Select/Special Topics in Classical Mechanics

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Module No. # 03

Lecture No. # 12

Polar Coordinates (ii)

Let us resume our discussion here on the coordinate systems. We now know what a vector is; how to describe it in a Cartesian coordinate system; how its components will transform when you rotate a coordinate system. Now, we will think about what is the most appropriate coordinate system to describe a vector. Because depending on the symmetry in which objects are laid out around us, we might want to choose a different coordinate system.

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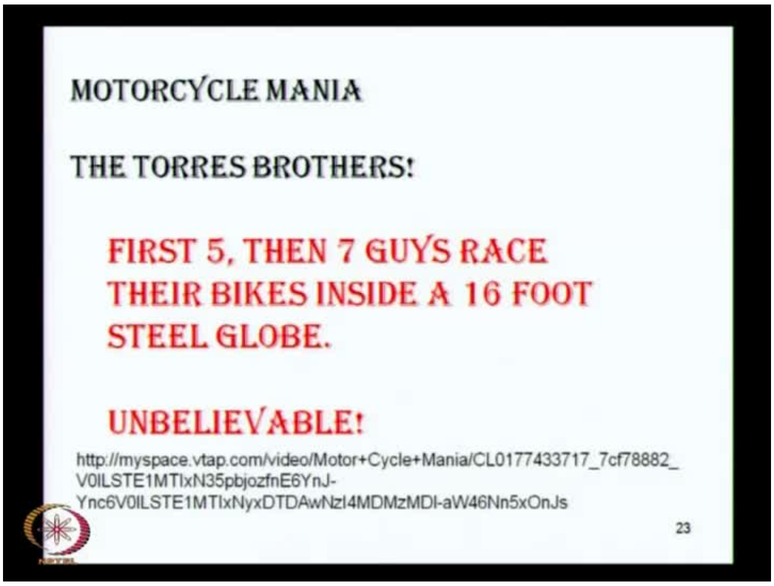

Before we do that, let me show you a movie called Motorcycle Mania. This is something like what you might have seen if you have seen the movie. The movie lovers amongst you would have seen Dhoom. Have you? Is there anybody here who have seen Dhoom - Dhoom 1, Dhoom 2? Now, you are going to see Dhoom 3 and this is by Torres brothers. This is available on the internet.

You first have five motorcyclists. They are all from this family; the Torres family and this is an unbelievable movie. I think before we get into hard core mathematics, let us have a little bit of fun. This you can see on the internet of course. This is the web link for it. You do not have to write down the internet link. If you just Google motorcycle mania with Torres brothers, you will get the link anyway. Google knows just about everything. So, you can get, but you can see it here and this is the one (Refer Slide Time: 01:48).

There are five of them inside this steel globe. Look at the (( )).

If you think this was it, you are mistaken. There is more (Refer Slide Time: 03:08).

What we saw in this movie was fun, but the important point is that if you have motion on the surface of a globe, then it is always equidistant from the center of the globe. If you wanted to keep track of any object, which is in motion on the surface of the globe, in the Cartesian coordinate system, you would need to keep track of three parameters: x, y and z. However, the square root of x square plus y square plus z square or r square is always constant. So, there is one constraint. You really do not need to keep track of all the three parameters. So, can we choose the coordinate system in which only two parameters are sufficient and the third is held constant?

Instead of a Cartesian coordinate system, a spherical polar coordinate system will be much more compact and adequate to describe motion in this. So, depending on the symmetry in which objects are laid out around us, you choose a different coordinate system.

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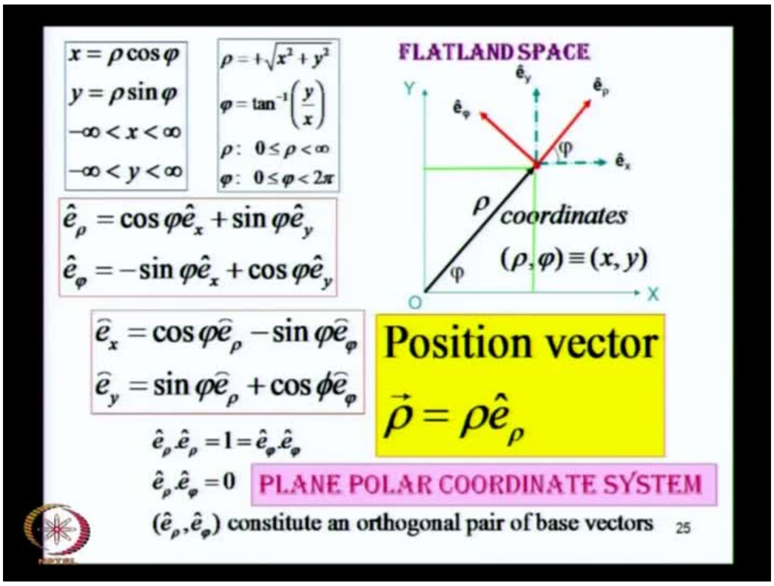

In the spirit of the flat land that we talked about earlier, we begin our discussion first in a flat world in which there are 2 degrees of freedom. You look at the position of an object; it is this red dot on the screen. You can describe the position of this red dot in terms of two parameters: It is x coordinate, which is this projection of this vector along the x-axis and the y coordinate, which is the corresponding projection on the y-axis. So, x and y give you complete information about where the subject is located. However, you can also choose two other parameters, which is the distance from the origin labeled by rho and the angle of this line with the x-axis. So, you need a reference direction, which is the direction of the x-axis so that you have a reference direction with reference to which the angles are measured.

The angular departure from the x-axis, which is called the azimuthal angle; this is the phi angle that you see in this picture. These two parameters: rho and phi also describe the exact location of this object. You can do it either way: by specifying either x and y or rho and phi. When you make use of rho and phi, you use coordinate system, which is called the plane polar coordinate system; plane polar because you are dealing with a plane; with a flat land. Phi is an angular parameter and it measures the departure of this line (Refer Slide Time: 07:38) from the x-axis, which is the azimuthal angle.

Now, let us look at it in another coordinate system because we have agreed that any coordinate system in which you have got a set of base vectors or bases, which is linearly independent will work. So, instead of choosing e x and e y as my base vectors, which are the base vectors respectively along the x-axis and the y-axis of the Cartesian coordinate system, I can choose any other pair of linearly independent vectors. So, instead of choosing this, I can choose this (Refer Slide Time: 08:28); it does not matter. I can choose this red vector (Refer Slide Time: 08:35) as one base vector and this one, which is linearly independent and orthogonal to it as another base vector. I can describe the location of any object in terms of this pair of vectors which is linearly independent and it also is orthogonal. So, it is a very convenient bases set and my bases can be this vector and this vector instead of the Cartesian unit vectors: e x and e y.

Now, how have I selected these red vectors? (Refer Slide Time: 09:09). This first vector is a direct vector of unit magnitude. So, it is a unit vector, which will belong to my bases set. This is along the direction of the position vector of this red object from the origin. So, from the origin, I construct the position vector and in the same direction, I have a unit vector, which is e rho. Then, I take the other vector, which is orthogonal to it. So, it is at 90 degrees, but it could be 90 degrees pointing this way or pointing in the opposite way. So, there are two directions that I can think of which are orthogonal to this first red vector (Refer Slide Time: 09:47). I choose the direction in which the azimuthal angle of phi would increase. So, azimuthal phi will obviously increase in this direction rather than in this direction. So, I choose the other vector so that there is no ambiguity in how I choose these two unit vectors.

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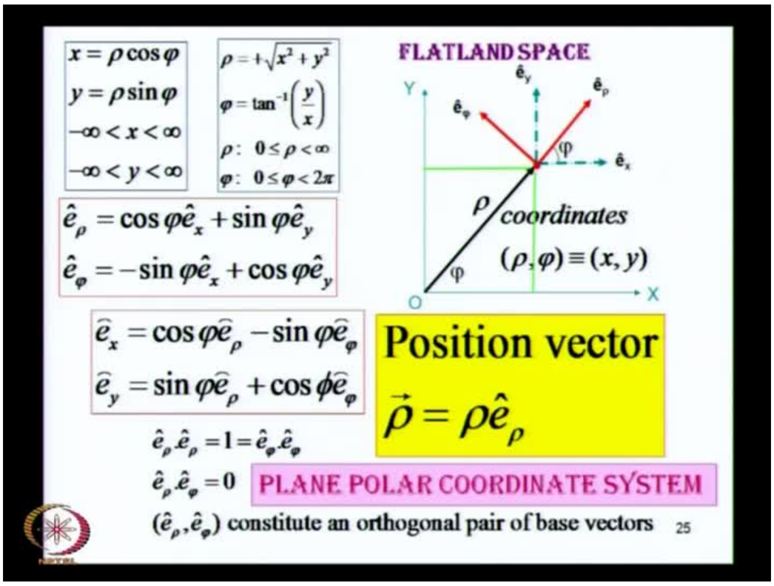

This first vector is called as e rho because it is along the radius vector or the position vector. The other one in which the azimuthal angle phi is increasing, is called as the unit vector, e phi. I have these two vectors, which are mutually orthogonal; they are of unit magnitude. So, they constitute an orthogonal pair of unit vectors and I can use that as the bases set.

The interesting thing is that if I want to have this red spot this object over here (Refer Slide Time: 10:45); However, instead of this position, if I want to have it over here; somewhere here (Refer Slide Time: 10:48) under the cursor, then the direction of e rho will obviously not be this. It will be along a position vector from here to here and then extended further here. Then, the direction of e phi will be from this point; in the direction in which phi is increasing. So, it will be in this direction (Refer Slide Time: 11:09). So, e rho and e phi are not constant vectors; they will change from point to point.

Whereas, e x and e y are constant vectors. They are always along the Cartesian X and Y axis respectively. So, e phi always points the direction in which the azimuthal angle increases from wherever that point is. These are the transformations (Refer Slide Time: 11:32) between the Cartesian coordinates x and y and the polar coordinates, which are called the plane polar coordinates. So, these are the equations of coordinate transformation. This is a range of the corresponding parameters: x and y; both go from minus infinity to plus infinity. However, rho and phi; rho, which is root of x square plus y square and phi is of tan inverse of y over x as you can see easily from this geometry. Rho can change from 0 to infinity; rho equal to 0 will give you the point of origin itself. Notice that the range of phi is from 0 to 2 pi, but the equality phi equal to 0 is included; the equality phi equal to 2 pi is excluded because phi equal to 2 pi would give you the same points. So, you have to avoid duplication of the points.

You can write the transformations between e x and e y because now you can think of e rho as some vector and you can always write it as linear of superposition of the base vectors. So, you write e rho as a superposition of e x and e y and these will come from the corresponding cosines (Refer Slide Time: 12:45). Do the same with e phi. These are the equations of transformation between e x and e y. From these, you can construct e rho and e phi and you can carry out the inverse transformations. From e rho and e phi you can get e x and e y (Refer Slide Time: 13:04). So, you can carry out these transformations very easily. What you have is known as the plane polar coordinate system with the base vectors and the transformation relations written exactly.

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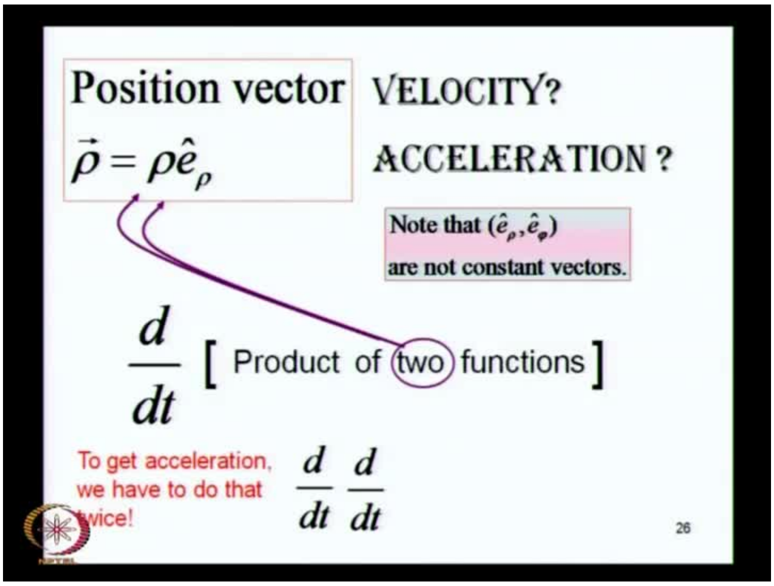

In this, the position vector is rho times e rho; rho is the distance. You have both the magnitude as well as the direction. Remember that however, e rho and e phi are not constant vectors; they will change from point to point. Therefore, you want to know what is the law, which governs this change? How do you determine that changes in these quantities; changes with respect to what?

Change is always with respect to something. Change is either with respect to time or with respect to an angle or with respect to a distance; or there is some independent parameter. You look for a change with respect to these independent parameters.

In Cartesian coordinate system, no matter which point in space you are talking about, the unit vectors e x and e y do not change. In other words, the derivative of e x with respect to x is always 0 because the unit vector e x is the constant with respect to x; it is also a constant with respect to y.

What about the derivative of e rho and e phi? Do they change with rho and phi? Will e rho change with rho? e rho will not change with rho because all along that line, the direction is always away from the center along the radial line and of unit magnitude. So, you know that e rho does not change with rho. You can say that del e rho by del rho, which is the derivative of e rho with respect to rho would vanish.

Now, I am using partial derivatives because there are two parameters to talk about: rho and phi. So, whenever we deal with these quantities, you have to keep track of how they vary with position and time. It is the time derivative of the position vector, which will give you the velocity. It is the time derivative of the velocity, which will give you the acceleration. It is these quantities that you have to work with when you set up an equation of motion. So, you have to take the time derivative of the position vector to get the velocity. When you take that time derivative of the position vector, but the position vector is now expressed in terms of the polar coordinate system rather than the Cartesian coordinate system. In which case, you must treat this as the product of two functions: rho is a scalar; e rho is a vector. You must ask - Does this change with time? You should also ask - If this changes with time?

In principle, if the object is moving; if this is the origin (Refer Slide Time: 16:16) and it moves only along the radial line in one direction, then e rho is not going to change. However, if it is having some haphazard motion, then e rho will change from time to time and you must take the time derivative of e rho with respect to time. In other words, you will have to take the differential of e rho with respect to time. So, you have to keep track of these things. To get acceleration, you have to do this process twice because it is the second time derivative.

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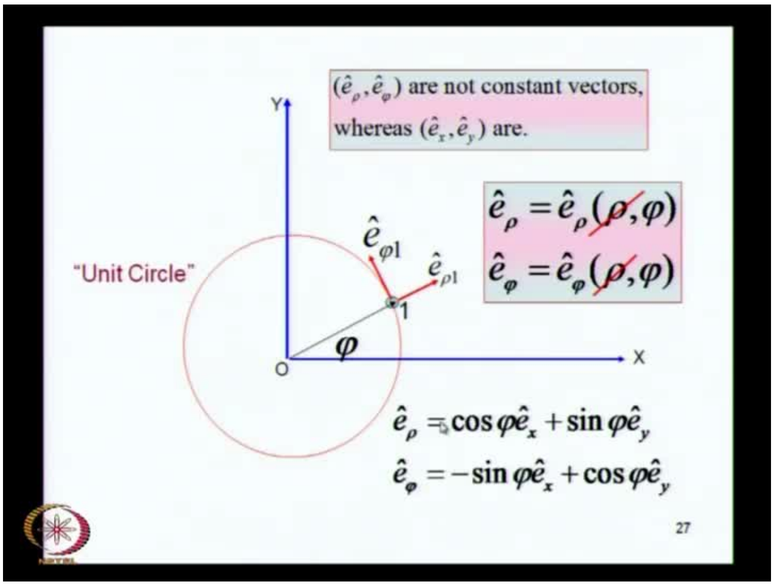

We know that e rho will not change with rho; e phi, which is always orthogonal to it in the direction in which the azimuthal angle is increasing, will also not change with rho. So, you can write two arguments for the sake of completeness, but actually neither e rho nor e phi change with rho. However, both change with respect to phi.

You construct what is called as the unit circle; you have the azimuthal angle and you have a point. Let us say this is at position 1 (Refer Slide Time: 17:25). At this position, the unit vector e rho is this. So, I subscript it with one corresponding to this point 1. The corresponding azimuthal unit vector is e phi with a subscript 1.

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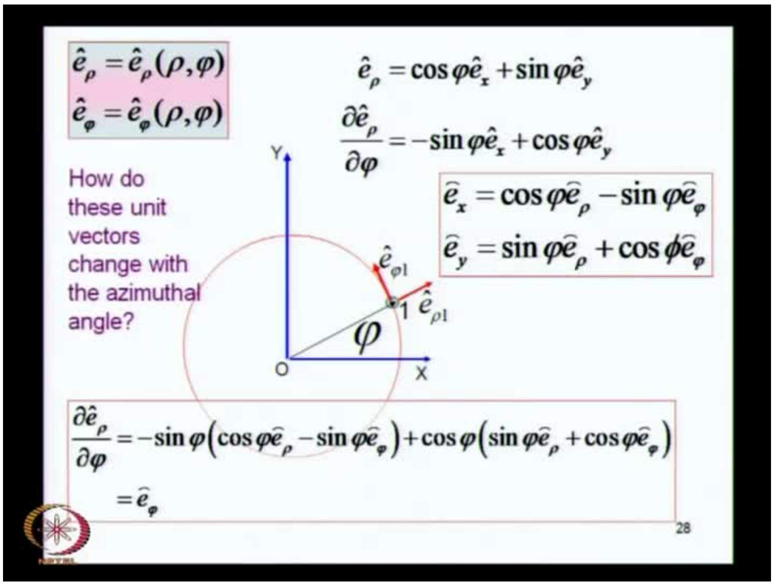

Now, this is the relation (Refer Slide Time: 17:41) that we have learned earlier. In general, these unit vectors will change from point to point. Each point must have these two parameters, but the dependence on rho of these unit vectors actually disappears. However, they do change with the azimuthal angle. So, let us ask - In what way they will change with the azimuthal angle? We need to find how e rho will change with phi. You have to take the derivative with respect to phi. This is the rate at which the unit vector e rho changes with phi.

You take the derivative of the right-hand side with respect to phi. The derivative of cosine phi will give you minus sine phi; the derivative of sine phi will give you cosine phi. So far so good, but this is some kind of a hybrid relation because some of its quantities are polar like sine phi, cosine phi, e rho phi, del e rho by del phi; all of these are polar parameters.

The reason this equation is a hybrid equation is because this is described (Refer Slide Time: 18:59) in terms of the unit vectors which are the Cartesian unit vectors rather than the polar unit vectors. So, you must transform this completely to the polar system. So, you can write e x and e y in terms of e rho and e phi; we have already done that through the inverse relations. Then, substitute for e x and by these quantities on the right-hand side. This is completely polar now. The right-hand side of these equations (Refer Slide

Time: 19:29) are completely polar and then the result will be expressed completely in polar parameters.

Now, you have del e rho del phi, which is this (Refer Slide Time: 19:41) minus sine phi, which comes over here; times e x. e x written in polar coordinate system is given by the right-hand side of this first equation here, which comes over here. Now, this has got terms in e rho and e phi over here (Refer Slide Time: 19:57) and also terms in e rho and e phi over here. So, if you combine all of them, you find that this minus sine phi cosine phi e rho cancels the plus sine phi cosine phi e rho. The remaining components: minus sine phi into minus sine phi, which is sine square phi and the product of these two cosine gives you cosine square phi. Then, the sine square phi and cosine square phi gives you the unity when added up.

This result (Refer Slide Time: 20:24) is that the rate at which the unit vector e rho changes with phi is equal to the unit vector e phi itself. It is a very simple derivation. Notice that the derivative of the unit vector with respect to phi is a vector; it is a direction in space. The derivative of e rho with respect to phi is along e phi because the derivative measures the change. A change is always orthogonal; it is always perpendicular to the quantity in which you are seeking the change; otherwise, it would not be a change at all. So, any change is always orthogonal to the quantity in which you are seeking this change. So, del e rho by del phi is equal to e phi. It is orthogonal to e rho.

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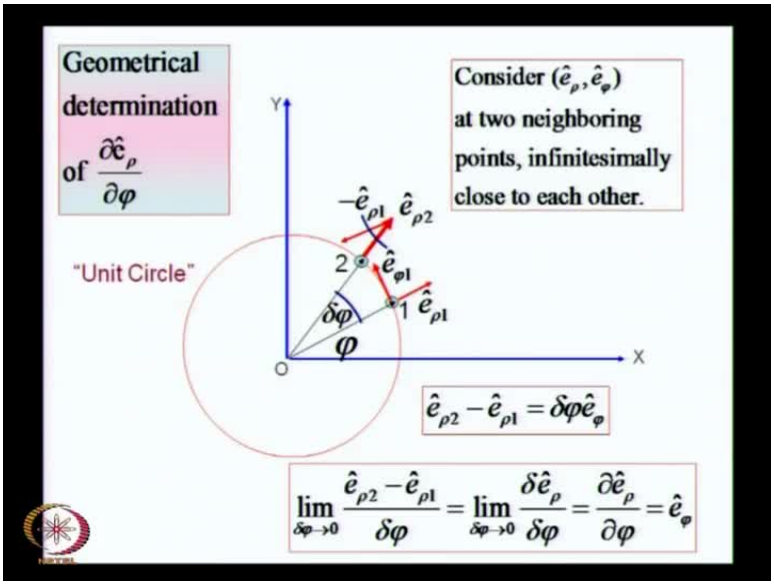

You can also get a geometrical determination rather than taking the derivatives of the sine and cosine functions. This is also a good exercise to do, which is by studying the geometry, you see that you have got a point 1. You construct the corresponding unit polar vectors e rho 1 and e phi 1 and move this point to adjacent point on the unit circle.

What is the change? The change is because of the change in phi. That is the principle cause. That is the independent degree of freedom with respect to which you are seeking a change. Now, at point 2, you construct the polar unit vectors. So, this is e rho 2; e phi 2 will be orthogonal to e rho 2 and in the direction in which phi is increasing. However, e rho 1 is this vector (Refer Slide Time: 22:23) and what I have done is to subtract from e rho 2 the vector e rho 1. So, this is e rho 1. This is minus e rho 1. It is parallel to it and directed oppositely. So, this difference gives me the change in the unit vector at these two adjacent points.

Whenever you take the derivative, what you do? Look at the difference in the value of the function divided by the difference in the independent parameter and take the limit that the denominator goes to 0. It is d y by d x, which is delta y by delta x in the limit delta x going to 0. So, that is how you define a derivative. You have this difference in the two unit vectors. That is from this geometry, you can immediately see that this (Refer Slide Time: 23:15) when you subtract from this vector, you will get a vector, which is in this direction. It will be in the direction of e phi and of magnitude delta phi because you are constructing this on a unit circle. So, this difference is equal to delta phi e phi.

Now, if you divide this quantity by delta phi (Refer Slide Time: 23:36) and take the limit delta phi going to 0, you get the corresponding derivative, which is a partial derivative of e rho with respect to phi. This derivative is equal to e phi as we have already seen in the previous slide. So, get exactly the same result as you certainly should.

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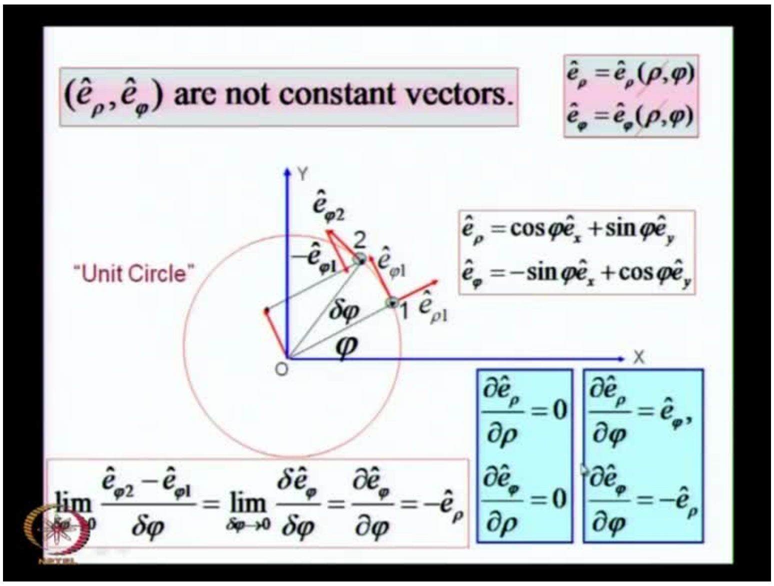

Then, you can do this for the other vectors for the difference in e phi at position 2 subtract from it e phi at position 1. Let us look at it again. First, you construct e phi at position 2 and subtract from it the e phi at position 1. If you see this difference vector, in the limit, the delta phi goes to 0. You can already see that it will be directed toward the origin of the coordinate system. So, it will be along minus e rho. As you would expect, a change in e phi will be along e rho. The change in the unit vector will always be orthogonal to the corresponding unit vector. However, it will be directed toward the center rather than away as you can see from this geometry. Indeed you get that result that the change in this e phi divided by delta phi in the limit delta phi going to 0 is equal to minus 0. So, you can get these derivatives of unit vectors.

One must get used to this idea because very often we do algebra; we make use of the Cartesian coordinate system in which the unit vectors are always held constant. It is important to keep track of the fact that we use coordinate systems very often depending on the symmetry in which objects are laid out around us. If we were to use the polar coordinate system, then the unit vectors will change from one point to another. If they are going to change, they will do so at a certain rate. That rate is given by the derivative of that unit vector with respect to the independent degrees of freedom. These independent degrees of freedom are rho and phi and we now have this. We can consolidate these results that the derivative of the unit vectors e rho and e phi with respect to rho are both 0. However, I am using partial derivatives because we do so while holding the azimuthal angle as a constant. The partial derivatives with respect to the azimuthal angle however do change; del e rho by del phi is equal to e phi and del e phi by del phi is equal to minus e rho. So, these are our consolidated results.

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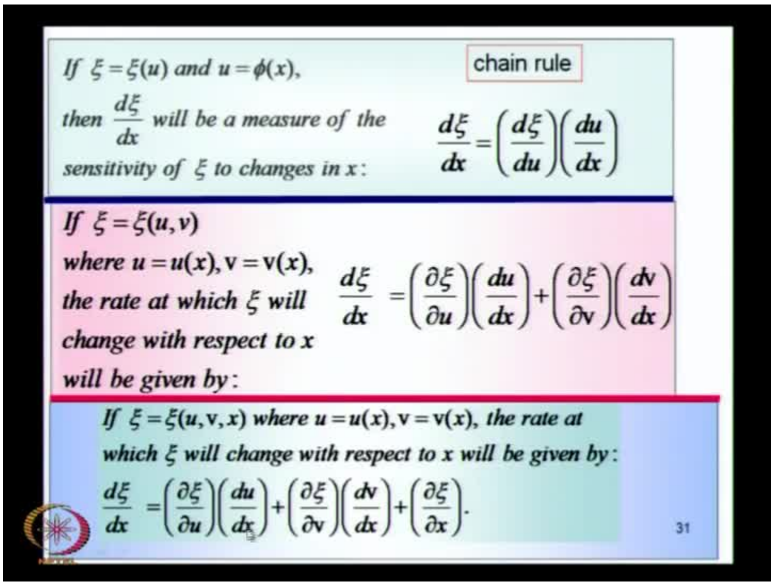

Now, we often have to make use of what is called as a chain rule. You would have used this in elementary calculus. That is, if you have a functions xi of u in which u itself is the function of x, then the change in xi with respect to x. This is because of change in xi with respect to u influenced by the change in u with respect to x. So, this is the chain rule that the rate at which xi changes with respect to x is given by the product of the rate at which the xi changes with respect to u with the rate at which u changes with respect to x.

Now, you may have a more complex dependence on x because xi may depend on u as well as on another parameter v; both u and v may change with respect to x. If that were to happen, then the rate of change of xi with respect to x will come from both the contributions. Why xi changes with respect to u and then what rate does it do so? That rate will be given by the partial derivative of xi with respect to u.

Now, you must use a partial derivative because there is a dependence that xi have on another parameter, namely v. So, this is the partial derivative of xi (Refer Slide Time: 27:45) with respect to u multiplied by the derivative of u with respect to x. Then, there may be a derivative of xi with respect to v multiplied by the rate at which v itself changes with x. So, this is a little more complex than the previous case.

However, who stops you from increasing the complexity? You can have a dependence of xi on x not because of its dependence on u as we had in the previous case. Yes, you may have that. In addition to that, you may have a dependence of xi on x through its dependence at v, but in addition to that, you may have a direct dependence on x; why not? If that is to happen, then the rate at which xi will change with respect to x will be determined by these three terms, (Refer Slide Time: 28:40) which must be summed out.

You must take the product of how xi changes with respect to u influenced by the rate at which u changes with x; how xi changes with respect to v influenced by how v changes with respect to x. So, that is the derivative of v with respect to x. Then, the derivative of xi with respect to x because of its direct dependence. This is sometimes called as the explicit dependence of xi on x, whereas over here, (Refer Slide Time: 29:13) the dependence of xi on x is called as the implicit dependence on x. So, this is the difference between an explicit dependence and an implicit dependence. When a function depends on a parameter directly, it is an explicit dependence. When it depends on some other quantity, which in turn depends on another independent parameter, then it is called an implicit dependence.

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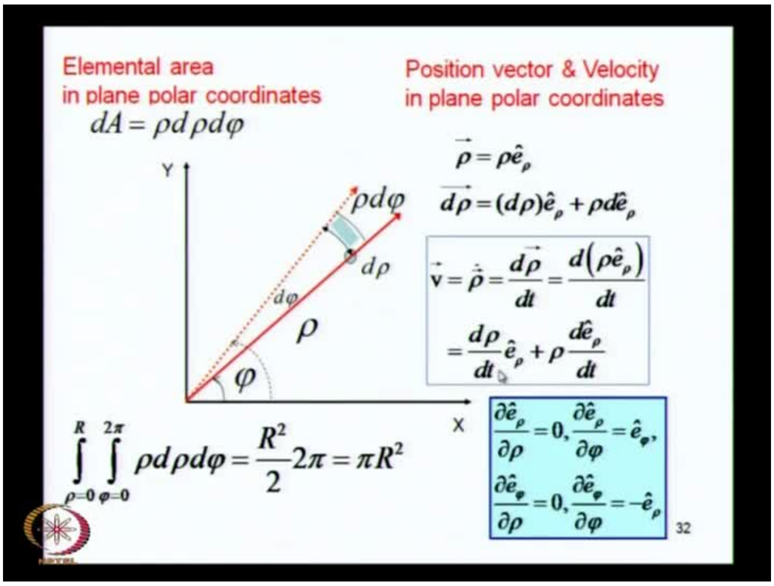

You have to worry about some of these things. You can now do all kinds of geometrical… You can construct geometrical objects like area. Here if you have to represent an area in the Cartesian coordinate system, the thing to do would be to construct elemental areas. These will be like rectangles in the flat space. Then, you can add up these rectangles to get the net area; integrate this.

Now, similarly, you can construct these elemental areas in the polar coordinate systems. These will not be rectangles, but these will be made up of a region of this flat land, which is sandwiched between an increment in the polar distance, rho. So, rho changes from rho to rho plus d rho through this little segment. The angle changes through phi and this arc (Refer Slide Time: 30:48) will have a length of rho d phi. So, this elemental area will be rho d rho d phi in the limit that both the increments in rho and the increments in the azimuthal angle shrink to 0. So, in the limit, you will get the elemental area.

You can always integrate over rho and d phi depending on what kind of an object you are dealing with. So, if you were to do it for a circle, you integrate this elemental area with rho going from 0 to the radius of the circle azimuthal angle phi going from 0 to 2 pi and you get the area of the circle. So, you can do this algebra in any coordinate system. You can do it in Cartesian coordinate system or polar coordinate system. You will need to learn to describe the position vectors, the velocity and accelerations in plane polar coordinate system.

If you want to determine the velocity, then the velocity is dr by dt. So, it is the rate of change of the position vector with respect to time. The position vector itself is this vector rho (Refer Slide Time: 31:57). So, you need to construct this d rho and divide it by dt, rather you want to take delta rho by delta t in the limit delta t going to 0.

The differential increment in rho will come because of the increment in rho, which is d rho times e rho plus rho times a change in e rho. This is because in increment, the object need not be always going along this line (Refer Slide Time: 32:26). If it were to go along some other line, then the corresponding azimuthal angle will be changing. So, the e rho unit vector will change from one point to another, if the azimuthal angle were to change. So, d rho will be equal to the sum of these two terms: (Refer Slide Time: 32:42) one of which is coming from the differential change in the magnitude of this and the other from its direction because e rho itself may change. We know how to determine this because we know the rate at which these unit vectors change. This is the differential increment in this unit vector that you are interested in finding. This is no problem because we already know the rate at which this unit vector changes with rho and phi; with rho it does not change, but with respect to phi it does.

You plug it in (Refer Slide Time: 33:20). Then, you can get the expression for the velocity, which is delta rho by delta t in the limit delta t going to 0. You get it by taking the derivative of this rho times e rho from these two terms. From the first term, you get d rho by dt times e rho and from the second term, you get the rate at which this unit vector changes with time.

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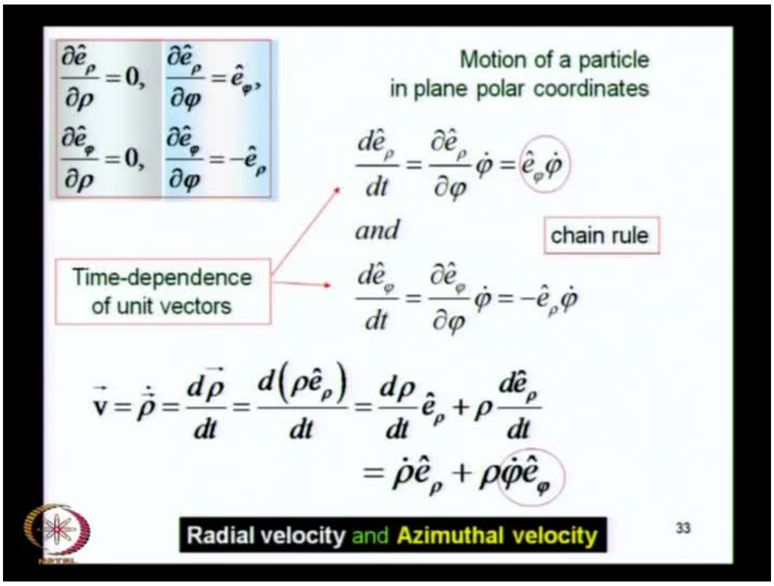

How do you get the rate at which this unit vector changes with time? We have found how it changes with respect to phi. This multiplied by the rate at which the angle phi itself changes with time d phi by dt. That is what I denote by a dot. When I put a dot on the variable, I am referring to a time derivative. So, phi dot is d phi by dt. The rate at which this unit vector e rho changes with time is del e rho; the rate at which it changes with phi because it is not going to change with rho; otherwise, even that would have contributed; that derivative is 0; times the rate at which the azimuthal angle changes with time. So, e phi times; we know this derivative that del e rho by del phi; we have already found out that this del e rho by del phi is equal to e phi. So, you have got e phi times phi dot, which is d phi by dt.

Likewise, you need to get the change in the derivative; you need to look at the time derivative of the unit vector e phi. This will be the rate at which e phi changes with phi;

it is not going to change with respect to rho, but with respect to phi. So, you take the derivative of e phi with respect to phi; scale it by the rate at which the azimuthal angle itself changes with respect to time, which is d phi by dt, which I write as phi dot. This rate (Refer Slide Time: 35:12) is equal to minus e rho. As we have determined earlier, del e phi by del phi is minus e rho and phi dot is over here. So, these are the corresponding rates. With this, you can now write the velocity completely in polar coordinates because you have got everything that you need.

Velocity is given by this expression (Refer Slide Time: 35:32) as we found from previous screen. This d e rho by dt is e phi times phi dot. So, we have to substitute this term over here (Refer Slide Time: 35:40). You can see that you have got a component along e rho and another component along e phi. So, this is called a radial component and this is called a azimuthal component. So, the velocity in polar coordinates will have two components: the radial component as well as the azimuthal component. You continue to express it as a linear superposition of two base vectors, which give you the linearly independent base vectors. These happen to be constants neither for all points of the space nor at all times, but so what? They are linearly independent and therefore, they give you complete pair of bases.

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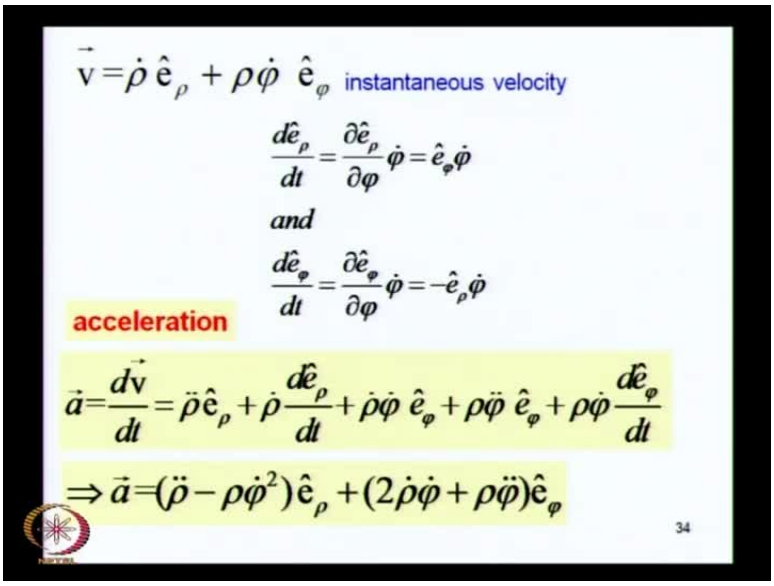

This is the instantaneous velocity. You can get acceleration by taking the second derivative. When you take the derivative of the velocity, you have to take the derivative of each of these terms; you have to look at this as the product of two functions rho dot and e rho. Here, this is a product of three functions rho, phi dot and e phi because e phi is not a constant. So, you must take the product, use the product rule while taking the differential. If you put all the terms together, combine the term in e rho and e phi and stack them together. You can do this algebra yourself. So, you do not have write this down, but, do have to work it out so that you develop confidence and comfort in using this.

You can see that the acceleration will also have a radial component as well as an azimuthal component. What will contribute to these components (Refer Slide Time: 37:33) are the changes in rho as well as changes in phi. Because you are looking at the second derivative, the second derivative will come from this term (Refer Slide Time: 37:46), but also from the first derivative of phi.

Mind you: The ultimate dimensions of each of these terms must be the same. This is one thing that you should always do to ensure that you have done it right. That is to check the dimensions of the physical quantity because if you make some silly mistake, which all of us are very capable of, you might write just 2 rho phi dot over here and forget about rho dot. If you did that as a careless mistake, you should immediately be able to spot it yourself before you take any step further. This is because the dimension of rho dot phi dot will not be the same as the dimensions of rho phi dot. Rho dot will have a dimension t inverse, which is different from that of rho itself. The dimension of every term will be l t to minus 2. This is the acceleration. So, for velocity, it is l t to minus 1 and for acceleration, it will be l t to minus 2. So, always develop a habit that the moment you look at a term, you pick up its dimensions in your mind and make sure that you have not made any careless mistakes.

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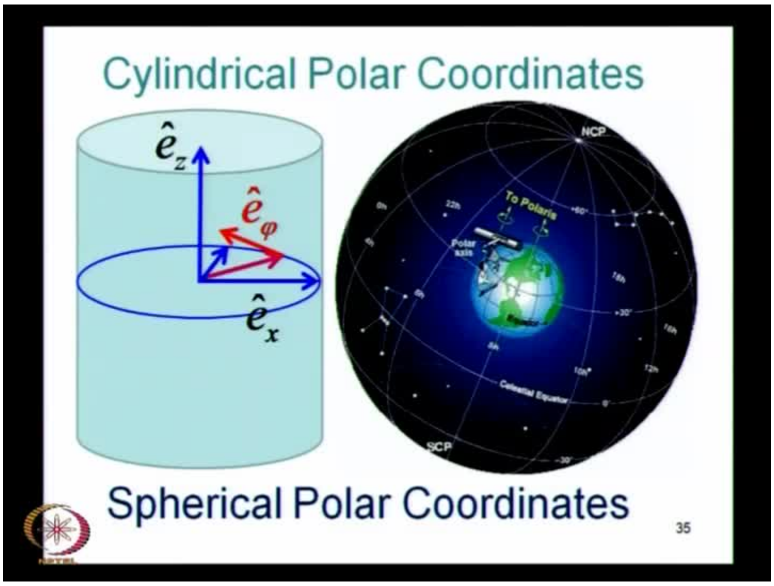

Now, we will go from the flat land to the three-dimensional world. We had e rho and e phi in the flat world. Now, you add the third axis, which is perpendicular to these two; that is, e z. That is what generates the so called cylindrical polar coordinates system. In the cylindrical polar coordinates system, the base vectors are e rho, e phi and e z. This forms a right-handed triangle. Just like e x cross e y gives you e z, you have e rho cross e phi, which gives you e z always. Then, you have a corresponding… You know that if you change these in succession, you will get other two cross products giving you the other two base unit vectors for the coordinate system.

This is the cylindrical polar coordinate system, which you can develop very easily by simply adding a third direction. In this third direction, the unit vector along the z-axis is a constant vector. So, not every unit vector must change from point to point. In this cylindrical polar coordinates, e rho and e phi change from point to point, but even e rho does not change if you go just along the radial line.

You can also have a spherical polar coordinate system because obviously, the symmetry of a cylinder and the symmetry of a sphere are different. So, depending on what kind of symmetry you are looking at; depending on how objects are laid out around you as I keep saying; depending on the symmetry that is involved in the motion that you are observing or analyzing, you can always choose an appropriate coordinate system so that you can minimize the degrees of freedom that you want to keep explicit track of. This is because, one over the other like you saw in Dhoom 3, the distance from the origin of a point on the globe is always held constant. So, you do not want to worry about it all the time. It is there; not that it is not there, but you do not have to worry about it.

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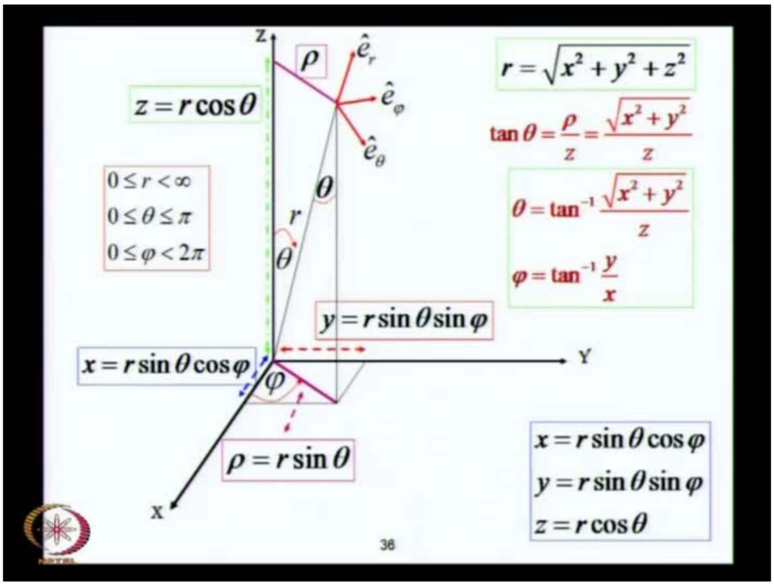


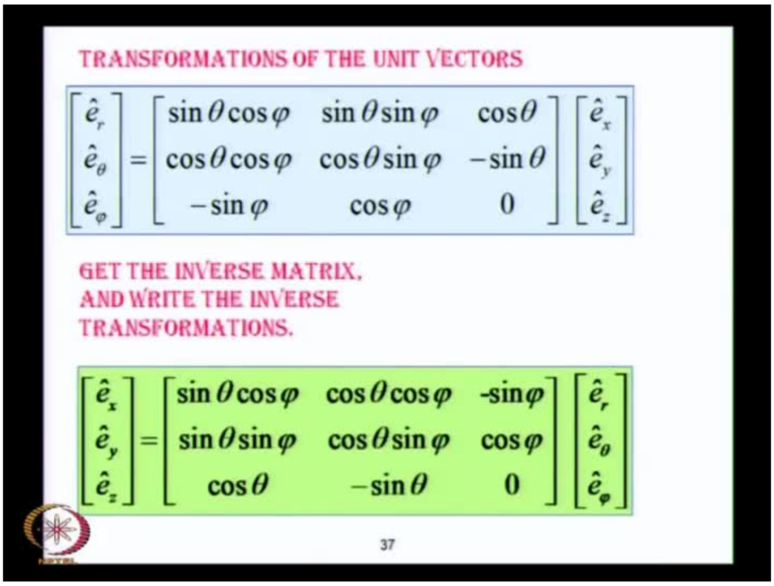
This is the spherical polar coordinate system in which it is defined with respect to the Cartesian coordinate system as we did in the cylindrical polar or the plane polar coordinate system. Here each point is described by three independent degrees of freedom. Instead of x, y, z, these are r, theta and phi. r is the distance from the origin; theta is the polar angle, which is the orientation of this point with respect to the z axis. So, this is called as the polar axis. With respect to the polar axis, what is the orientation of this point? This is measured by theta. Obviously, theta will go from 0 to pi because it can be either oriented along the z-axis or other points in space will be oriented opposite to that. So, theta will change from 0 to pi and phi is the same as the azimuthal angle of the cylindrical polar coordinates system. However, instead of measuring it with respect to the x-axis alone, you measure it with respect to the z-x plane because this is the three-dimensional world that we are now working with.

You take the z-x plane as your reference plane. With reference to this plane, what is the angular departure of this point? A point in the z-x plane itself; any point in the z-x plane will have phi equal to 0, whereas this point (Refer Slide Time: 42:58) that we have chosen to discuss in this figure, has got a departure from the z-x plane through an azimuthal angle, which is measured by phi. You can see that in this plane, phi can take a round trip and it can take all values from zero to 2 pi. So, this is the range of phi, which goes from 0 to 2 pi.

This is your spherical polar coordinates system. You can measure this distance (Refer Slide Time: 43:26) given by the purple line from the z-axis. This is the same rho as you made use of it in the plane polar coordinates system because this line is completely equal to this line (Refer Slide Time: 43:39). What is this in terms of this spherical polar coordinates? This distance is r; this angle is theta. So, rho will be nothing but r sine theta. So, you can carry out transformation from one coordinate system to the other. You can go from x y z to rho phi z; you can go over to r theta phi by carrying out these transformations. So, rho is equal to r sine theta; rho itself is square root of x square plus y square. So, you can always go from the Cartesian to the cylindrical polar to spherical polar and carry out these transformation back and forth any which way depending on what is going to make your mathematics the easiest or the least cumbersome.

However, any coordinate system will work because there is nothing secret about one or the other. So, these are the relations; x is this distance, (Refer Slide Time: 44:31) which is projection of this line along the unit vector e x. This will be rho cosine phi. So, this is r sine theta cosine phi. Then, this will be rho sine phi. So, this is r sine theta sine phi. This distance, which is z in your Cartesian coordinates system, will be nothing but r cosine theta. So, you can carry out these transformations and write them and consolidate them over here in these relations that x, y, z are respectively given by these three relations (Refer Slide Time: 45:15). You can also carry out the inverse transformations and write r, theta and phi in terms of the Cartesian coordinates, which are the inverse transformations.

(Refer Slide Time: 45:30)



You can also carry out the transformations of the unit vectors. We will do it directly by writing it in a matrix form because we know that there it is much more convenient to use this matrix form. We have done this already for the plane polar coordinates system. So, I will not work out the details step-by-step, but I will leave it as an exercise, which I hope at least some of you will do some part of it if not all. So, people call me optimists and for good reason.

This is a set of unit vectors: e x, e y, e z. These are the Cartesian unit vectors. You can get the polar unit vectors: e r, e theta, e phi. This is the unit vector in the direction in which the distance r would increase. This is the (Refer Slide Time: 46:19) polar unit vector; it is the direction in which the polar angle theta would increase. This is the azimuthal unit vector; it is along the direction in which the azimuthal angle phi would increase. They are all of unit magnitude and they all constitute a right-handed triangle. So, e r cross e theta will give you e phi; e theta cross e phi will give you e r.

You can use any set of unit vectors, any coordinate system depending on the geometry. Depending on the symmetry of the problem, you can choose an appropriate coordinates system. You can get the inverse relations. You can do so by simply getting the inverse of this matrix (Refer Slide Time: 47:06). Vector algebra and matrix algebra are all integral parts of doing physics. These need not be looked at as mathematical exercises because this is all part of getting the velocity. A velocity is physics and so is this. This is because you are going to get the velocity by taking the derivative of the position vector. If you describe the position vector in this spherical polar coordinate system, you must take the derivatives of the corresponding unit vectors. So, you have to work with these inverse transformations and you have to work with the derivatives of these unit vectors.

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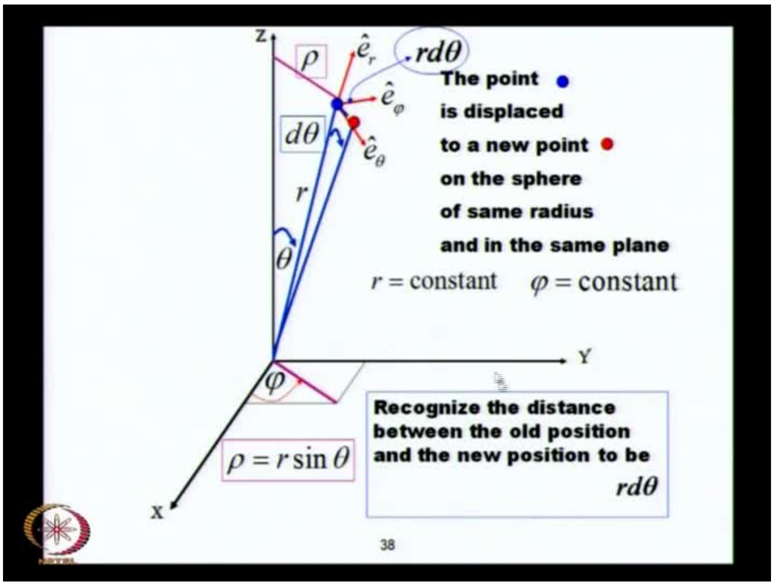

Let us see how these vectors change from point to point. You begin with a point, which is the blue point. If you displace it to a new point, which is this red point and you pick this point, the second point to be on the surface of a sphere; the Dhoom 3 globe if you like, it is on the surface of a sphere of the same radius. You also keep it in the same plane. So, this is a plane between this purple line (Refer Slide Time: 48:20) and this purple line so that the azimuthal angle is not changed. That is the idea.

The only thing that is changed is the polar angle theta. So, there are three degrees of freedom: r, theta and phi of which you have neither changed r, the radial distance from the origin nor changed the azimuthal angle, which is phi. The only thing you have changed is the polar angle theta. Now, you can ask - At what rate do the unit vectors change with respect to the polar angle theta keeping the other two parameters fixed? So, you will work again with partial derivatives. So, r is held constant, phi is held constant and you can see that this distance (Refer Slide Time: 49:07) is r d theta because this is the increment in the angle theta. You recognize this distance to be r d theta.

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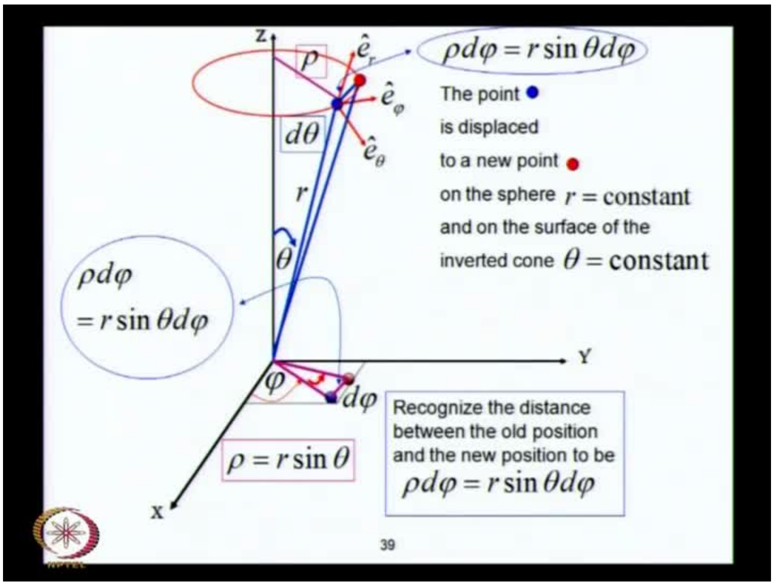
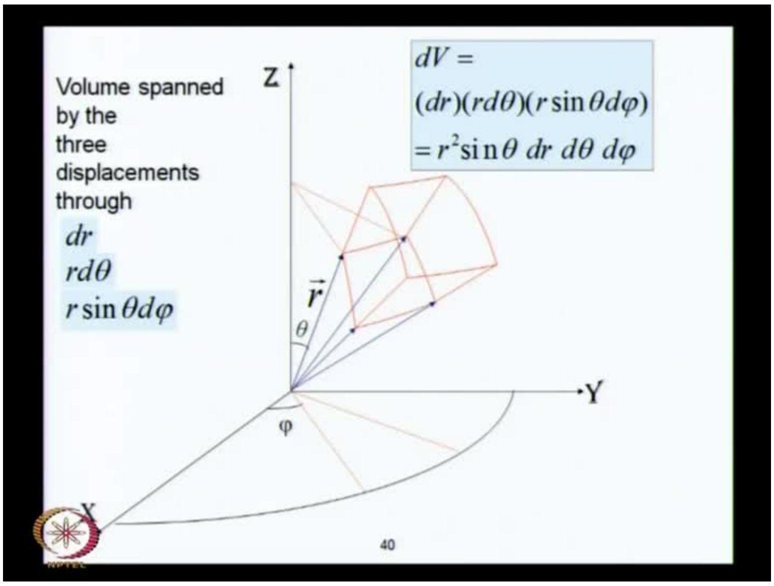


Likewise, you can get the projections on different directions. If you want to move it not along this globe, but if you think of a cone, rather an inverted cone with this as the base and its vertex at the origin, then you have a cone, which is made up of this circle. This red circle would constitute the base of the cone and its vertex at the origin. You move this point on the rim of this cone from one point to another.

Now, what you have done is that you have not changed theta because all points on the rim of this cone are at the same polar angle theta. However, what you have changed is the azimuthal angle phi. They are also at the same distance from the origin. So, there is no change with respect to r either. They are on the cross-section; they are on the intersection of the surface and the cone. So, if you were to take a spherical surface at a distance r and intersect it by the rim of this cone, then what is changing on that rim is neither r nor theta, but phi alone. Then, you can take derivatives with respect to the azimuthal angle phi.

You can do this geometry; I think it is a very important exercise. I do strongly urge you to do this as part of your homework, sit down with a piece of paper, construct these diagrams and ask yourself at what rate do these unit vectors change with respect to r, theta and phi. Now, these figures that I have drawn for you should suggest you how to get these derivatives.

(Refer Slide Time: 51:11)



You can also construct volume elements. This volume element in the spherical polar coordinates system will be made up of a part of space, which is contained inside increments in r, which is along dr. Then, there will be an increment along this, which is r d theta and then an increment along this, which is rho d phi, which is nothing but r sine theta d phi.

The product of this (Refer Slide Time: 51:53) will give you the corresponding volume element. Make sure that the volume element has got dimensions of l cube so that you have not missed out any parameter. Here you have got l square coming from r square; theta and phi are dimensionless, but you have dr over here, which will give you the third dimension. So, the volume element does have the dimension of l cube. So, always keep track of the dimensions.

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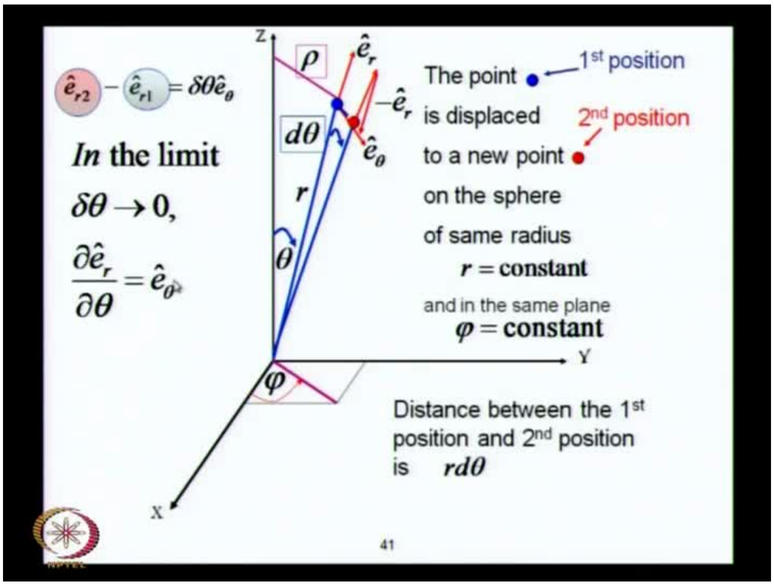

It is a good idea and you can construct the rate at which the unit vectors change. Here you are looking at the change in unit vector because of a change in the polar angle. So, this is the exercise that I had asked you to do, but you can do some of the exercise over here so that I do not take any risk that you do not do it at all.

There is this difference between the unit vector at position 2; subtract from it the unit vector / radial vector at position 1. So, this is the unit vector. You draw a vector, which is parallel to it, but in the opposite direction. You see that it is in the direction of e theta and its magnitude is delta theta e theta. Can you see that?

This change in the two unit vectors: the unit vector e r at two adjacent points. In these 2 adjacent points, (Refer Slide Time: 53:15) I have chosen these adjacent points to be separated by the polar angle theta alone; not by r and not by phi. I carry out this change only in a fixed plane; a plane, which is at a fixed azimuthal angle with respect to the z-x plane. If you look at this and if you just divide this by delta theta and take the limit delta theta going to 0, you will get the rate at which the unit vector e r changes with respect to theta. It will be orthogonal to this. However, knowing that is not enough because there are two directions, which are orthogonal to e r: one is e theta and the other is e phi; besides there is a plus and minus sign to worry about. So, you really have to work it out explicitly. So, this is what gives you the rate at which the unit vector e r changes with respect to theta. This is (Refer Slide Time: 54:15) equal to e theta.

Then, you can get the rate at which the other vectors e theta and e phi also change with respect to theta. Now, I will let you do this as homework. Please do it.

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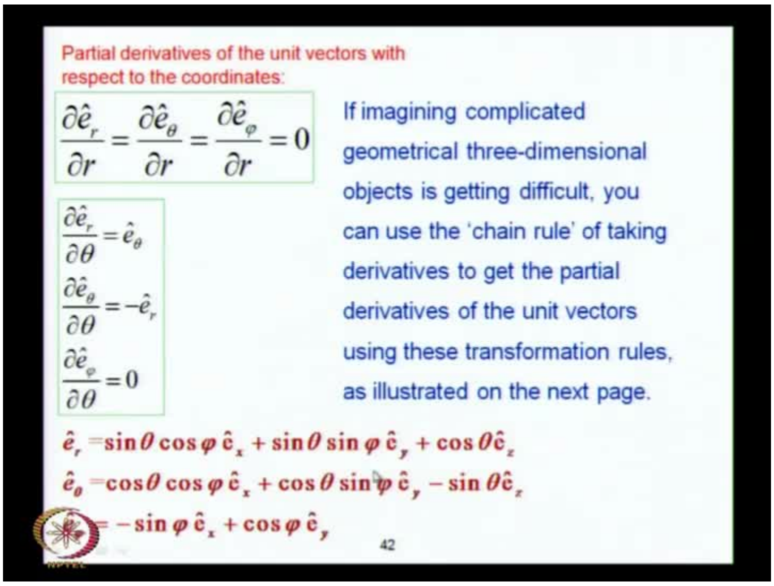

These are the results. The rate at which the unit vector e r changes with theta is e theta as we already found. The rate at which e theta changes with theta is minus e r and the rate at which e theta changes with theta is 0 because it is not going to change; the azimuthal unit vector will not change with theta.

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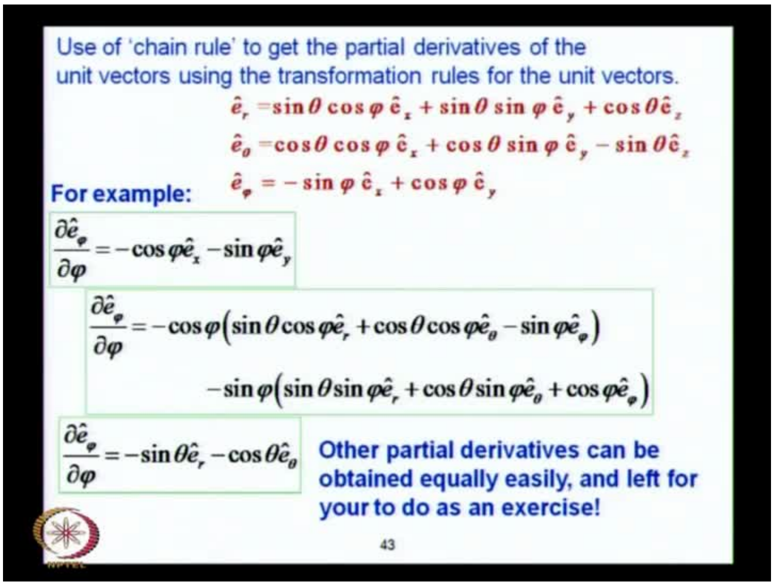

Now, you can write the transformation equations from e x, e y, e z to e r, e theta, e phi. To get the derivatives of the unit vectors, you can always use the chain rule. In doing this geometry sometimes worries up, but good. It is tedious, but good. So, you should do that.

On the other hand, using the chain rule is very straightforward. You do not have to draw complicated figures because after all you have to depict these three-dimensional features in the flat land. Sometimes it becomes a little cumbersome. You can then do this by using the chain rules because you can write the unit vectors in terms of the Cartesian unit vectors e x, e y and e z. Then, you can take the derivative of e r with respect to phi. So, you have to take the derivative of each of these terms with respect to phi, but e x, e y, e z are constants. So, their derivatives with respect to phi would vanish. So, you get a result. However, now you have got a hybrid kind of quantity because you have got a polar creature on the left; on the right, you have got a mixed creature. The coefficients are polar, but the unit vectors are Cartesian. However, you know how to transform the Cartesian unit vectors to the polar unit vectors.

You can use those transformation relations and write everything in terms of the corresponding polar quantities. So, you can write these e x and e y in terms of e r, e theta, e phi using the transformation relations. Now, you have got a result, which has got only polar quantities. So, this is a purely polar relation. You can combine the terms and you find that some of these terms are cancelled already. So, you can see that the component along e phi is already cancelled. Look at this component along e phi (Refer Slide Time: 57:04), it is a product of minus sine phi with minus cosine phi. That is, plus cosine phi sine phi; Over here (Refer Slide Time: 57:11) the component along e phi is also cosine phi sine phi, but it is with the minus sign. So, when you combine all the terms, some other components may kill each other and then you can get rather simple relations.

In getting this relation; in getting the derivative of e phi with respect to phi, we have not plotted any figures; we have not drawn any figures; we are not thinking of three-dimensional changes in a flat land. That is sometimes difficult to do, but you can do it easily by using the chain rule. Essentially, you get the same results and you can get other partial derivatives also in a similar fashion.

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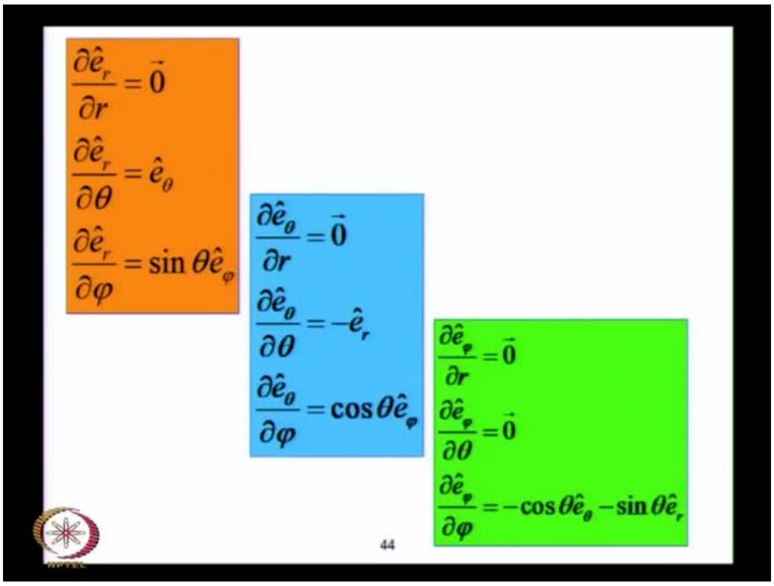

These are the net results, which are consolidated here in the tri color background. So, you have the rate at which the unit vectors e r, e theta and e phi change with respect to r, theta and phi. This is the rate at which the e r changes with respect to r, (Refer Slide Time: 58:18) theta and phi. This is the rate at which the unit vector e theta changes with respect to r, theta and phi. This is the (Refer Slide Time: 58:30) third unit vector, namely the azimuthal unit vector.

This tells us how this one changes with respect to r, theta and phi. So, you do not have to write this out, but you have to derive them yourself. Hopefully, you know how to derive it; it is very simple.

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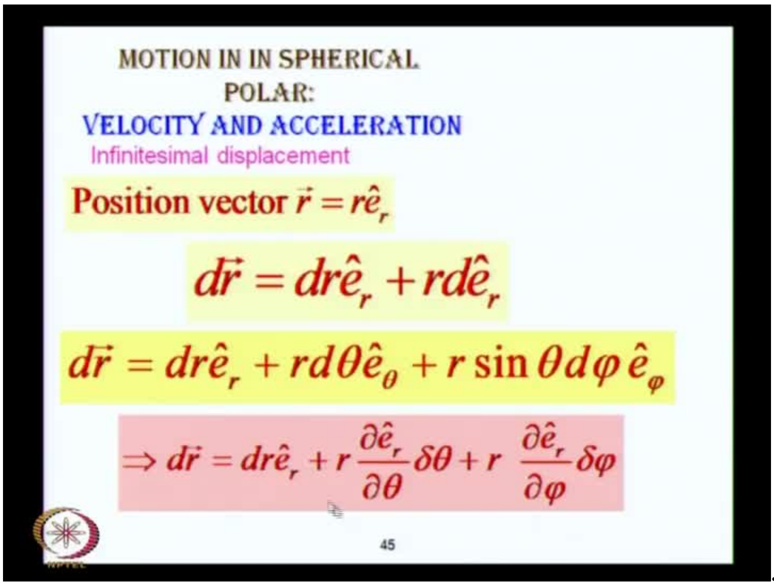

How would you describe motion in spherical polar coordinates system? Position vector is just a distance times the radial unit vector. To get the velocity, you require r dot, which is dr by d t, which is delta r by delta t in the limit delta going to 0. So, you have to take the differential increment in r. This will come from the change in r as well as the change in e r because these unit vectors are not constants unlike the Cartesian unit vectors.

Now, you can get d e r, (Refer Slide Time: 59:27) which will have components along e theta as well as e phi, but not along e r. Change in unit vector is always orthogonal to it; that is all we know. However, whether it will have a component only along e theta or only along e phi or along both; these are matters of details to be actually worked out and we have figured out how to do it. So, this is your differential increment in the position vector. To get these changes, you have to get the rate at which the unit vectors change with respect to theta and phi.

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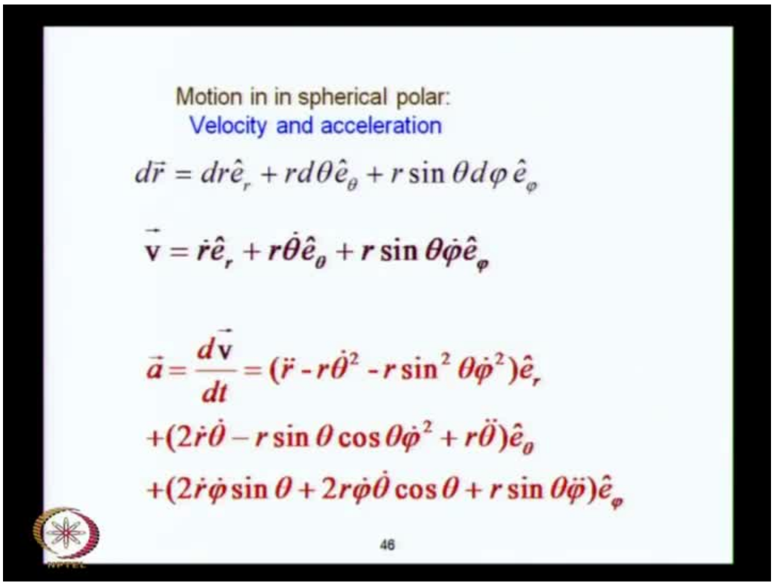

You can write all of these relations and get the corresponding velocities by dividing this by delta t; taking the limit delta t going to 0. You have the velocity, which will have a component along the radial, the polar and the azimuthal directions. So, these corresponding components are called as the radial component, the polar component and the azimuthal component of the velocity.

Likewise, you can get the second derivative. Now, you have plenty of terms because look at this (Refer Slide Time: 1:00:43) - this is already a product of two functions. This is the product of three functions; this is the product of one, two, three and four functions. You have to take the derivative of product of these functions using the same rule that the derivative of a product f and g with respect to t is f times d g by dt plus g times d f by dt. So, this is the same kind of rule that you use no matter how many functions multiply each other in the argument of the derivative differential operator. So, there are plenty of terms. You combine all the components along the unit vector e r, e theta and e phi and you get the net expression for acceleration. There are plenty of term over here (Refer Slide Time: 1:01:28) and we are going to use this machinery when we solve the equations of motion in different coordinate systems. Depending on the symmetry, which is of interest to us, the results will be different.

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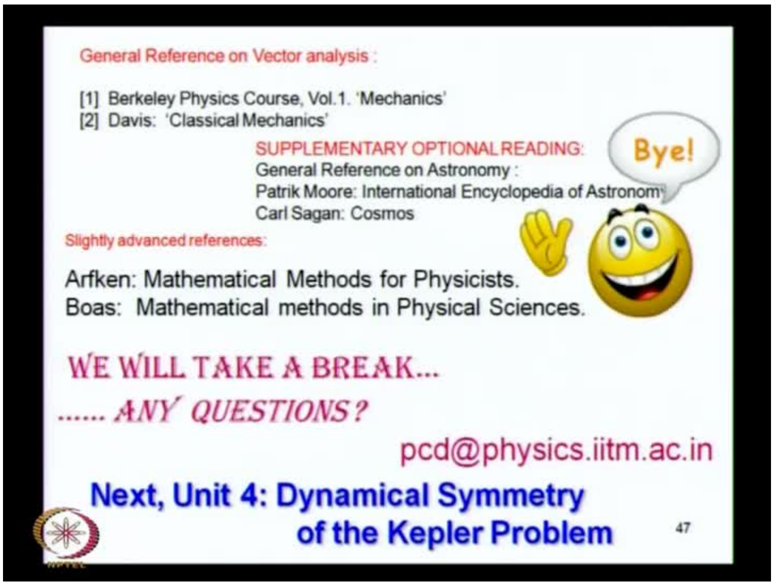

I will conclude this class today. If there are any questions, I will be happy to take. I could suggest a few references for this. The Berkeley Physics Course and Classical Mechanics by Davis are good sources for basic notions of Vector Algebra. Patrick Moore's Astronomy is a nice book to read; not to learn about calculus of vectors or anything like that, but just to appreciate the orientations of different objects in the world around us. Remember that it is observations in the universe; it is astronomy, which really inspired physical observations and is at the root of development of physics itself. So, Patrick Moore's book is something, which I enjoyed very much. Reading about and looking at the stars, doing star gazing and learning about the constitutions and so on. At a more advanced level, you can read Mathematical Methods for Physicists by Arfken or by Mary Boas.

Further, we will meet for unit IV in our next class in which we will work with the Kepler problem, but we will work with specific aspect of the Kepler problem, namely what is known as the Dynamical Symmetry of the Kepler problem. So, let me not jump ahead of ourselves at this point; we will wait on it till we meet for the next unit in this course.