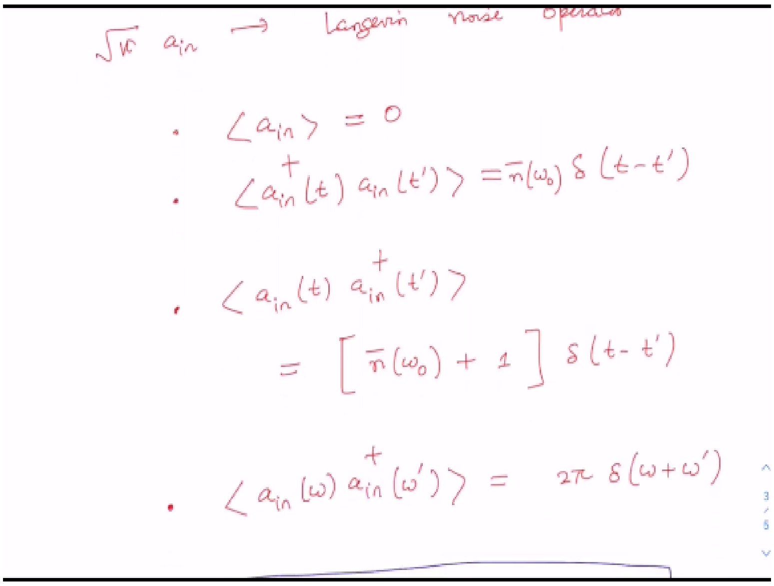


And this is a Langevin noise because it satisfy the characteristic of Langevin, quantum Langevin noise. And we got couple of very useful relations.

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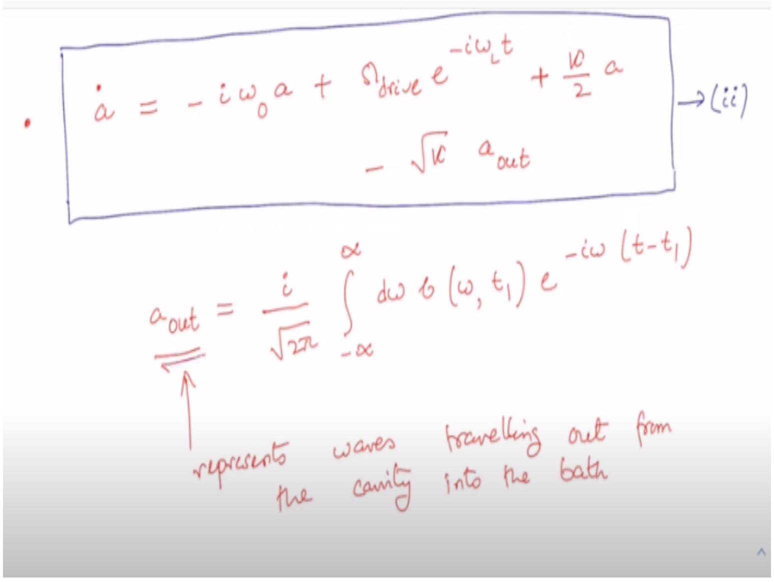


Then we also got a solution for the bath mode going from a final time t 1 to some instant of time t. That is we now went in the backward direction and that resulted in this equation, Heisenberg equation of motion for the optical mode. Here this quantity a out represents waves traveling out from the cavity into the bath.

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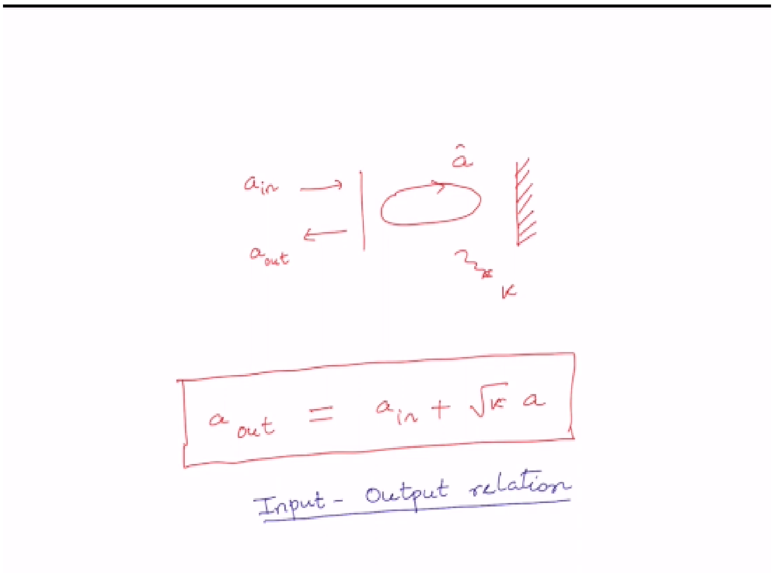


So using this equations and formalism, finally we obtained a very important relation known as the input-output relation where we can have this output field in terms of the input field and what is there in the inside the cavity or the cavity mode.

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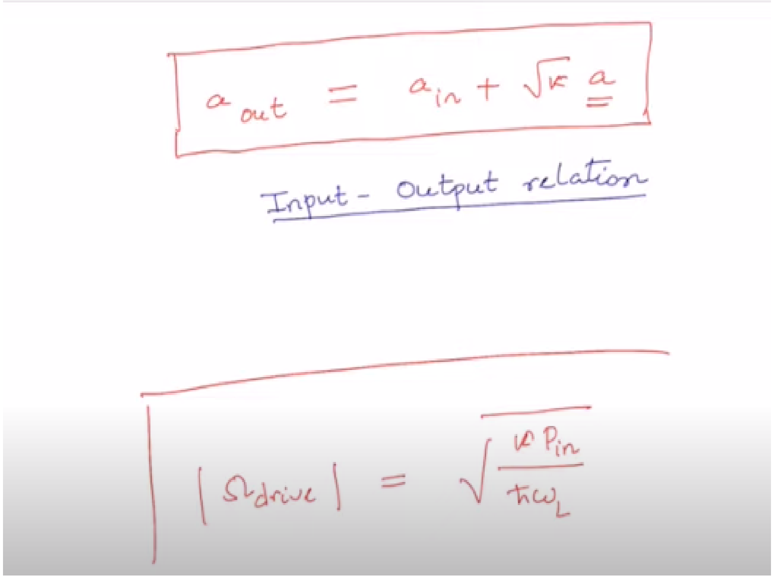


We finally applied this input-output relation to work out the drive amplitude of the laser and we got this particular expression.

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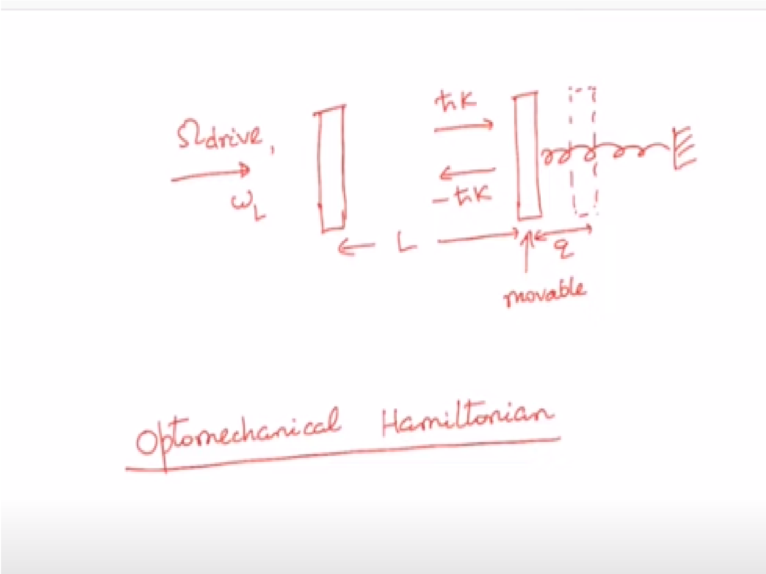


Now let us consider the typical optomechanical system which is basically a Fabry-Perot cavity, but with one of the mirrors movable, say let us this particular mirror is movable. So say it can get displaced to this position. When it is not displaced the length of the cavity is say L and this displacement let me denote it by q. Earlier we denoted it by x. And so this mirror is movable.

And also this cavity is driven by laser with amplitude omega drive and laser frequency omega L. And we know that the reversal of momenta in every round trip, say if a photon hits the mirror with momentum h cross k, so it get reversed and its momentum is minus h cross a. This reversal of momenta in every round trip is the origin of radiation pressure force, which we discussed earlier in this module.

The radiation pressure force is at the root of optomechanical light matter interaction. Now let me talk about the optomechanical Hamiltonian, which in an earlier class we actually guessed it. But let us do it more formally now and we will do it in some more details. So let us now write down the optomechanical Hamiltonian, quantum Hamiltonian. We will assume that the motion of this movable mirror is very slow.

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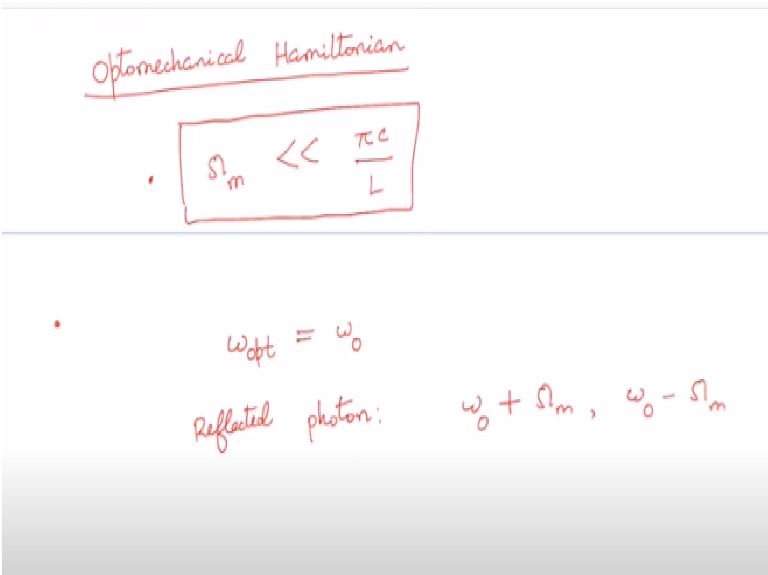


And the mechanical oscillator frequency, let me denote the mechanical oscillator frequency by omega m here with the suffix m for the mechanics. This mechanical oscillator frequency is say much smaller than the free spectral range. And free spectral range is given as pi c by L which also we discussed in an earlier class. Now what happens is this that and why this assumption is made, it will be clear to you.

Actually when an oscillating mirror oscillates, it can convert incident photons with frequency, the cavity photon frequency say omega opt. Let me now denote it by simply omega of 0. When this photon hits the movable mirror, the reflected photon can have a frequency that would be modified.

And it would be modified if we just confine ourselves to the first side bands then the reflected photon will have frequency omega 0 plus omega m or it may have a frequency omega 0 minus omega m. As you know it may be either blue shifted or it may be red shifted. So this condition here that this mechanical frequency has to be much less than the free spectral range.

This ensures that the moving mirror will scatter very few photons from the originally occupied mode at frequency omega 0 to the nearest neighboring cavity resonance, which by definition we know that it is a free spectral range away.

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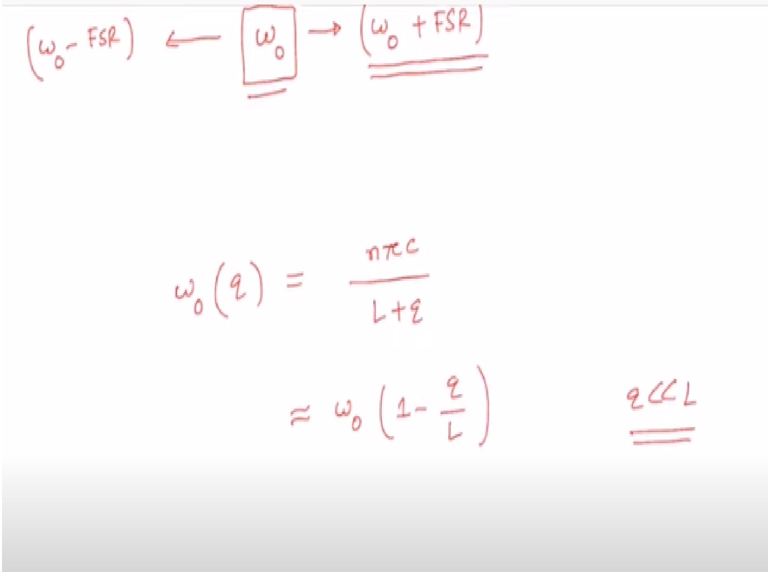


So because we want to confine ourselves to one particular cavity mode only that is say omega 0, other optical mode are situated from here by the so called free spectral range. The others optical mode would be separated, the second one from here would be in this direction. It would be situated to one free spectral range away or the other one is also similarly, it would be omega 0 minus free spectral range.

So if we take this particular condition, then we can ignore all optical modes except the one with frequency omega 0. However, as you know because of the length of the cavity is not, it is not constant, omega 0 is, this omega 0 is slightly modified. Now it is a function of say position of the oscillating mirror, movable mirror and we earlier we discussed it. This is n pi c by L plus q.

The resonance frequency would now depend on the displacement q, which if this displacement is much smaller than the length of the cavity, under that condition we can write this expression as omega 0 1 minus q by L.

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$$\hat{H} = \hbar\omega_0(q)\,\hat{a}^\dagger\hat{a} + \frac{\hat{p}^2}{2m} + \frac{1}{2}m\Omega_m^2\hat{q}^2$$

$$= \hbar\omega_0\,\hat{a}^\dagger\hat{a} - \hbar\omega_0\frac{q}{L}\,\hat{a}^\dagger\hat{a} + \frac{\hat{p}^2}{2m} + \frac{1}{2}m\Omega_m^2\hat{q}^2$$

$$g_0 = \frac{\omega_0}{L}$$

$$\hat{H} = \hbar\omega_0\,a^\dagger a - \underbrace{\hbar g_0\,q}_{}\,\hat{a}^\dagger\hat{a} + \frac{\hat{p}^2}{2m} + \frac{1}{2}\Omega_m^2\hat{q}^2$$

So the Hamiltonian in the optomechanical system without inclusion of noise and external laser drive can be written very simply. That would be h cross omega 0, which is a function of position of the movable mirror. This is the photon energy plus the mechanical oscillator energy which is a harmonic oscillator. So p square by twice m plus half m, m is the mass of the movable mirror.

And omega m is its resonance frequency, omega m square q square. All these are operators. This can be also written as now if I put this, break it up then I will have h cross omega 0 a dagger a minus h cross omega 0 q by L a dagger a. And I have this. This is the energy of the, this part is the energy of the movable mirror. So this we can actually write in a another form.

If I defined a quantity constant parameter say g 0, this I defined as omega 0 by L, then I can write this expression as H is equal to h cross omega 0 a dagger a minus h cross g 0 into q a dagger a plus p square by twice m plus half m omega m square capital omega m square q square. Now this particular term here it is proportional to the as you can see it is proportional to the number of photons in the cavity mode as well as it is proportional it is multiplied by the mechanical amplitude q.

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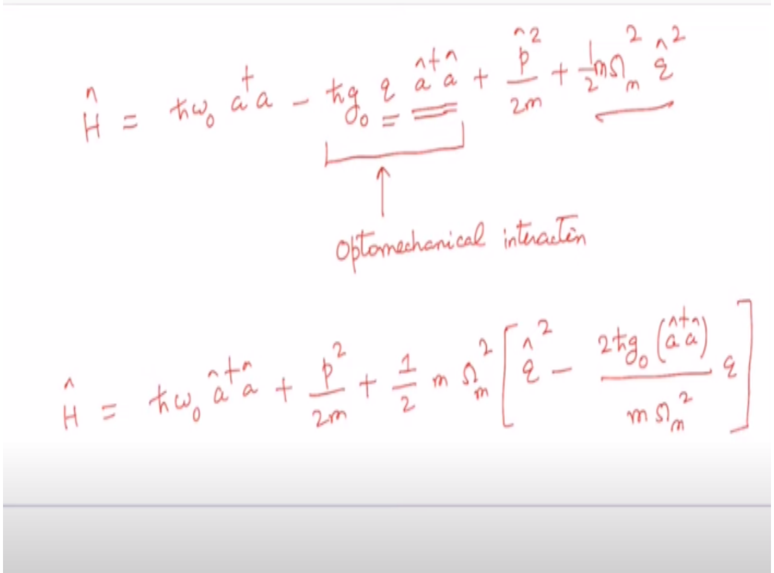


And this describe as you know, it is described the optomechanical interaction. So this term describe the optomechanical interaction. This Hamiltonian could be written in different form as well. So let me write that because this will teach us something useful. We can write it as h cross omega 0 a dagger a, all these are operators, plus p square by twice m plus half m.

Here m I have to write, m omega m square say let me take q square, this term here. Then minus twice h cross g 0 a dagger a into q divided by m into omega m square. So that is what I write here.

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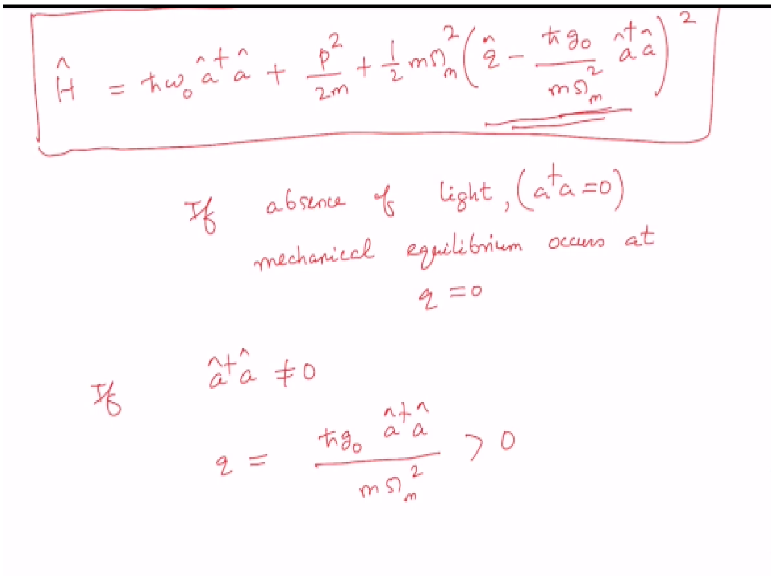


Then, I can write it as, very easy to show, that I can write it as a dagger a h cross omega 0 a dagger a plus p square by twice m plus half m omega m square into q minus h cross g 0 divided by m into omega m square a dagger a okay whole square. So this is very simple to show. Now this particular form clearly shows that if there is no light, there is absence of light, if absence of light, absence of light is there, that means, a dagger a is equal to 0, right?

If there is no light that means a dagger a is equal to 0. The mechanical equilibrium as you can see, equilibrium occurs at q is equal to 0. Because this term would not be there and you will have the mechanical equilibrium occurs at q is equal to 0.

On the other hand, if there is light that means if a dagger a is not equal to 0, then that is in the presence of light the equilibrium point is shifted to q is equal to, it is very clear from this expression, it is shifted to h cross g 0 a dagger a divided by m omega m square which is greater than 0. This makes sense physically because as the radiation pressure acts to the right.

The Hamiltonian is, this particular Hamiltonian which we have started with, this Hamiltonian is also written in a little bit different form and in a very familiar form you are already aware.

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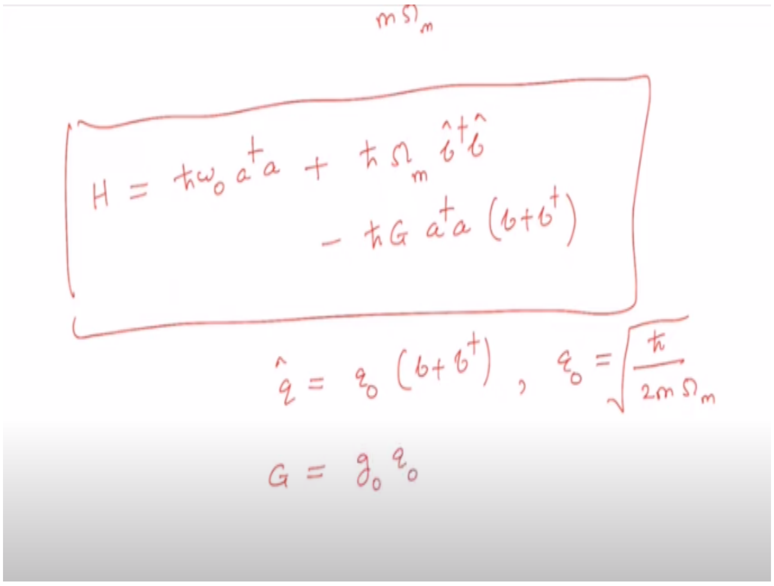


We can write it as H is equal to h cross omega 0. This is we are going to write in terms of the annihilation and creation operator of the mechanical oscillator. So it would h cross omega m b dagger b and the optomechanical interaction part you can write as h cross G a dagger a b plus b dagger. This is what I write. While I have written it what we have used is this. We have used q.

The position coordinate operator we write it as q 0 b plus b dagger where q 0 is the zero point fluctuation and this is h cross divided by twice m omega m under root. And this capital G which is known as the optomechanical coupling constant or coupling parameter is g 0 into the zero point fluctuation q 0. Let us now explore the Eigenstates and Eigenvalues of this Hamiltonian.

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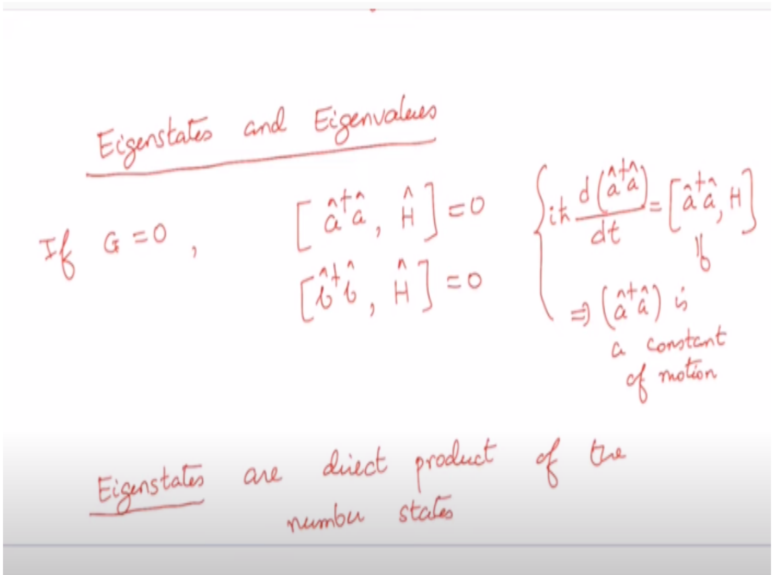


Eigenstates and Eigenvalues. Firstly, let me consider the case when there is no coupling between the optics and the mechanics. If G is equal to capital G that is the coupling parameter is 0. Then you will see that the number of photons as well as the number of phonons is conserved for this Hamiltonian because you can show very easily. It is very straightforward if you just look at the Hamiltonian here.

Then the commutation relation between a dagger a and H. This would be equal to zero and similarly, you will get that b dagger b and its commutation with the Hamiltonian would be 0. And that is why we say that the phonon number as well as the photon number is a constant of motion. And this conclusion I am making based on this Heisenberg equation.

You see from the Heisenberg equation I have d a dagger a dt is equal to communication between a dagger a H. In fact here i h cross is also there. So if this is equal to 0, this implies that a dagger a is a constant of motion, is a constant of motion or it is conserved. And similar equation we can write for b dagger b.

So because of this, it is easy to conclude that the Eigenstates in this case when there is no coupling between the mechanics in the light mode, the Eigenstates are nothing but direct product, Eigenstates are simply Eigenstates are direct product of the number states corresponding to photons and the phonons.

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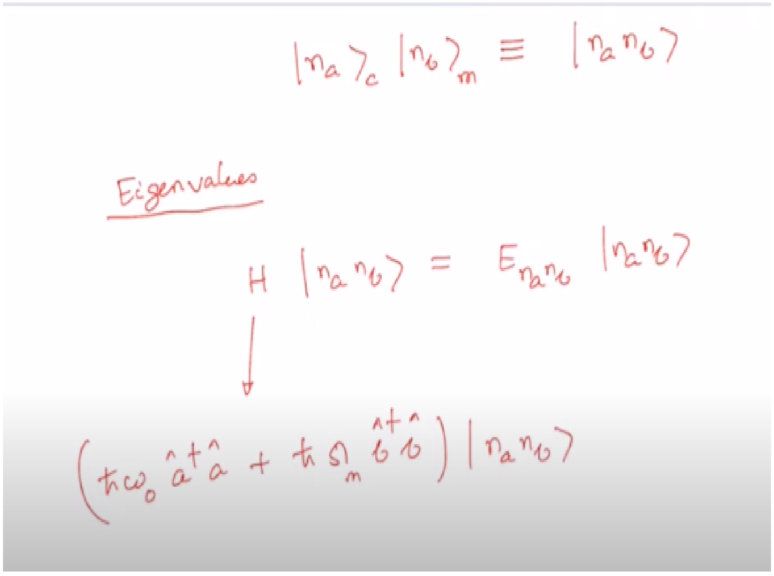


And this we can write as n a, this is for the cavity mode or optical mode and n b which is for the mechanical mode or mechanics and this in shorthand notation let me write it is as n a n b. So this is the Eigenstate. Now what about the Eigenvalues. It is very easy to work out. Eigenvalues can be worked out just by solving this Eigenvalue equation. So H n a n b is equal to say the Eigenvalue is E n a n b, its energy.

So this is what we have to solve. So first of all let me put the Hamiltonian. That is h cross omega 0 a dagger a plus h cross omega m b dagger b. Now here we are considering that capital G the coupling parameter is 0.

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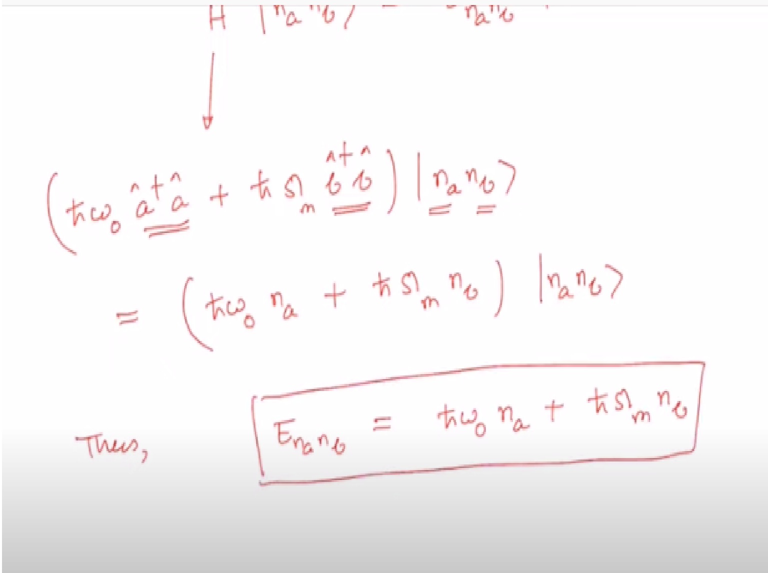


So now if we work it out, a dagger a will operate on n a, b dagger b this is the number operator operating on n b. So we will get simply h cross omega 0 n a plus h cross omega m n b. And this would be n a n b. So clearly therefore we can write that E n a n b the Eigenvalue is simply h cross omega 0 n a plus h cross omega m n b, okay.

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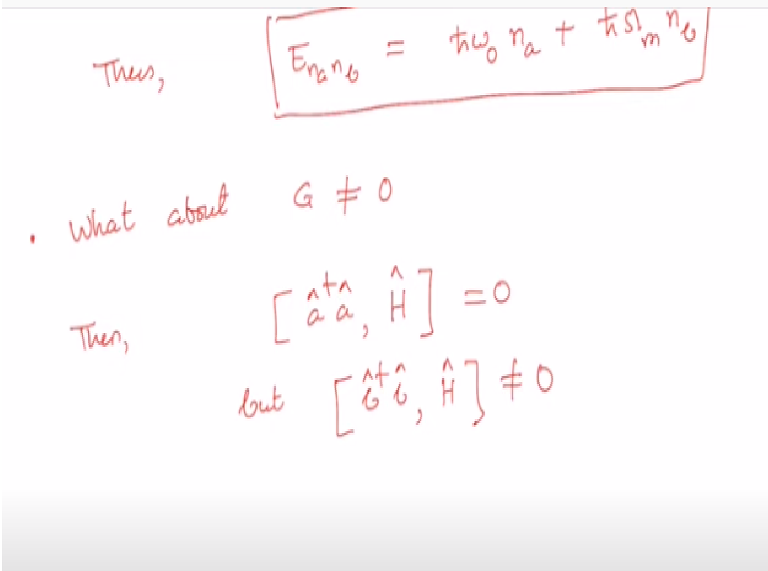


Now what about the case when this G the coupling is nonzero. In this case, then you can immediately see that the commutation relation between a dagger a number operator for the cavity mode and the Hamiltonian, this is equal to 0, but this commutation relation between b dagger b and the Hamiltonian is not equal to 0.

So it means that the optical number state is still an Eigenstate of the optomechanical Hamiltonian, but the mechanical number state is not, okay. So what we can say further is that since this particular term in the Hamiltonian, this term in the Hamiltonian, now G is nonzero corresponds to the displacement of the mechanical oscillator due to the effect of the optical force, so we may expect that the mechanical Eigenstate is a displaced number state.

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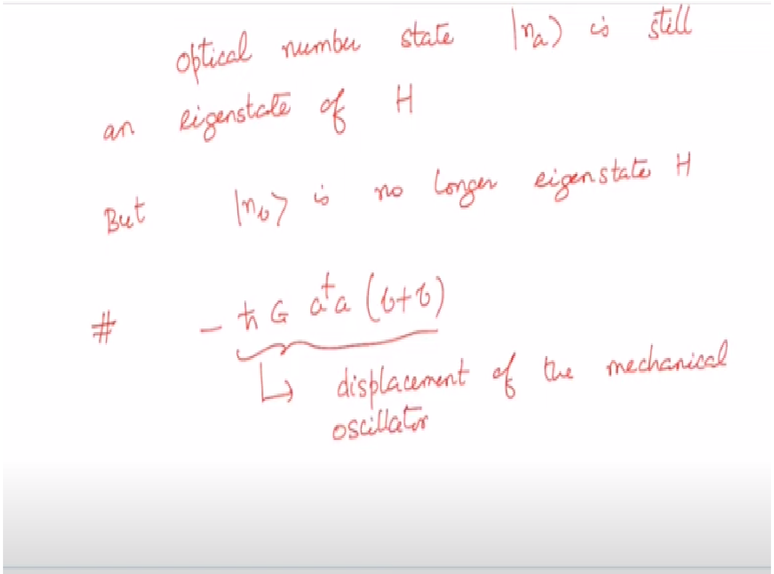


Let me write here again the conclusion from this facts is that optical number state that is n a, we can say that is still is still an Eigenstate, is still an Eigenstate of the Hamiltonian but and n b is no longer Eigenstate of H.

But now we have this particular term based on which we can conclude or assume that because this corresponds to displacement, this whole term refers to the fact that displacement of the mechanical mode or the mechanical oscillator, displacement of the mechanical oscillator by an optical force.

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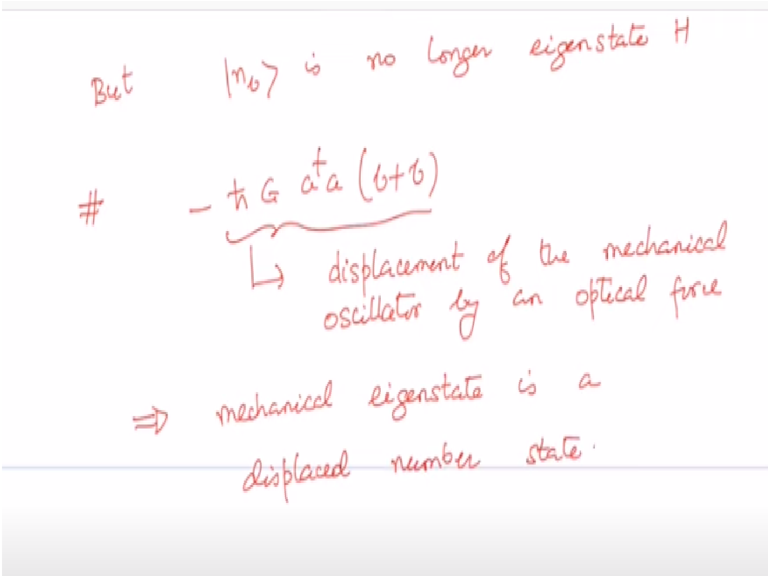


And so we can say that mechanical Eigenstate, this means that the mechanical Eigenstate is a displaced number state.

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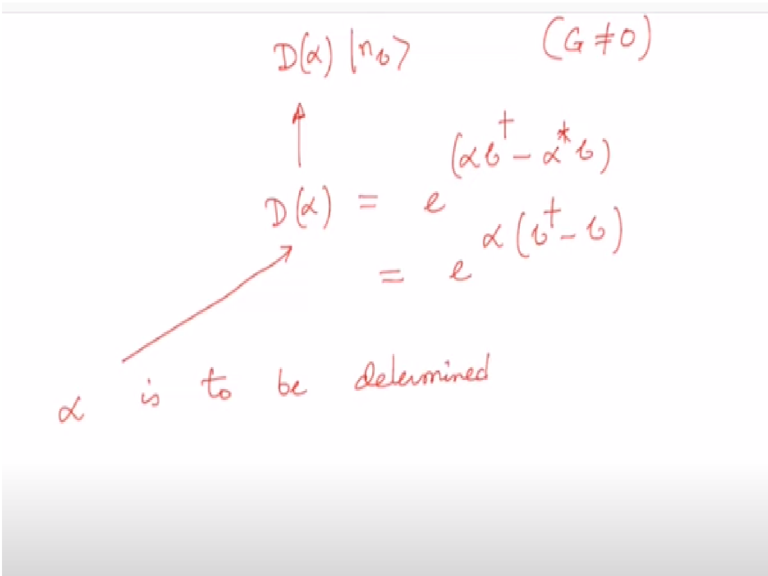


And what I mean by that is our original number state is n b, mechanical number state. Now it gets displaced when G is no longer 0. Therefore, by the way, you know that this is the displacement operator and we have discussed about it earlier in module 1. It is e to the power alpha. For mechanical oscillator it will be e to power alpha b dagger minus alpha star b. We can take alpha to be a real quantity.

Then we can write it e to the power alpha b dagger minus b and this parameter alpha is to be determined and we will see how it is to be determined.

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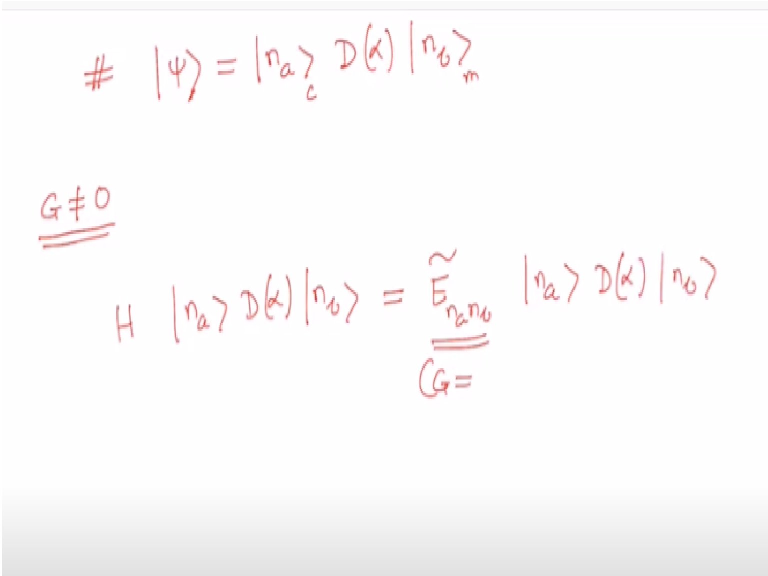


And actually, what is the core idea here is that now we are considering our Eigenstate as n a direct product of the number state for photons and the displaced state of the phonon is our Eigenstate, let us say psi. So this is the Eigenstate we can assume. Let us now set the Eigenvalue equation for G is equal to not 0. So in this case we can then set the Eigenvalue equation in this form.

The Eigenstate is n a and direct product of the number state for a photon and the displaced state of the mechanical phonon and this is equal to E n a n b. And this E n a n b that I am writing is the Eigenvalue. But here, do not get confused with the earlier E n a E n b. That is here, this is we have worked out for G is equal to 0. But this E n a, let me actually put a tilde sign so that you do not get confused.

And this is what we now need to work out. And this is little bit lengthy calculation, but very straightforward. Let me tell you how to do that. We have to work out this equation first as we have done it for G is equal to 0 case.

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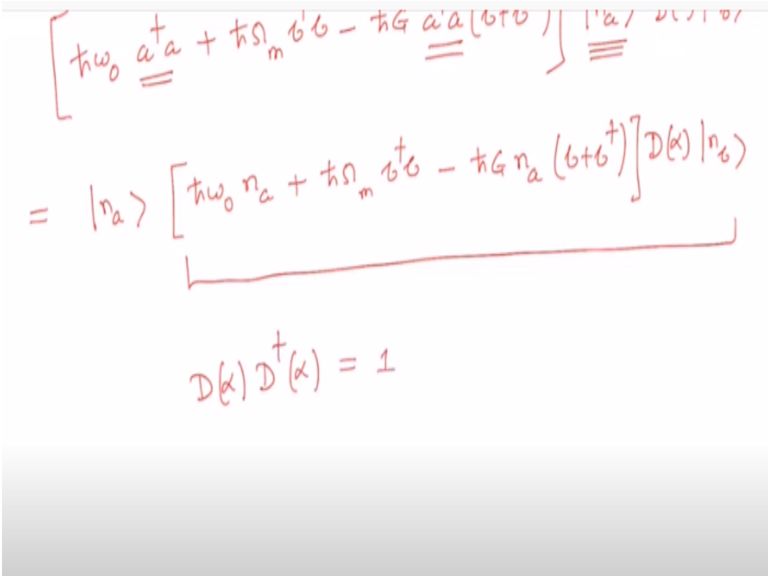


So Hamiltonian here we have h cross omega 0 a dagger a plus h cross omega m b dagger b. And now we have minus h cross G a dagger a b plus b dagger. So this is our Hamiltonian. It is operating on the number state and displaced mechanical state. So we will get from here because a dagger a will operate on this. Similarly, this a dagger will operate on n a number state for the photon.

So this will lead us to I will get a number because of that and n a I can take it out, you will understand once I write it. You will have h cross omega 0, this is just a number n a plus h cross omega m b dagger b minus h cross G. This would be a number n a and I have b plus b dagger. And I have here D alpha n b. This is what we have to work now, work it out. And we can work it out using this relations.

This maybe we can do it in problem solving session. But it is very easy. This relation already we know from our module 1, d alpha d dagger alpha is equal to 1.

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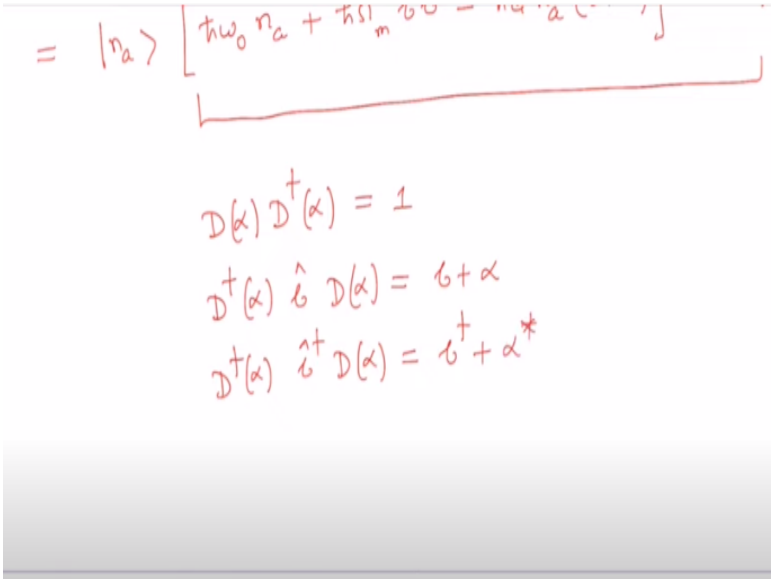


And this also we know that d dagger alpha b d alpha is equal to b plus alpha and d dagger alpha b dagger d alpha is equal to b dagger plus alpha star.

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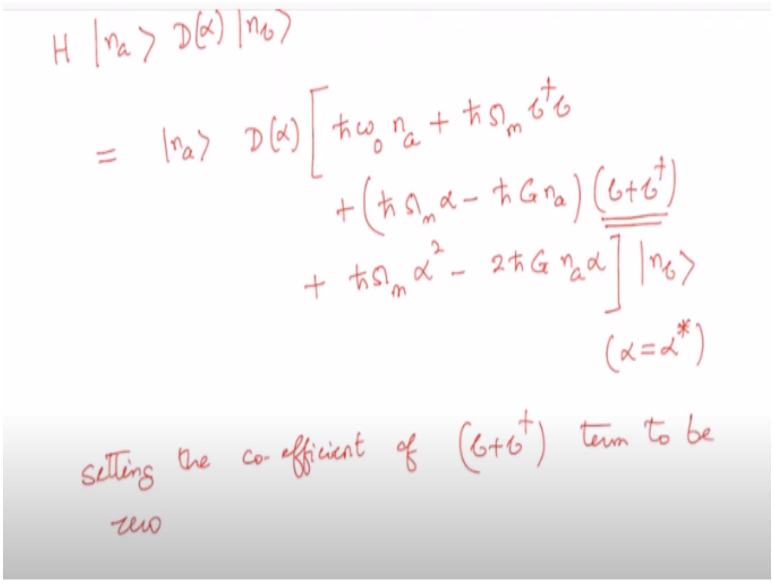


So using this relations, we can show that H n a D alpha n b is equal to n a d alpha. And here we can show that this will get a term like this, expression like this. We will have h cross omega 0 n a plus h cross omega m b dagger b plus h cross omega m alpha minus h cross G n a. It is the coefficient of b plus b dagger. Then we will have terms h cross omega m alpha square minus twice h cross G n a alpha and here I have n b.

I encourage you to do this while we have done it. We take alpha is equal to alpha star, alpha to be real. Now setting the coefficients of b plus b dagger as 0, setting the coefficient of b plus b dagger term to be 0, we can get the value of the parameter alpha.

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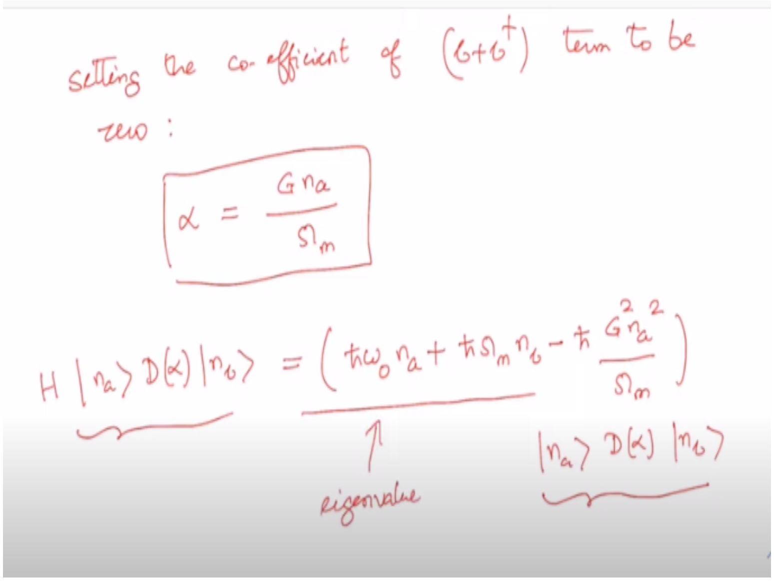


We will get alpha is equal to G into n a divided by omega m. Now using this value, we can then have H n a D alpha n b. This is our Eigenstate for the system when G is not equal to 0. This would be n a D alpha. In fact, if you put the parameter for alpha, this value if you put it what you are going to get is let me just write down the final expression.

You will get h cross omega 0 n a plus h cross omega n b minus h cross G square n a square divided by omega m. And you will have n a D alpha n b. So this is your Eigenstate and this is your Eigenvalue, right? This is your Eigenvalue.

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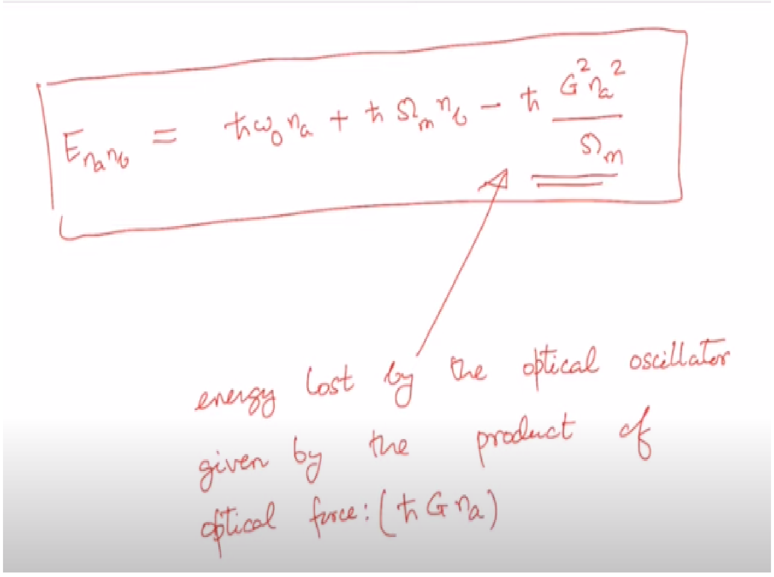


So we recognize that the Eigenvalue E n a n b when G is nonzero is equal to h cross omega 0 n a plus h cross capital omega m n b minus h cross G square n a square divided by omega m. So what we see here is this that the difference from G is equal to 0 is this last term. So this last term accounts for the shift from G is equal to capital G is equal to 0 energy levels and it can be interpreted as the energy lost.

This can be interpreted as the energy lost by the optical oscillator, by the optical oscillator because the optical force, because of optical force, because it displaced the optical force is spent in displacing the mechanical oscillator given by energy lost by the optical oscillator given by the product, as you can see here product of optical force. And optical force if you see that this is a h cross G into n a. This is the optical force.

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energy lost by the optical oscillator
given by the product of
optical force: $(\hbar G n_a)$ and shift
in the equilibrium position of the
mechanical oscillator

$$\alpha = \frac{G n_a}{\Omega_m}$$

$$\hbar \frac{G^2 n_a^2}{\Omega_m} = (\hbar G n_a)\left(\frac{G n_a}{\Omega_m}\right)$$

And shift in the mechanical equilibrium position and shift in the equilibrium position of the mechanical oscillator and this shift is given by the parameter alpha and alpha is equal to G n a divided by omega m. So let me again write this thing. So it clearly we have this last term as h cross G square n a square divided by omega m.

This I can write into two parts. One term is h cross G n a. That is the optical force. And then we have this parameter alpha which is G n a divided by omega m. And this is the displacement and this is the force. So it has the dimension of energy overall. So I hope you get the idea here. Let me show you something very interesting now.

We can very easily get a useful form of the Hamiltonian which will deliver what we have just concluded about the energy Eigenvalues.

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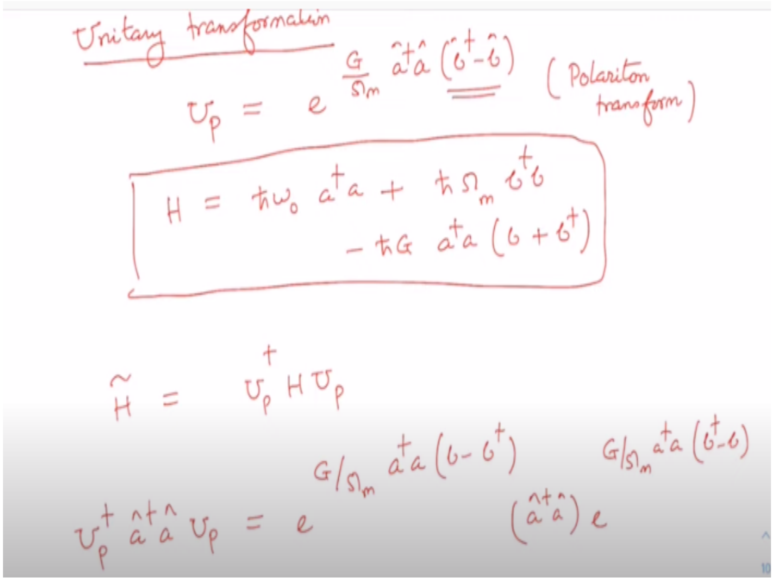


We can use a unitary transformation known as polariton transformation as defined as U p is equal to e to the power capital G by omega m a dagger a b dagger minus b. And if we apply this unitary transformation to our Hamiltonian, let me write down the Hamiltonian once again. So we have our Hamiltonian H is equal to h cross omega 0 a dagger a plus h cross omega m b dagger b minus h cross G a dagger a into b plus b dagger.

So if we apply this in polariton transform, this is called polariton transform. If we apply it, then we will get a transform Hamiltonian. So we will have U p dagger U p. We can work it out and several terms would be there. First of all let us for example, quickly work it out U p dagger a dagger a because this would be here as you can see from this Hamiltonian.

We have to apply it from both sides with U p dagger a dagger a U p and here I have e to the power G by omega m a dagger a. And we will have here it as b minus b dagger because this is b dagger minus b. I am taking the U p dagger. So it is b minus b dagger. And here a dagger a and here I have G by omega m a dagger a b dagger minus b, alright.

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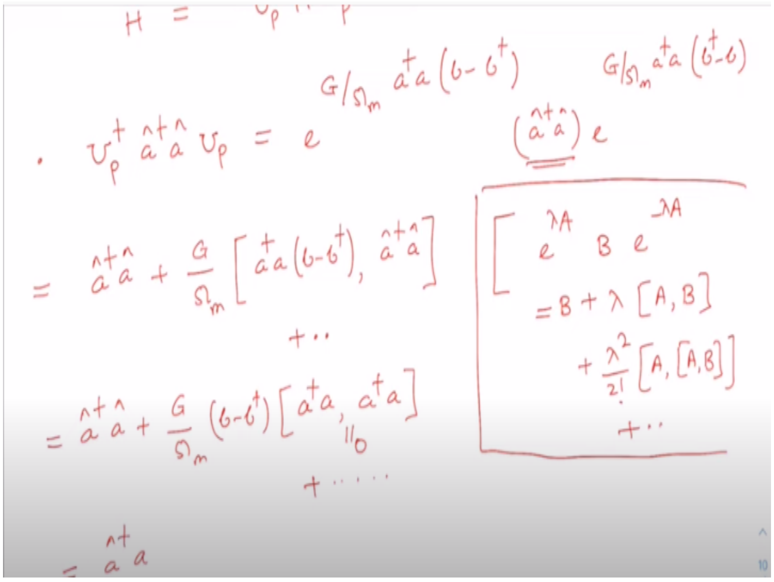

So to work it out let us recall this formula, Baker Hausdroff formula, e to the power A B e to the power minus A and this we know that this would be B plus, in fact we can put a lambda sign here, lambda parameter there. Then it will be b plus lambda A, B plus lambda square by 2 factorial A, (A, B) and so on. If we apply this particular formula here then we will get it as our B is this, this whole operator let us take.

So this is a dagger a. Then the second term would be, let me take the lambda parameter as G by omega m. Then operator a is here, a dagger a b minus b dagger and here I have a dagger a and then we will have the other terms. Now as you can see from this term okay let me write another line here, a dagger a plus I have G by omega m.

I can take b minus b dagger outside and I have a dagger a, a dagger a and rest of the terms and you know this is equal to 0. And because this second term vanishes, so all other terms will vanish and we will be left out with only a dagger a, okay.

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$$U_p^\dagger \hat{a}^\dagger \hat{a}\, U_p = \hat{a}^\dagger \hat{a}$$

$$U_p^\dagger b\, U_p = e^{G/\Omega_m a^\dagger a (b - b^\dagger)}\, \hat{b}\, e^{G/\Omega_m a^\dagger a (b^\dagger - b)}$$

$$= \hat{b} + \frac{G}{\Omega_m} \hat{a}^\dagger \hat{a}$$

$$U_p^\dagger b^\dagger U_p = \hat{b}^\dagger + \frac{G}{\Omega_m} \hat{a}^\dagger \hat{a}$$

So what we get is that U p dagger a dagger a U p is simply a dagger a. In the similar fashion, we can work out terms like say U p dagger b U p which is e to the power G by omega m a dagger a b minus b dagger b e to the power G by omega m a dagger a. I have here b dagger minus b. If I work it out, it is very straightforward to work it out. If you do it you will get b plus G by omega m a dagger a, okay.

And also you can get U p dagger b dagger U p and this will give you b dagger plus G by omega m a dagger a. Now putting all these results in the Hamiltonian here okay, we will get our transform Hamiltonian as this.

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$$\tilde{H} = \hbar\omega_0 a^\dagger a + \hbar\Omega_m \left(b^\dagger + \frac{G}{\Omega_m} a^\dagger a\right)\left(b + \frac{G}{\Omega_m} a^\dagger a\right)$$

$$- \hbar G a^\dagger a \left(b + \frac{G}{\Omega_m} a^\dagger a\right)$$

$$- \hbar G a^\dagger a \left(b^\dagger + \frac{G}{\Omega_m} a^\dagger a\right)$$

$$= \hbar\omega_0 a^\dagger a + \hbar\Omega_m b^\dagger b - \hbar\frac{G^2}{\Omega_m}\left(\hat{a}^\dagger \hat{a}\right)^2$$

Let me write the full expression then I will simplify it. I will get h cross omega 0 a dagger a plus h cross omega m b dagger plus capital G by omega m a dagger a into b plus G by omega m a dagger a. Then we will have a term like h cross G a dagger a b plus G by omega m a dagger a and I will have minus h cross G a dagger a b dagger plus G by omega m a dagger a.

If I open it up and then do the simplification, then finally I will get h cross omega 0 a dagger a plus h cross omega m b dagger b minus h cross G square by omega m a dagger a whole square, okay. Now it is, from this Hamiltonian it is straightforward to get the expression for the energy Eigenvalue.

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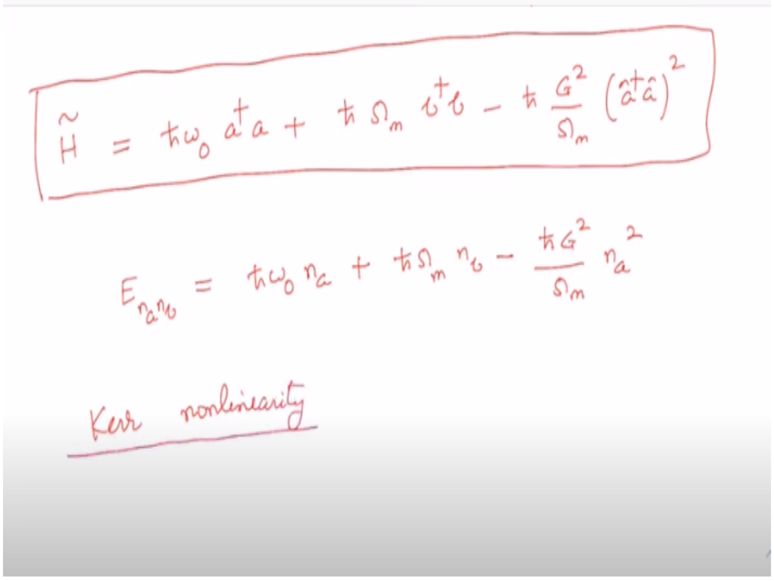


And you will get it immediately the energy Eigenvalue when G is nonzero, capital this coupling is there between the optics and the mechanics you will have h cross omega 0 n a plus h cross omega m n b minus h cross G square by omega m n a square, okay. Now from here we see that the effect of the optomechanical interaction is to make the harmonic optical oscillator anharmonic.

And this form of because the harmonic oscillator is no longer linear harmonic oscillator, it has become nonlinear, and this form of nonlinearity is known as Kerr nonlinearity or Kerr nonlinearity. It is called Kerr nonlinearity. And because of this optomechanical interaction and optomechanical system is known to be inherently nonlinear because of this.

And by the way maybe you know that a Kerr medium is the one where the optical path length depends on the optical intensity. Let us understand it a little bit more.

From this Hamiltonian, from this Hamiltonian we can get the equation for the optical mode.

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$$\dot{a} = \frac{1}{i\hbar}\left[a, \tilde{H}\right]$$

$$= -i\omega_0 a + i\frac{G^2}{\Omega_m}\left(2a^\dagger a + 1\right)a$$

$$\left[a, \left(a^\dagger a\right)^2\right]$$

$$= \left[a, a^\dagger a\, a^\dagger a\right]$$

And we can write a dot is equal to 1 by i h cross the commutation between a and this Hamiltonian. And if you work it out, it will straightaway you will get it as minus i omega 0 A plus i G square by omega m and you will get 2 a dagger a plus 1 into a. By the way, let me quickly show you how I have arrived at this particular expression. Because I have to work out the commutation relation between a and a dagger a whole square.

So just let me show here a, a dagger a whole square the commutation would be I can write it as a, a dagger a, a dagger a, which can be broken down into two parts.

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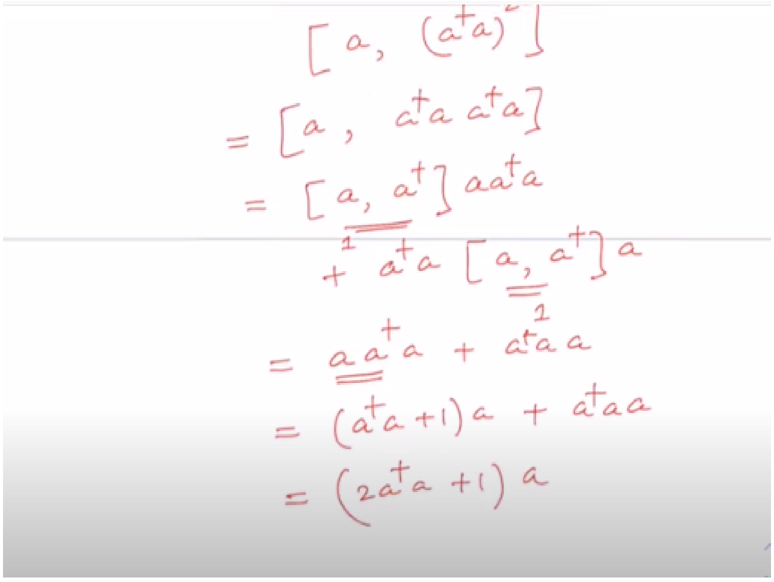


I have say a, a dagger, aa dagger a plus I have a dagger a, a, a dagger a. And because a, a dagger is equal to 1, okay. From here you see I will get a a dagger a plus a dagger aa. Now another thing I can do, I can write a a dagger as a dagger a plus 1 into a plus a dagger aa. From here you see that I will get 2 a dagger a plus 1 into a. That is how I obtained this particular expression.

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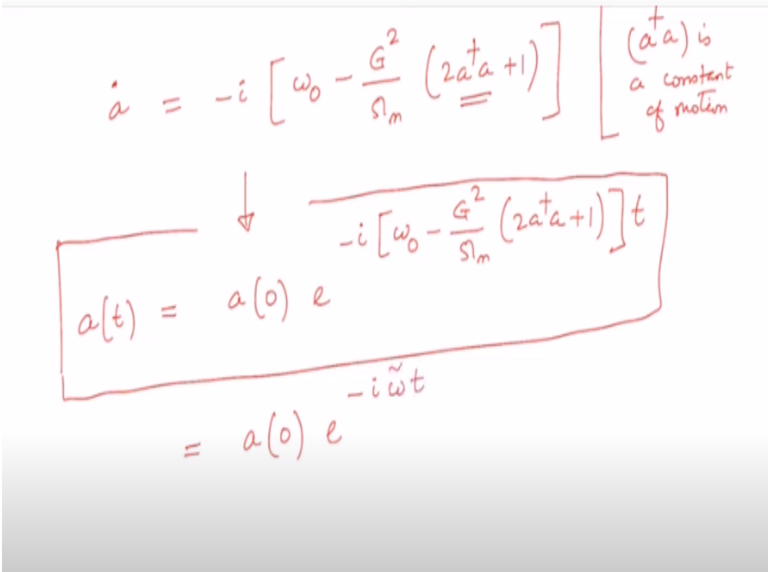


So therefore, what I have here is a dot is equal to I can write it as minus i omega 0 minus G square by omega m twice a dagger a plus 1. This can be easily solved because we know that a dagger a that is the photon number is a constant of motion. So we can easily write a solution for this optical mode and that would be a of t is equal to a of 0 e to the power minus i omega 0 minus G square by omega m twice a dagger a plus 1 into t.

Let me again tell you that a dagger a is a constant of motion or it is a conserved quantity. So that is the reason I can express the solution in this particular form. Or in fact I can write it this way also, a 0 e to the power minus this frequency is slightly modified because of the presence of the coupling. That is say minus i omega tilde t.

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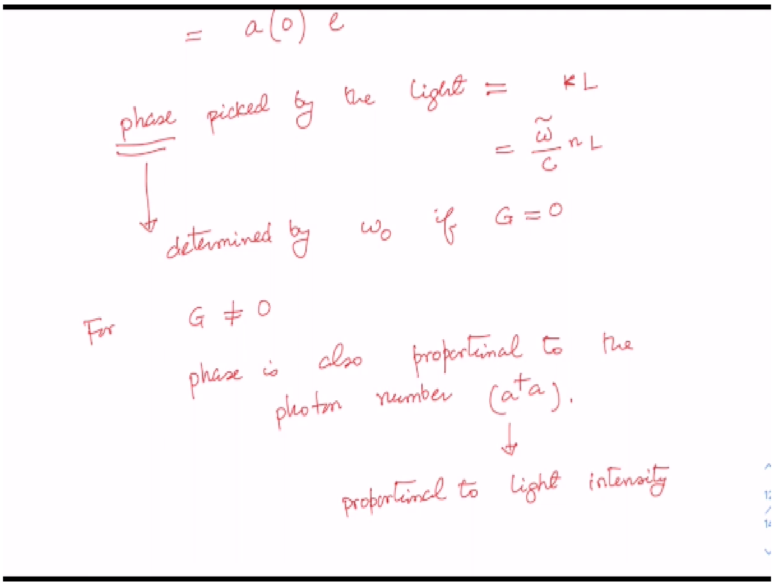


So this implies that the phase picked by the light mode is equal to propagation vector of the light field into L. Do not get confused with kappa, this is simply k, propagation vector which is equal to omega tilde by c into the refractive index into length of the cavity if the refractive index is generally let me say 1. Anyway, so this is what the phase is, phase as seen by the light mode.

And you will see that this phase is determined by the optical frequency omega 0, if capital G is equal to 0. This is determined, this phase is determined by omega 0 optical frequency if G is equal to, okay it is G is equal to 0. Then this term would not be, this particular term would not be there.

So we will have when it is G is equal to 0 for coupling when the coupling is there, phase is also proportional to the photon number a dagger a as you can see from this expression here, right. And that is the, and this photon number is again proportional to light intensity. And as I said earlier, that a Kerr medium is the one where the optical path length depends on optical intensity.

So far we discussed an ideal situation by considering the optical mode and the mechanical oscillator only.

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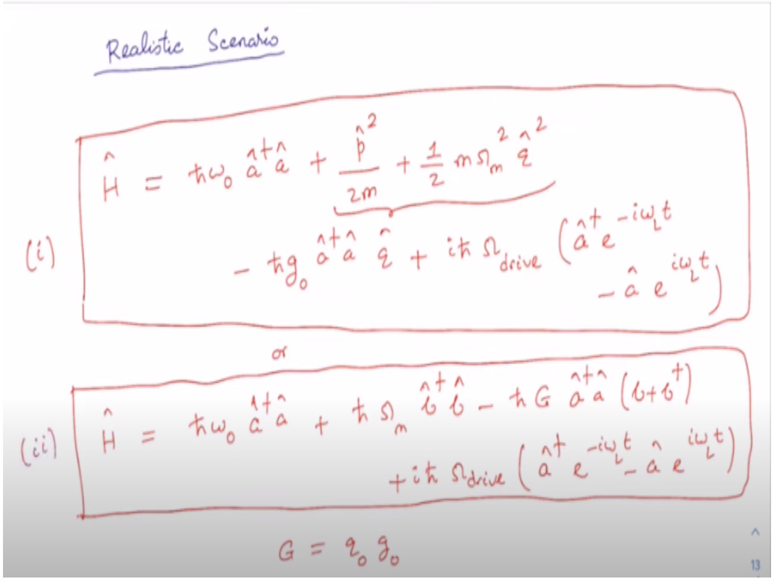


Now let us consider a realistic scenario by considering the external laser drive as well. In this case the quantum Hamiltonian would take this particular form, so we have to add the external laser drive now. So our Hamiltonian is h cross omega 0 a dagger a that takes the optical mode into account.

Then we have this mechanical oscillator p square by twice m plus half m omega m square q square and then optomechanical interaction term is taken into account by this particular term minus h cross G 0 a dagger a q, the coordinate of the mechanical oscillator. Then this term that we are now adding is the laser drive.

And this would be this laser drive has amplitude omega drive and it is a dagger e to the power minus i omega L t. Omega L is the laser frequency and this particular term has to be Hermitian. So we are adding this term as well. So this one we can also write in the in terms of the creation and annihilation operator of the mechanical oscillator. And in that case it would become h cross omega 0 a dagger a plus h cross omega m.

Now we have here b dagger b. So this we are now replacing it by the creation and annihilation operator. Then here we will have minus h cross capital G a dagger a b plus b dagger. By the way, you may recall that this capital G is equal to we have this q 0 into g 0. q 0 is the zero point fluctuation and of course we are having this term also, plus i h cross omega drive a dagger e to the power minus i omega L t minus a e to the power i omega L t.

So these are the two forms of Hamiltonian that we are now going to consider. So let us say this is equation 1 and say let us this is equation number 2.

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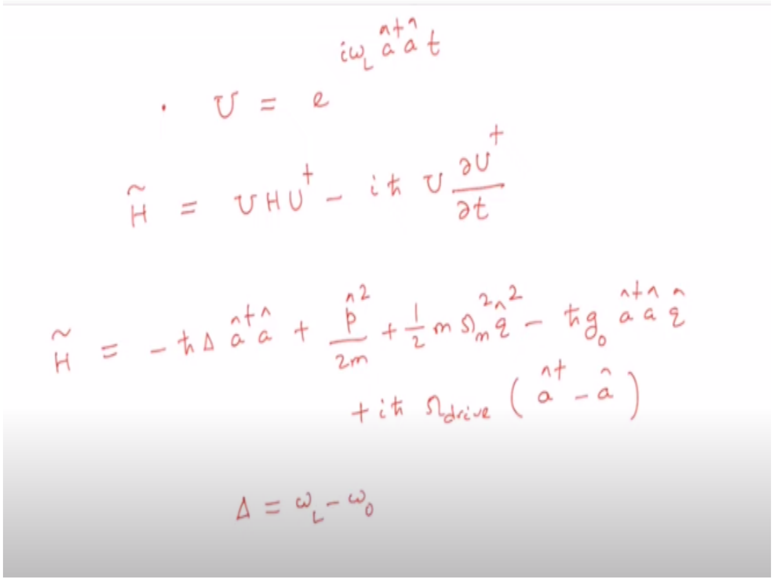


So as we did in the classical regime, we can get rid of the explicit time dependence by going over to a frame of reference rotating with the laser frequency omega L, which amounts to applying a unitary transformation say U is equal to e to the power i omega L a dagger a t. So if we make this unitary transformation, then we will be able to get rid of this explicit time dependence that is there in the external drive term here.

And if we do that, basically our Hamiltonian would get transformed into a new Hamiltonian by this transformation that is H tilde this is the new Hamiltonian. It would be U H U dagger minus i h cross U del U dagger del t. We have done similar things earlier. So if you do that, make this transformation, we will obtain H tilde is equal to minus h cross delta a dagger a plus p square by twice m plus half m omega m square q square.

Then we have minus h cross g 0 a dagger a q. And we have i h cross omega drive into a dagger minus a. So we are now getting rid of this explicit time dependence where this delta is the detuning parameter. That is omega L minus omega 0.

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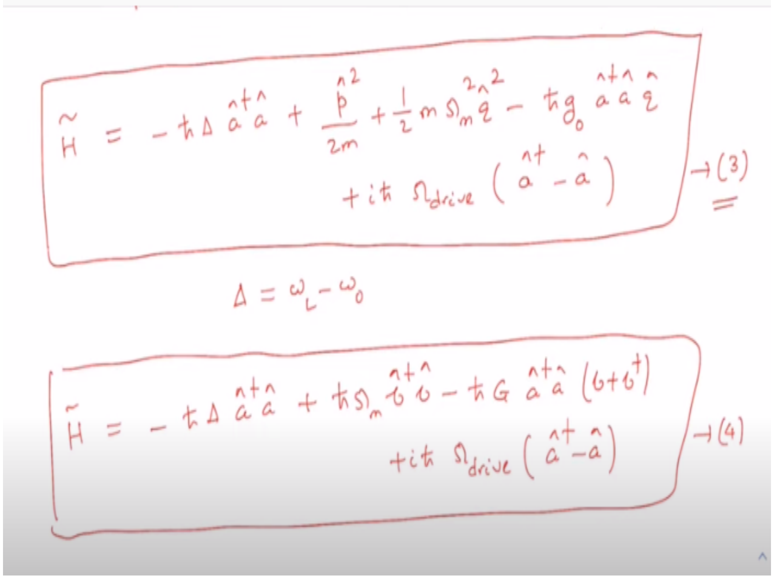


This we can write in terms of the creation and annihilation operator as well. So let me first term it as my equation number say 3. And we will have here in terms of creation and annihilation operator of the mechanical oscillator, I can rewrite this as minus h cross delta a dagger a plus h cross omega m b dagger b minus h cross G a dagger a b plus b dagger plus i h cross omega drive a dagger minus a.

So let me take it as my equation number 4, okay. Now let us consider the effects of surrounding environment on the model of this optomechanical Fabry-Perot cavity. So to do that first let us write the Heisenberg equation for various variables here, various operators. Let me consider this equation 3 and from this equation we can write down the Heisenberg equations.

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$$\dot{\hat{q}} = \frac{1}{i\hbar}\left[\hat{q}, \hat{H}\right]$$

$$\dot{\hat{p}} = \frac{1}{i\hbar}\left[\hat{p}, \hat{H}\right]$$

$$\dot{\hat{a}} = \frac{1}{i\hbar}\left[\hat{a}, \hat{H}\right]$$

$$\dot{\hat{a}} = i\Delta\hat{a} + ig_0\hat{q}\hat{a} + \Omega_{drive}$$

$$\dot{\hat{q}} = \frac{\hat{p}}{m}$$

For example, say we have this q dot, q dot is equal to 1 by i h cross. This q is an operator, q dot position operator for the mechanical oscillator. That will be q h, it is very easy to calculate. Similarly, we have p dot is equal to 1 by i h cross p H. This we have to calculate. And we know how to calculate a dot is equal to for the optical mode 1 by i h cross a H.

In fact it is H tilde here because we are now taking it in the new transform Hamiltonian new rotating form, but frame but, now rather than writing it H tilde we will take it again write it as simply H. So if we calculate it, then we are going to get these equations. For example, for this optical mode we will have a dot is equal to i into delta a plus i g 0 q into a.

Then plus omega drive and q dot is equal to p by m. All these are operators. So all of them are operators.

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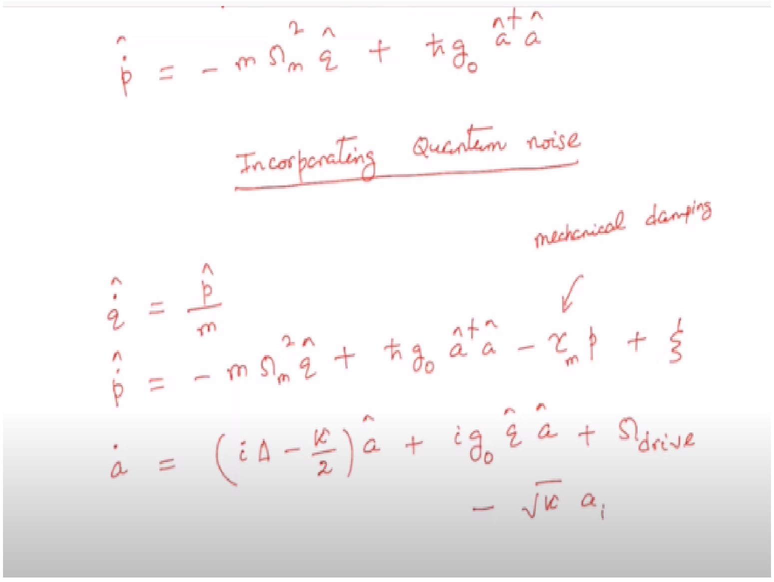


And then we have p dot is equal to minus m omega m square q plus h cross g 0 a dagger a. Now based on our discussion on quantum Langevin noise in previous lectures, we can now write down the following quantum Langevin equation for the cavity optomechanics. Now we have to take into account the quantum noise. So incorporating quantum noise, we can write down these equations, Heisenberg equations as follows.

We have say q dot is equal to p by m. We have p dot is equal to minus m omega m square q plus h cross g 0 a dagger a minus now say the mechanical oscillator has damping. It is represented by some gamma m. So gamma m is the mechanical, the mechanical damping. And then we are having this Langevin noise as well.

And for the optical mode we have a dot is equal to i delta and say it decays at the rate kappa by 2, amplitude decays at the rate kappa by 2. So we have this term here plus i g 0 q into a, okay. All these are again operators and we have this omega drive. And also we have this vacuum fluctuation that is has to be incorporated or has to be added.

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$$a_{in} = \tilde{a}_{in}(t)\, e^{i\omega_L t}$$

The noise terms have zero-mean value:

$$\langle a_{in}(t) \rangle = \langle \tilde{a}_{in}(t)\, e^{i\omega_L t} \rangle = \underbrace{\langle \tilde{a}_{in}(t) \rangle}_{\sim 0} e^{i\omega_L t} = 0$$

$$\langle \xi \rangle = 0$$

$$\langle a_{in}(t)\, a_{in}(t') \rangle = \delta(t-t')$$

So here a in is equal to a in tilde if you look at our last class e to the power i omega L t. And the noise terms has, the noise terms have zero mean value. So we will have say a in expectation value of this noise would be it will be a in tilde t e to the power i omega L t. And we can write it as expectation value of a in e to the power i omega L t. And this is equal to 0.

So therefore, this mean is 0. And similarly, the mean of this Langevin noise is also 0. On the other hand also we know the corresponding time correlations. For example, for this input noise from vacuum fluctuation, the time correlation will be a in t, a in t dash. That is equal to delta t minus t dash. You can refer to the previous class, last class where we discussed all these things.

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The noise terms have

$$\langle a_{in}(t) \rangle = \langle \tilde{a}_{in}(t)\, e^{i\omega_L t} \rangle = \underbrace{\langle \tilde{a}_{in}(t) \rangle}_{\sim 0} e^{i\omega_L t} = 0$$

$$\langle \xi \rangle = 0$$

$$\langle a_{in}(t)\, a_{in}(t') \rangle = \delta(t-t')$$

Also, $\langle a_{in}(\omega)\, a_{in}^{\dagger}(\omega') \rangle = 2\pi\, \delta(\omega+\omega$

Also we have the correlation function in the frequency domain as well. We have worked it out also there a in omega a in dagger omega dash is equal to 2 pi delta omega plus omega dash. Let me stop here for today. In this lecture, we have discussed the quantum mechanical Hamiltonian for the cavity optomechanical system. And we saw why a quantum optomechanical system is inherently nonlinear.

Also we have worked out the quantum Langevin equation in the context of a cavity optomechanical system. In the next lecture, we are going to discuss the linearized quantum optomechanics and also we will find out the quantum limit for the ground state cooling of an optomechanical system. So see you in the next class. Thank you.