Select / Special Topics in Classical Mechanics

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Module No. # 04

Lecture No. # 13

Dynamical Symmetry in the Kepler Problem (i)

Greetings, we will discuss unit 4 in this course. This will be on the Dynamical Symmetry in the Kepler problem. We will have a continuous discussion on symmetry and conservation laws with specific reference to the dynamical symmetry in the Kepler problem.

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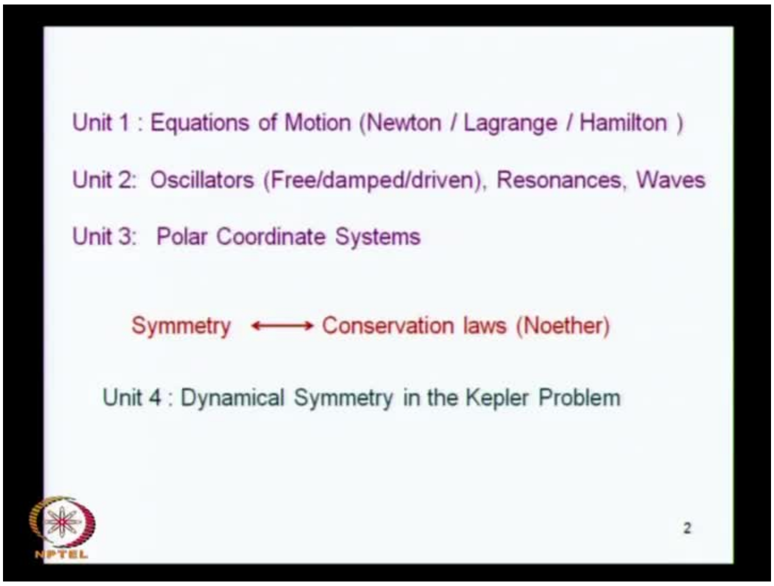

For those who joined the course late, let me mention that in unit 1, we studied the equations of motion - the Newton, Lagrange and the Hamilton equation of motion. We studied the oscillators and we studied the polar coordinate systems in unit 3. We will be using the polar coordinates in today's class.

We have already had some acquaintance with connection between symmetry and conservation laws. We met this in the first unit, where we illustrated this in a few cases.

This unit number 4 is specifically designed to discuss the dynamical symmetry. The dynamical symmetry is also called as the accidental symmetry in the two-body Kepler problem.

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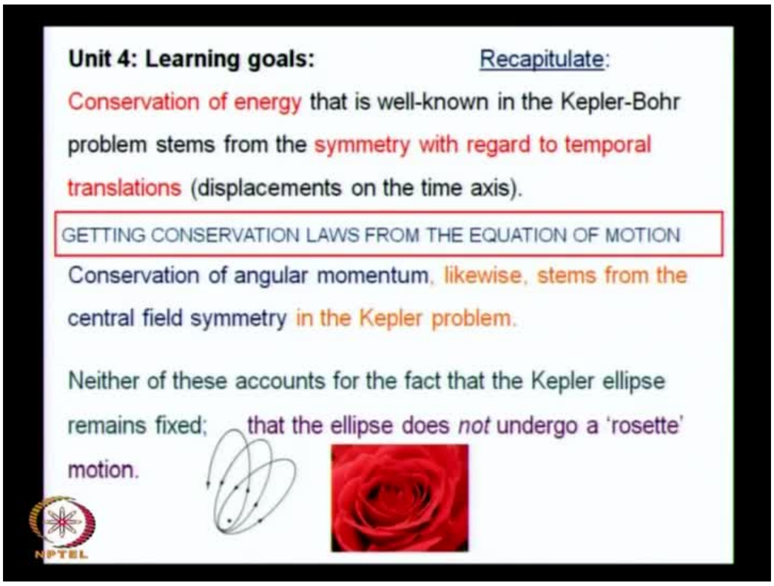

First, we will recapitulate little bit about the connections between conservation and symmetry and associate conservation of an energy with symmetry about temporal evolutions. So, we will very quickly recapitulate this, but we will see it coming in a slightly different context from what we have seen earlier. What we will do is to find how to determine, what physical quantities are conserved, if you know what the equation of motion is.

Given an equation of motion, how do you get a conservation law? That is something we will discuss. As conservation of energy comes from the symmetry with respect to temporal evolution over a passage of time, the conservation of angular momentum is connected with the central symmetry. We will also get this from the equation of motion. We will ask that in the Kepler problem. You have these elliptic orbits that you see here. This solution is quite correct, but why do the ellipses not press? Why does it not go from one to the other in what is called as the Rosette motion because if you look at this from a distance, it will look like the petals of the rose. It turns out that neither the conservation of energy nor the conservation of angular momentum is sufficient to explain the fact that the ellipse does not actually presets and the ellipse does not undergo or rest at motion.

So, we will try to understand what is the origin of the fact that the ellipse itself is fixed and it does not presses.

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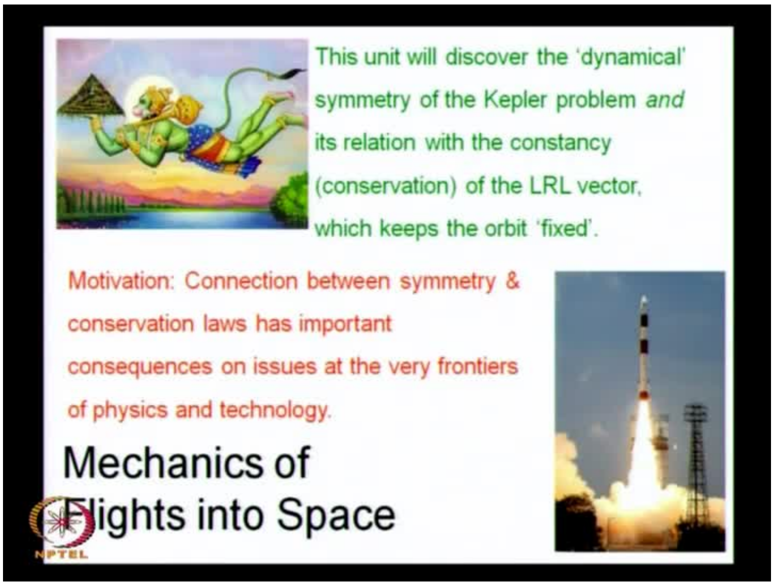

This is connected to what is known as the dynamic symmetry of the Kepler problem. There is the vector quantity, which is conserved and it is along the major axis of the ellipse, so that remain fixed. This vector is called as the LRL vector - the Laplace-Runge-Lenz vector. We will study this. The motivation to study this topic is to explore further connections between symmetry and conservation laws. This has very important and deep consequences on the foundations of physics, on the very frontiers of physics and also in technology. We need to study the mechanics of flights into space and all of these things have to come together, when it comes to you. For example, you know the rocket technology.

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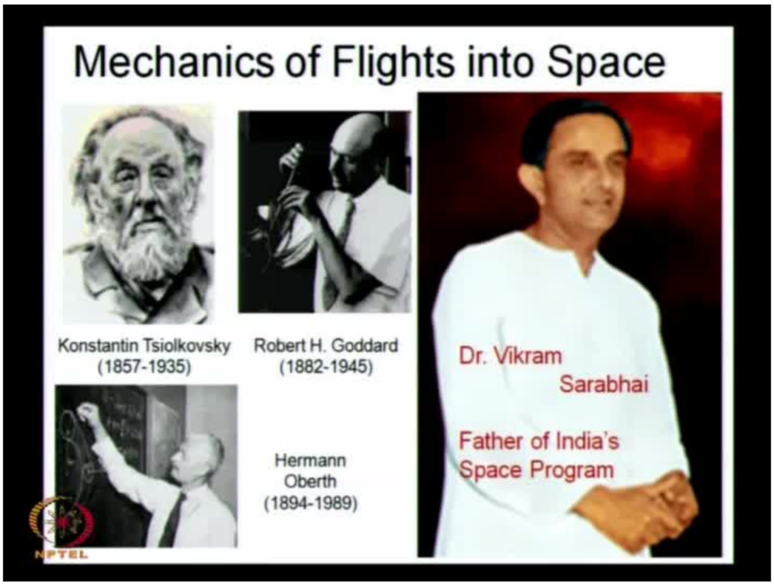

This is particularly important for us because we are now in this space age and there many scientist, engineers, technologists and entrepreneurs amongst us. These are the people behind the rocket science technology. The Russian, however you pronounce it, the first name sounds little easier, which is Konstantin followed by family name and then Robert Goddard the American and Hermann Oberth, who was the German. These are some of the first pioneers of the rocket technology. In our own context, we have Doctor Vikram Sarabhai, who is regarded as the father of India's space program.

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This is especially important for us. ISRO has a large number of space related projects, in which we fired rockets and satellites into space. It is obviously very important for fundamental physics as well as for high technology. We will study these topics in detail.

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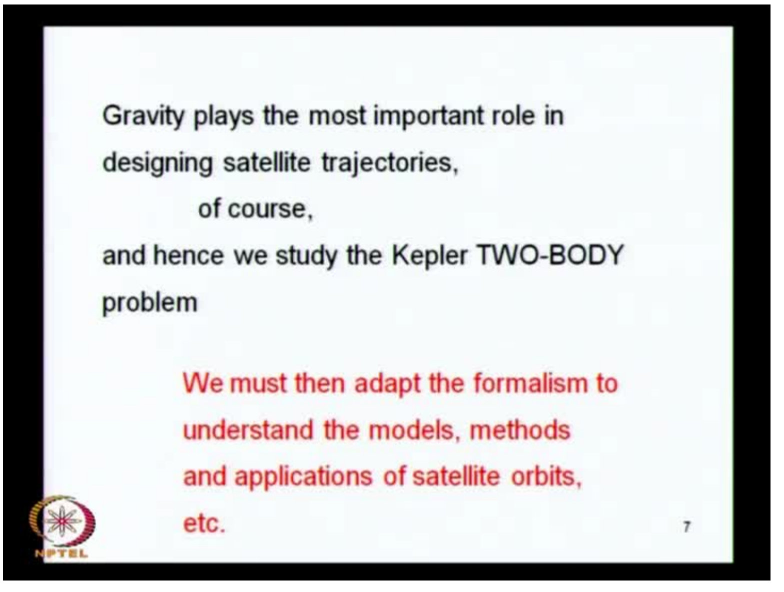

Of course, gravity plays the most important role in all of this. So, we will begin our discussion with the two-Body Kepler problem, in which the two bodies have a gravitational interaction between them. As one study this subject further, one will need to adapt the formalism to study satellite, orbits and other things.

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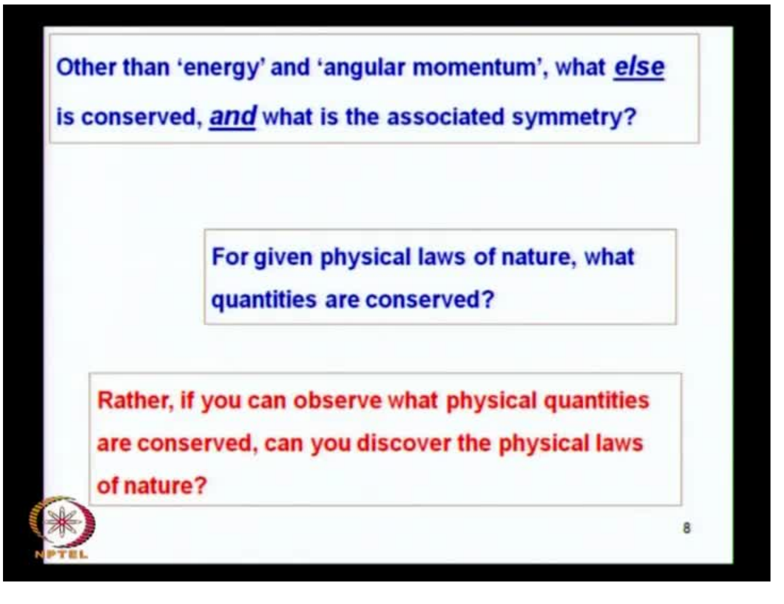

We begin this discussion by raising a question that in the Kepler problem. Other than the energy and angular momentum, what else is conserved? Is there anything else, which is conserved in addition to energy and angular momentum? We also ask a question that for a given physical of nature and you know what the physical law is and then can you determine what quantities will be conserved? How do you determine this? What is the mechanism of discovering the conserved quantities, if the law of nature the physical law is already known?

We can also ask the inverse question that if you observe what physical quantities are conserved, then from this, can you deduce what the law of nature is. It is the inverse question and ultimately one of the major goals of physics is to discover the laws of nature. You could propose it and you could discover it through observation. We use the laws of nature to find what physical quantities will be conserved. All ask an inverse question from the observation about the conservation of physical quantities, how do you get a law of nature? So, we will discuss some of these issues in this unit.

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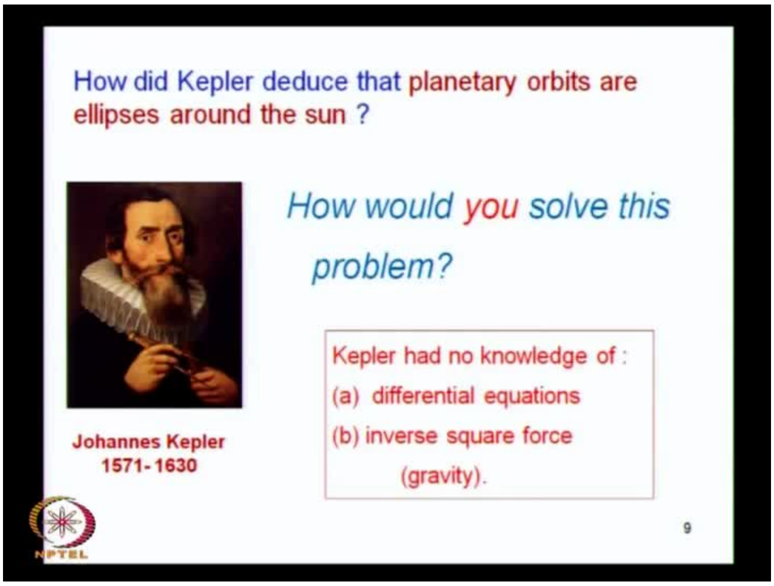

We begin by asking this question, how did Kepler deduce that planetary orbits are ellipses around the sun? Looking at this Kepler's picture, what bothers me more is not how and why the planetary orbits go around the sun, but why this piece goes around his neck. This is something, which bothers me and it worries me about Kepler, even though I know that he died in 1630. This is the question and I think you can try to answer it first by asking yourself how would you find what is that trajectory of a planet, if it is known that the interaction between the planet and the sun is given by the law of gravity, which is G m 1 m 2 by R square and that is the force of attraction between two masses.

So, how would you solve this problem? I think what you might do is set up the equation of motion, which is f a equal to m a. You know F, which is G into m 1 m 2 by R square. You know the force, magnitude, direction and differential equation. So, you will solve it. From the solution of the differential equation, you will get the trajectory and that is all you will solve.

Now, Kepler did not know differential equations and Kepler did not know that the force of attraction between two masses is G m 1 m 2 by R square and that is the reason the question that is raised at the top of this slide is how did Kepler deduce the planetary orbits or ellipse around the sun so obviously he did not use the technique which you would use today

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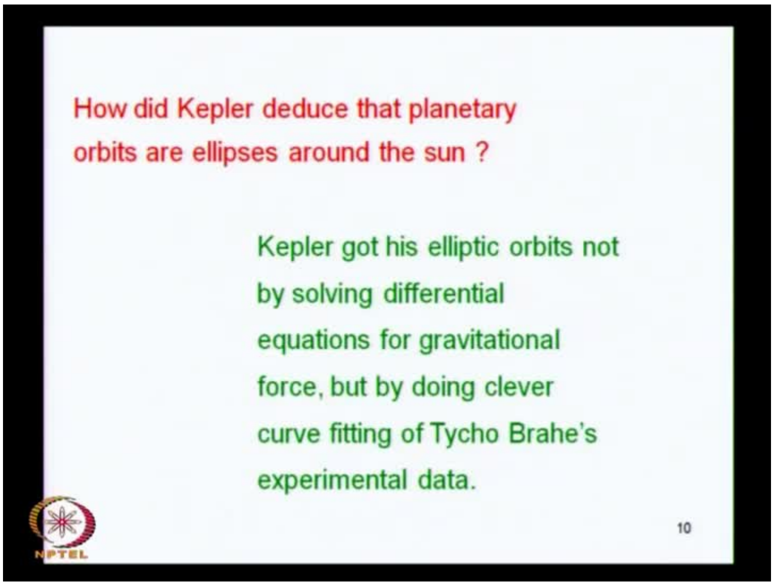

What Kepler did was to make use of observation, which was tabulated by Tycho Brahe. Tycho Brahe was a brilliant astronomer. He recorded the positions of various planets and told how these positions change with time during the course of year. From these positions, what Kepler did was a very clever curve fitting. He deduced that if the positions of the planets have to be what they were reported in Tycho Brahe's very careful observations, then the orbits would have to be ellipses. This was a conclusion deduced based on very accurate and careful observations recorded by Tycho Brahe.

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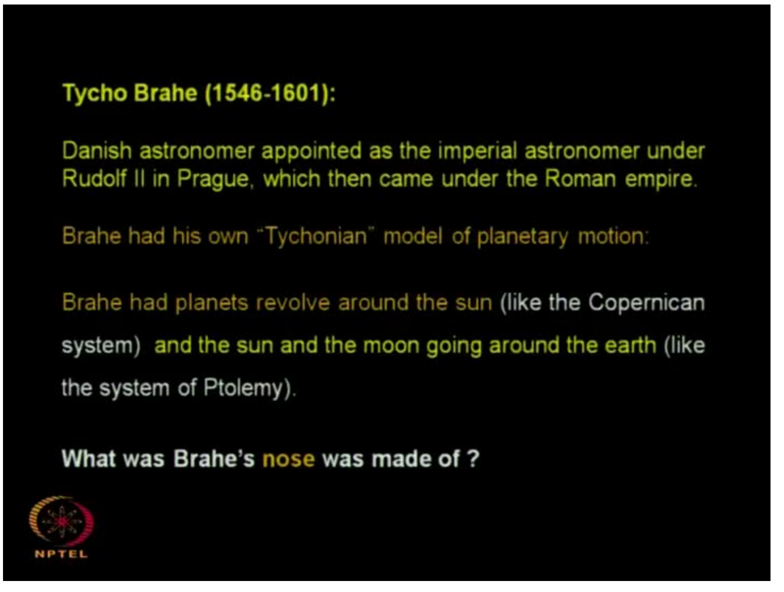

Now, Tycho Brahe was a very brilliant astronomer and he was from Denmark. In fact, he had his own Tychonian module of the planetary motion. We mentioned the Copernican system Toulmin system in one of our earlier units. What Brahe did was to come with the planetary system, which was his own, in which he presume that the planets goes around the sun. He had the sun and moon going around the earth, just like Toulmin's system. So, it was a combination of the Copernican and the Toulmin system.

Although, this has nothing to do with the topic, it is something, which I thought I would ask you. If you are asked, what Tycho Brahe's nose was made of? May be some of you know that answer. It was made of gold and the reason it was made a gold is because he lost it in sword duel with another student. They wanted to decide between them that who was a better mathematician? You are very competitive, but please use some other ways of settling, who is the better mathematician.

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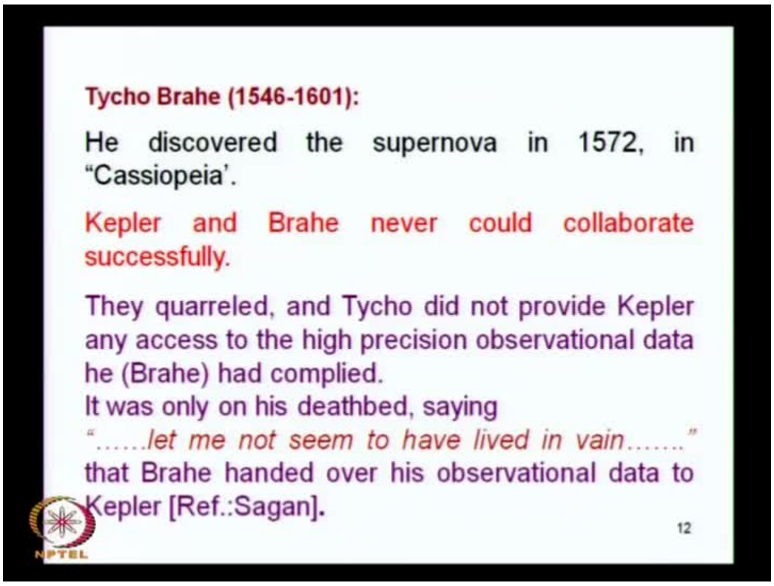

He discovered the core supernova in 1572, in the consolation Cassiopeia. Kepler needs access to Brahe's observations because it was by doing clever curve fitting to Tycho Brahe's data that Kepler finally got the ellipse. It was not easy because Brahe would refuse to share his data with Kepler, he was very possessive of his data. He did not want to share it with anybody. They quarreled and only on Tycho Brahe's death bed, he finally gave his data to Kepler saying that let me not seem to have lived in vain. Saying this, he finally handed over the data to Kepler and then Keplar was able to deduced that the planetary trajectories will have to be ellipse.

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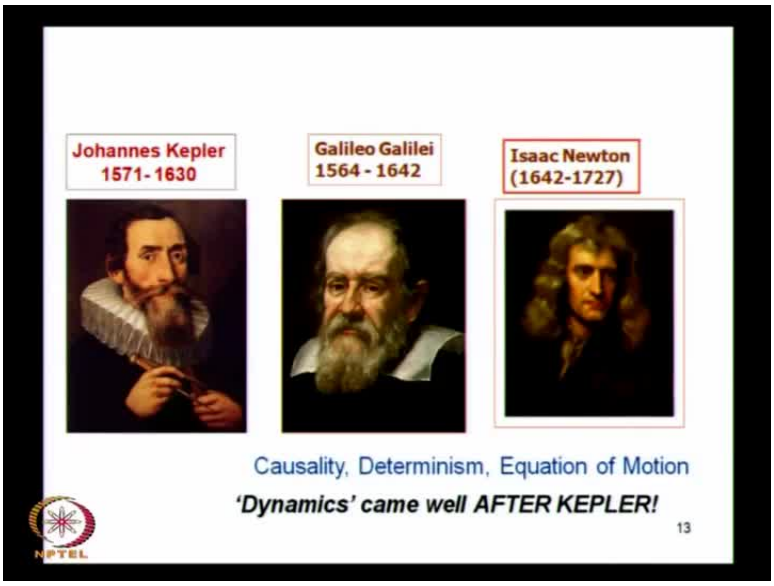

Now, the solution you would get by solving the differential equation and plugging F equal to G into m 1 m 2 by R square would also be correct. This is dynamics, but this came well after Kepler. It came though the development of equations of motions through Galileo's law of inertia followed by Newton’s laws of motion, but then also it required the knowledge of gravity.

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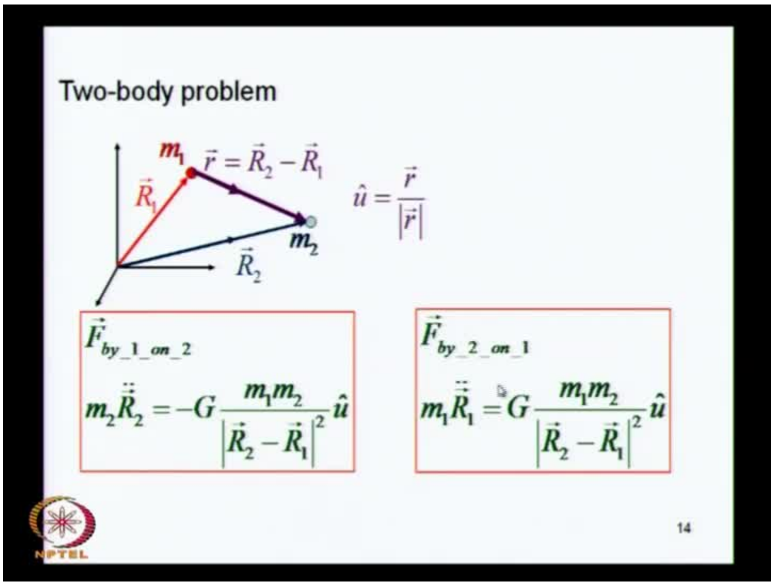

This is something for which the two-body problem has to be solved. So, let us quickly go through this. You have two masses m 1 and m 2 and these refer to some coordinate system. With respect to this coordinate system, you have the respective position vectors R 1 and R 2. You have a relatively different vector, which is R 2 minus R 1. There is a direction, along which the interaction between m 1 and m 2 takes place. This unit vector - r vector divided by its length, gives this direction. So, you get only direction.

The force by the mass 1 on the mass 2 is given by mass times acceleration. This is the force and this is gravity G m 1 m 2 by R square. Likewise, force by 2 on 1 is again mass times acceleration for the object one. I have defined the unit vector along the direction, in which I have shown from m 1 m 2. This is why this force is along plus u and whereas this force is along minus u because they are towards each other.

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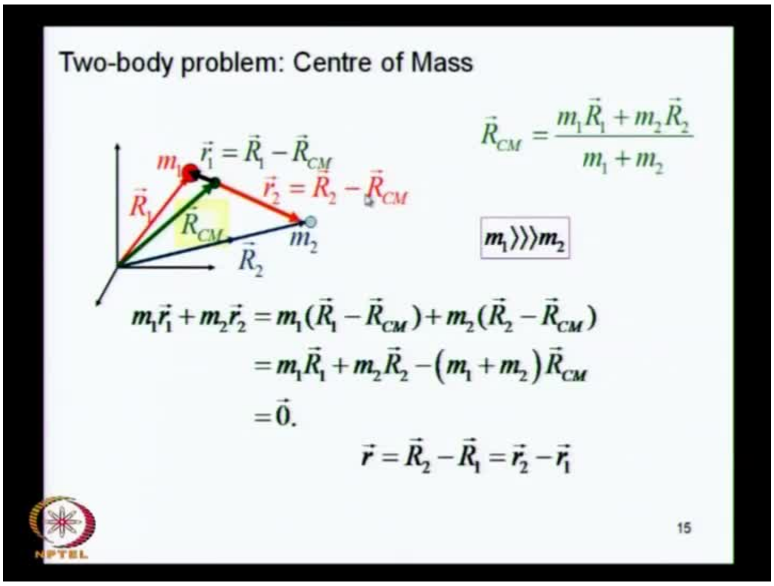

It also useful to find the centre of mass, which is given by the weighted position vectors and weighted by their respective masses. You define a centre of mass, which is in between and closer to the larger mass. We develop in approximation, when one of the masses is must larger than the other for the solar system, for any planet in the solar system or for any satellite of any planet in the solar system. It holds further for moon, the earth system and for artificial satellites, which you will launch in due course of time. This is the basic algebra, which defines the center of mass coordinate system and it is nothing very special about it.

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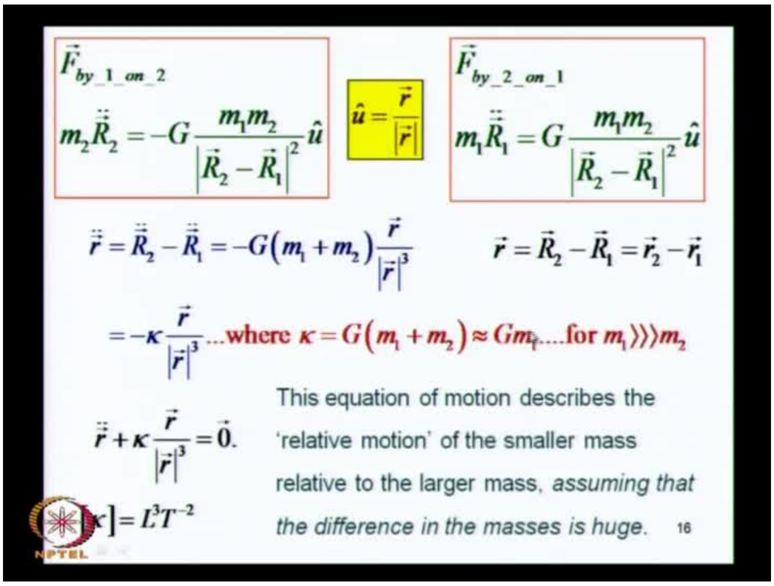

From this, you can get an expression for the acceleration. So, you can develop any equation of motion for the planet in the field of the centre of mass by determining acceleration. Since this relative coordinate is just the difference of these two position vectors, it is second derivative and it will give you the corresponding accelerations. From these two equations, you can just subtract this R 1 double dot from R 2 double dot and get the equation of motion. So, this is very simple and straightforward equation of motion. This has a magnitude of the distance between that two and this is the cube of the distance. So, essentially you are getting an inverse square of force.

This is your equation of motion and from this equation of motion you can get this solution. How will you get the orbit? You have to solve the differential equation and from the solution of the differential equation, you will get the actual trajectory - the orbit. We have to set it up under the assumption that one of the two masses is much larger than the other. It is a perfectly good approximation in the context, which we are discussing this. Notice that the constant kappa is determined by the irreversible constant G and the two masses are approximated to G times m 1. Here, m 1 is much larger than m 2 and so this is the constant. It is determined completely by the irreversible constant and the mass of the object 1 has the dimensions of L cube T to the minus 2 and keep track of this. In some books and papers, you will find that some other units have been used. Sometimes, these terms have been scaled by some other mass constants. So, you may see slightly different expression. Kappa has the dimension as L cube T to the minus 2

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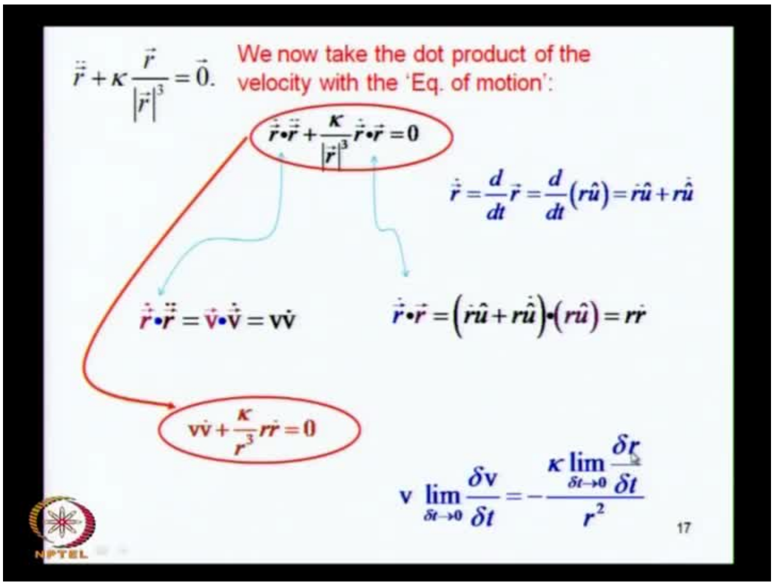

Now, this is the equation of motion and what will now do is to take the dot product of the equation of motion with velocity. Now, why do we do that? What are you going to learn by playing with the equation of motion by carrying out simple straightforward algebra? You can actually discover what physical quantities are conserved. So, you do not have to learn conservation principles at different postulate. They come from the equation of motion. Given the law of nature, you can find what physical quantities will be conserved. You could ask inverse question that if you know what physical quantities are conserved, can you get the law of nature.

So, we will first acquaint ourselves or reacquaint ourselves with how you get the conserved quantities from the laws of nature. So, the law of nature is built into the equation of motion because we have plugged in the gravitational law. That is the law of nature that we have plugged in. There is a physical law that we have invoked. We have plugged it in and from that we get the equation of motion. What we do now is to play with this equation of motion. So, what we do is to write the equation of motion and compose this scale of product or the dot product of the velocity with the equation of motion. This is exactly what we have done. We have this equation of motion and take the projection of this on the velocity. It is r dot dot r double dot and this is a scalar product. Likewise, this is r dot dot r from the second term. On the right hand side, you will get the number 0. So, this is the dot product of the velocity vector with the equation of motion.

We know what the velocity is. The velocity is the time derivative of the position vector. On the position vector, it has the magnitude and the direction of the position vector, but the direction of the position vector is not a constant quantity. So, you must take into account that time derivative of the direction is u dot. So, u is the unit vector giving the direction. It is time derivative, which is indicated by u dot. It is the rate of change of the unit vectors and we will need this, when we deal with the velocity.

Now, this term r dot dot r is the scalar product of velocity. In the position vector, you can easily see that it is nothing but rr dot. You will need the dot product of u dot with u, but we have already discussed this, when we studied the plane polar or the cylindrical polar or the spherical polar of coordinate system and the changes in the unit vectors. Change will always be orthogonal to the unit vector itself, which is why it changes. Otherwise, if it has a component, how would that be a change? We have shown this explicitly in the case of the polar unit vectors and we do change from point to point.

So, this scalar product is given by the product of these two scalars r and r dot. Likewise, we need this one also and this one is the scalar product; this is velocity and this is the rate of change of velocity, which is acceleration. This is the scalar product of the velocity with acceleration.

In this case, it turns out to be rr dot and this will turn out to be vv dot for the same reason. So, you have a simpler form and you can rewrite this to get vv dot for the first term and the kappa over r cube times r r dot for the second term. This has come straight out of the equation by motion by taking its projection on the velocity. Now, what you get out of it is an extremely important quantity in physics. You will recognize that vv dot is v times this derivative of velocity with respect to time on the right hand side. Move these terms to the right with the minus sin and so it is rr dot over r cube. I have got r over r square r over r cube, it gives me 1 over r square. I have r dot, which is delta r by delta t and I am changing the limit as delta t going to 0.

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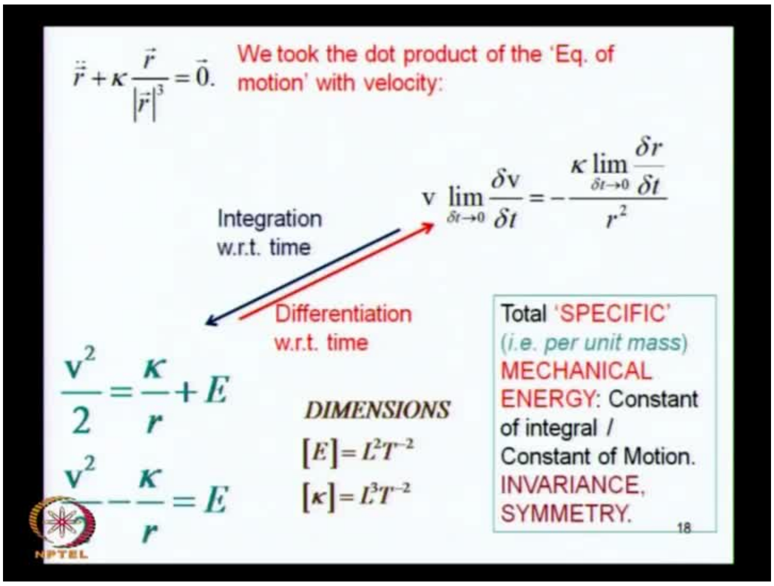

Having got this, if you now integrate this equation with respect to time, you will easily get the integral of the left hand side; it is v square over 2. The integral on the right hand side is kappa over r. It is straight one-step integration and of course, there will be an constant of integration. I merge the two constants of integration coming from integration of this. Integration of this into a single constant of integration is written as E.

Now, this is an amazing quantity because what you have discovered from this is the conservation of energy. This is the constant of integration; it is the constant and therefore, a conserved quantity. We have got this E by integrating this quantity over here. If you differentiate this, you go back to this. These are inverse processes and the dimensions of E come from the dimensions of that. We have chosen for kappa, which is L square T to the power minus 2. It is the dimension of energy that we normally talk about. It also has a mass and so m L square T to the minus 2 is the usual dimension of energy.

Here, I have written the dimension of energy as L square T to the minus 2. This is what I will be calling as the specific energy per unit mass. So, mass has been taken out and that is a reason you do not see m explicitly over here. That is the reason I alerted you to the fact that keep track of the dimensions because in some of the box, you will read E equal to m v square by 2, which is the kinetic energy. The potential energy will be kappa over … All have the dimensions of m L square T to the minus 2 and that will give it different dimensions for kappa and keep track of that. So, this is what is called as specific mechanical energy, which is energy per unit mass rather than the energy itself. I will be dealing with what I shall refer to as the specific mechanical energy. For those who do the rocket science, this is a more common term to use.

How did we get it? We started with equation of motion. We just played with that and we just did algebra. We took the projection of equation of the motion on the velocity and then we did calculus, we did integration. From the integration, we discovered a conserved quantity; namely the energy. Essentially, we got the conservation of energy from the equation of motion. We also see that as we arrived at this conservation principle, which is the conservation of energy. It is the integration with respect to time and differentiation with respect to time, which are inverse process. These were intrinsically built for an analysis.

This should remind us that we are meeting a conservation principle, which obviously comes from the fact that the derivative of this energy is with respect to time. It goes to 0 and there is a symmetry with respect to temporal evolution that as time changes from yesterday to today to tomorrow or from t 1 to t 2, the law has not changed. So, there is asymmetry with respect to time and it is this symmetry, which is connected with the conservation principles. So, this is a theme that we have carried in our discussion in some of these unit, especially in unit 1. Again in unit 4, there is an intimate connection between symmetry and conservation laws.

Here, we see that by doing this integration with respect to time. We discover the law of conservation of energy. We have actually deduced it from the equation of motion by just doing algebra. All we did was to take the projection of the equation of motion on the velocity and then integrated this.

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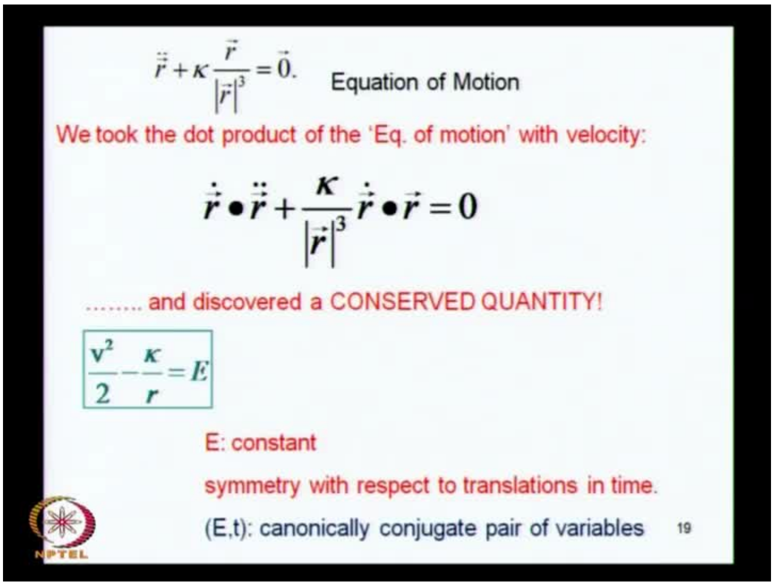

Now, let us see if this method really works. Can we get other conservation laws just by playing with the equation of motion? Look at the equation of motion, in the previous case; we have taken the dot product of the equation of motion with velocity. It leads us to a conserved quantity, namely the specific mechanical energy. We saw this in the context of the connection between symmetry and conservation law, in the context of the Lagrangian or the Hamiltonian formulation that we discussed in unit 1. We recognize that the conserved quantity energy is canonical conjugate to the variable t, which really does not appear in the Lagrangian. So, there is a symmetry with respect to time and that results in the conservation of energy.

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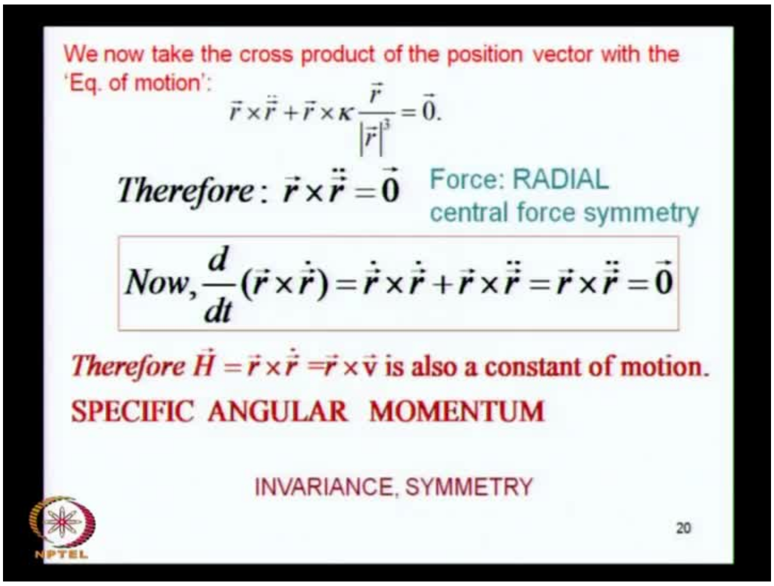

Now, instead of taking the dot product, let us take the cross product and play with it. We have the equation of motion and I am taking the cross product. We will find yet another conservation law with this wonderful technique. All we have done here is to take the cross product not with velocity, but with the position vector.

In the previous case, we constructed this scalar product and the dot product with the velocity. Now, we take position vector and compose the cross product with equation of motion. So, equation of motion has two terms. I take the cross product of each of those terms with position vector and get a vector equation. Now, if you see the second term, it is the cross product of r with r. So, cross product of two co-linear vector will vanish. It means that the first term vanishes. So, r cross the acceleration will go to 0.

Acceleration is along the direction of the force. Force has a radial symmetry and there is the symmetry, which have become manifest. These are the beautiful things in physics that the symmetry of the force become manifest in this particular context and that the force is radial force. As the result of this, we are going to discover another conserved quantity. So, we have got the cross product of the position vector with acceleration, which becomes the null vector.

Now, we use a simple algebraic identity, which is the time derivative of a cross product of position. The velocity vector will be the time derivative of the first vector. It is the cross product with the second vector plus the first vector and the cross product with the time derivative of the second vector. So, this is just an identity that we have made use of and this obviously vanishes. This is the cross product of velocity with itself and now you have this, which is the same thing as the force being radium. So, this is the identity, which is completely consistent with the fact the forces are the radial vector.

Now, if this is true, then r cross velocity. If this is the time derivative of the position vector, this is the time derivative of a cross product. What is the cross product of this? This is the cross product of the position vector r with the velocity of r dot and the time derivative of the cross product of the position and velocity. This time derivative goes to 0 and if a time derivative of any quantity goes to 0, then that quantity is conserved.

So, the cross product of position and velocity is a constant of motion and that is angular momentum, rather it is the specific angular momentum. Angular momentum is r cross p. P is momentum and it is mass times velocity, but we are developing these equations in units of mass. So, angular momentum is r cross p. Thus, specific angle of momentum is r cross v. So, the specific angular momentum is conserved. If you multiply this by mass, you have the conservation of angular moment.

Angular momentum conservation has again come from the equation of motion by simply composing its cross product with the position vector and doing simple algebra. This is really very interesting. We are dealing with the specific angular momentum rather than the angular momentum itself. So, once again we make a connection between conservation and symmetry. The quantity, which is conserved is the angular momentum or the specific angular momentum. The symmetry is a radial or the central force symmetry on the force, so that is this connection between symmetry and conservation law that we meet again.

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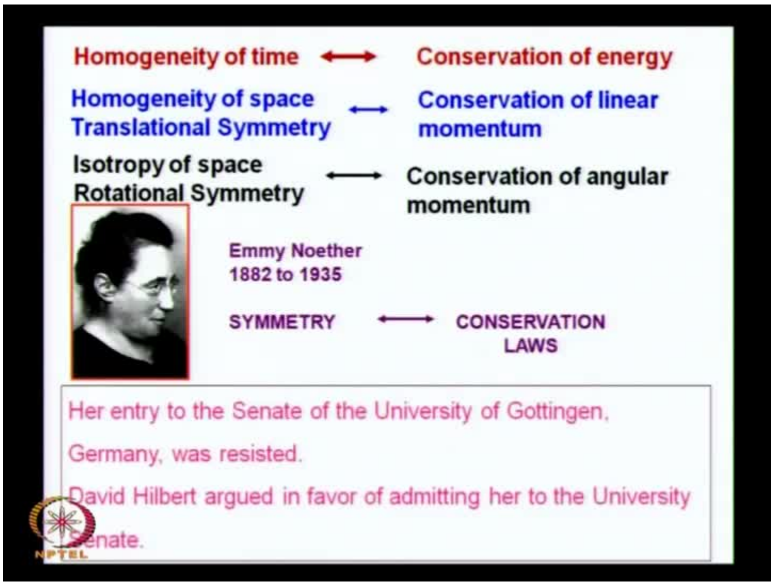

We discussed this at some length in unit 1 that the homogeneity of time temporal evolution. If it leads to homogeneous equations of motion, we do not change with time. You get the conservation of energy and homogeneity of space, in which translational symmetry is evolved. It leads to conservation of linear momentum, isotropy of space, rotational symmetry and central field symmetry. You get conservation of angular momentum and this is the essential content of the Noether's theorem, which we discussed in unit 1 to some extent.

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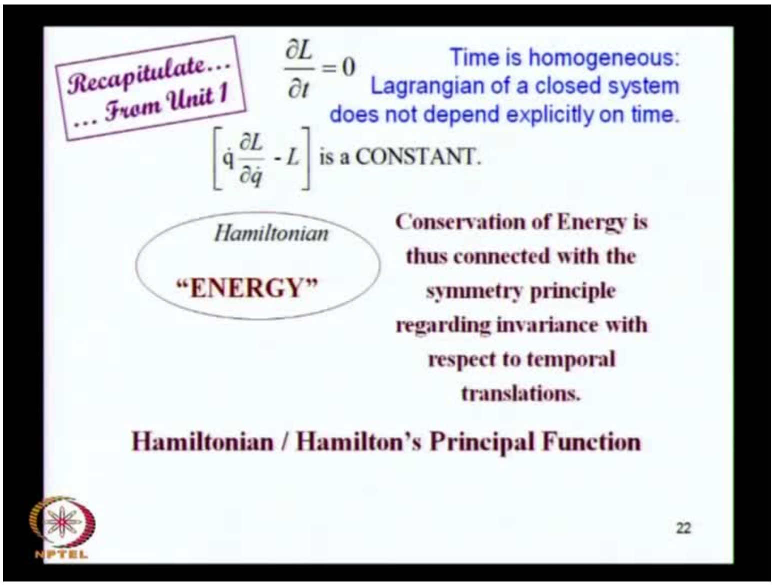

Now, let me remind you, as it is a very important feature - the invariance of the Lagrangian with respect to time. The Lagrangian and Hamiltonian formulation lead to the conservation of energy.

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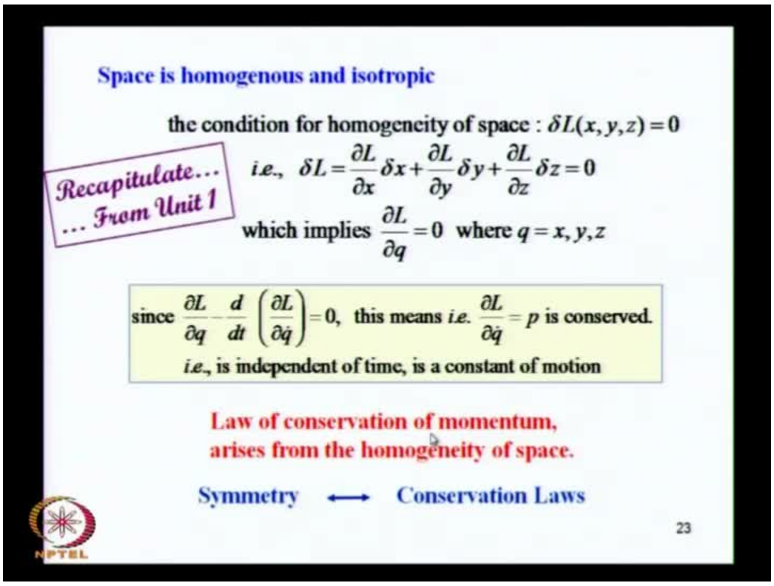

Likewise, if you consider displacements in space and this again, we discussed at some length in unit 1. So, I will not spend any time in discussing this any further, but I will just remind you. You have seen these slides and equations in unit 1. I would not spend my time on that. I will just remind you that because the space is homogeneous; from the Lagrangian's equation, you find that the conservational momentum comes from the homogeneity of space.

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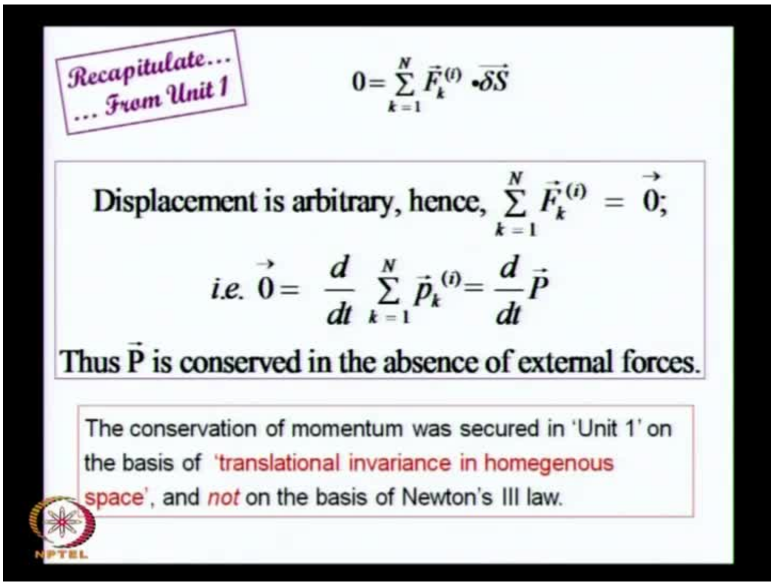

We also studied in unit 1 that if you consider virtual displacement of a system of particles, which is interacting within itself, then the net virtual work done is 0. If this displacement is arbitrary, then the sum of these internal forces must vanish. It let us to recognize that the linear momentum is the conserved quantity, which again comes from the homogeneity of space. So, this came from the discovery of Newton's third law. Instead of learning it as a law of nature, we could actually get it from the properties of translational invariance in homogeneous space.

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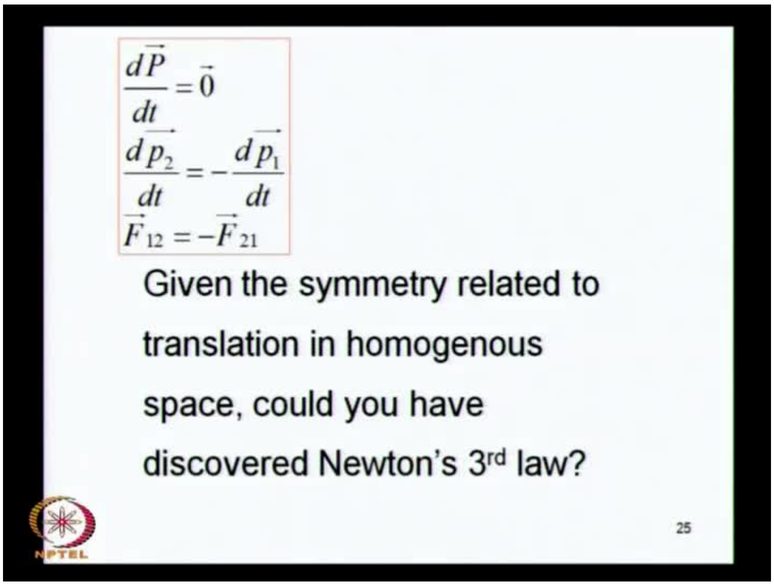

We discussed the possibility that given the symmetry related to translation in homogeneous space, could we discover Newton's third law? The answer is yes. So, this is one way of getting to the… We have discussed how to get conservation principles from the equation of motion, for which you need the law of nature and the physical law. Giving the physical law, you find the conservation principles. are what quantities are conserved or you could ask if you know what quantity is are conserved how would you find the law of nature what is the physical law

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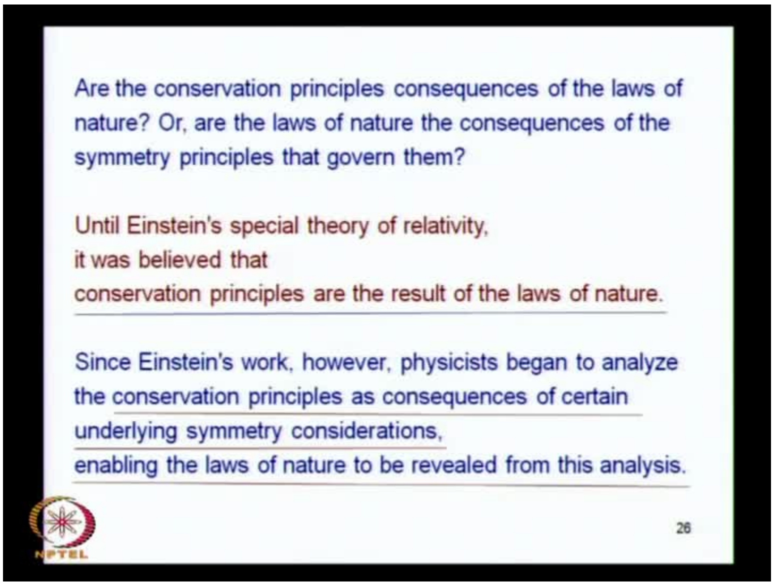

So, this question is very nicely framed in this particular fashion. Are the conservation principles consequences of the laws of nature? Or, are the laws of nature the consequences of the symmetry principles that govern them? The thinking in the physics community has changed with Einstein. Until Einstein came up with the special theory of relativity, one believed that the conservation principles are the result of the laws of nature, which is what we did in the previous two cases.

We began with the equation of motion. From that we deduced what quantities are conserved, but Einstein began to look at the conservation principles as consequence of the underlying symmetry. Using this connection, they learnt to discover what are the physical laws, which govern them. So, this suggest a recipe of discovering laws of nature, which is one of the primary goals of physics.

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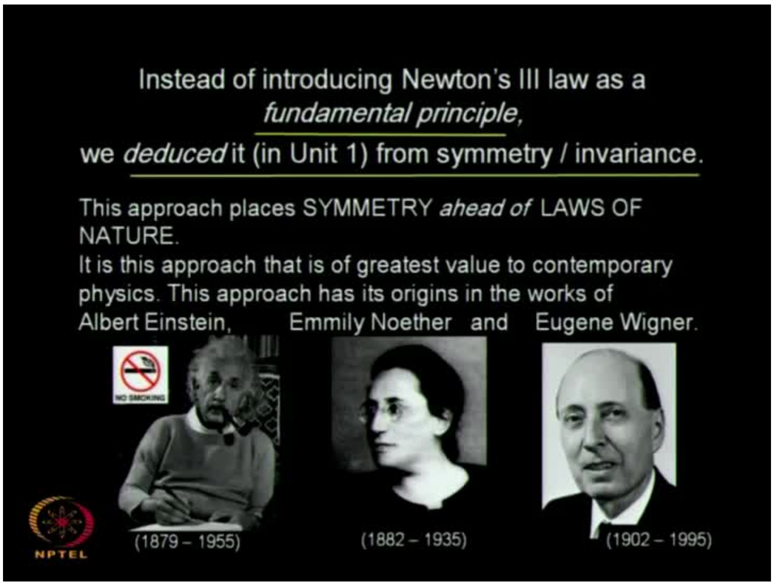

I illustrated this in the case of the Newton's third law, which we did in detail in unit 1. We will quickly recapitulate that we deduced Newton's third law from symmetry and invariance, rather than the other way around. This approach places symmetry ahead of the laws of nature. The people who contributed to this process began with Einstein and this is contained in a very famous theorem known as the Noether's theorem. Next, Eugene Wigner, who seriously illustrated these ideas using group theoretical methods, but those are obviously beyond the scope of this course.

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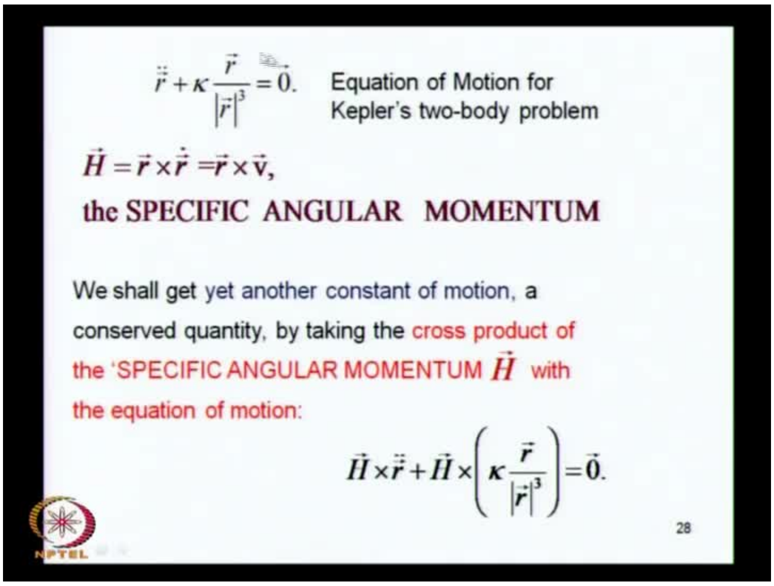

Now, let us play with the equation of motion a little bit further. We have already discovered very nice feature out of it. Now, we have this specific angular momentum, which is the conserved quantity. Now, what we will do is construct the cross product of the specific angular momentum with the equation of motion. Again, we are playing the equation of motion.

We will begin with the equation of motion. It took its projection of velocity and got the conservation of energy. We began with equation of motion and took the cross product with the position vector. We discovered the conservation of energy, conservation angle of momentum. Now, we began with the equation of motion. We constructed the cross product of the equation of motion with the specific angle of momentum and ask ourselves- is this going to lead to any new conserved quantity? If it does, what is the associated symmetry and that is the question that we will now address. We are used to the fact that if there is conserved quantity, there will be an associated symmetry and vice versa.

We also got use to the idea that given the equation of motion, we can discover new conserved quantities. So, we play some more and think about constructing the cross product of the equation of motion with H, which is the specific angular momentum. Will it lead us to another conserved quantity? Yes or maybe, no. Let us do this and find out, but let us also keep this in mind that if we do run into another conserved quantity, what kind of symmetry will be associated with this. That is the another question that we shall need to deal with.

Let us construct the cross product of the equation of motions. So, this is the equation of motion. We construct the cross product of this equation of motion with the specific angular momentum, which I write as H. H cross r double dot is the first term and H cross r multiplied by this kappa over r cube coming from the second term equal to a null vector is a result that we get. This is just algebra and this is the vector algebra

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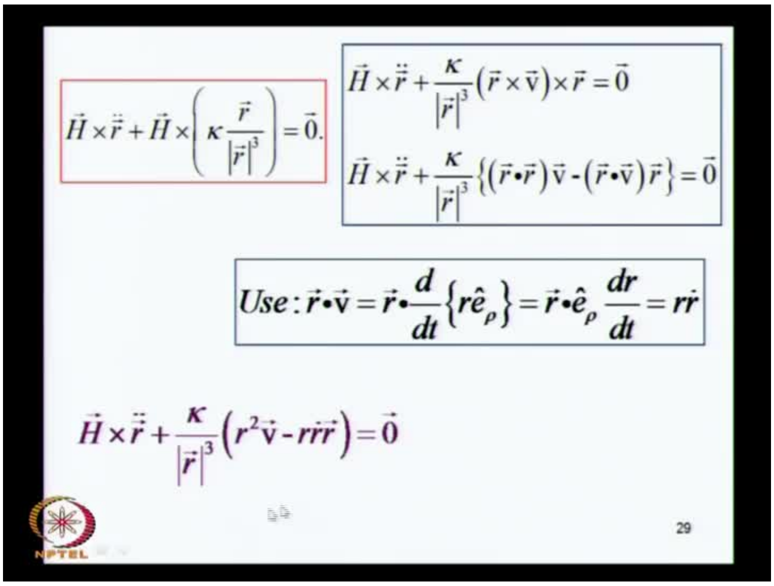

This is what we have. We have composed the cross product of the equation of motion with the specific angular momentum H itself. I only use the definition of H, which is the angular momentum, rather than the specific angular momentum, which is r cross v. I have read it in this, instead of H. I have used r cross v over here and now, I got the cross product of r cross v with another vector - r cross v cross r. So, this is the famous bac-cab rule.

You can expand this cross product, in which there are three vectors and this cross product will give you the r dot r coming from the outer product. This is the one, which is in the middle and comes here as minus the dot product of this outer vector with the adjacent vector, which is r dot v times the remote vector, which is r. So, this is the famous bac-cab rule, which I have used over here. I have simply rewritten this, after expanding this vector product.

Now, what we need is this r dot v r dot r. Of course, we already know r square and we need r dot v. So, let us determine this and v is the time derivative of the position vector, which is the magnitude times direction and this will be rr dot. As we have seen in the previous case, this is rr dot dr by dt, which is the rate of change of the magnitude of the position vector. Now, what if you combine these results, you have H cross acceleration coming from the first term. In the second term, you have kappa over r cube and what you have in this bracket is r square v minus r dot v is this rr dot and this position vector r. So, this is just a simplification of this algebraic relationship that we have been dealing with.

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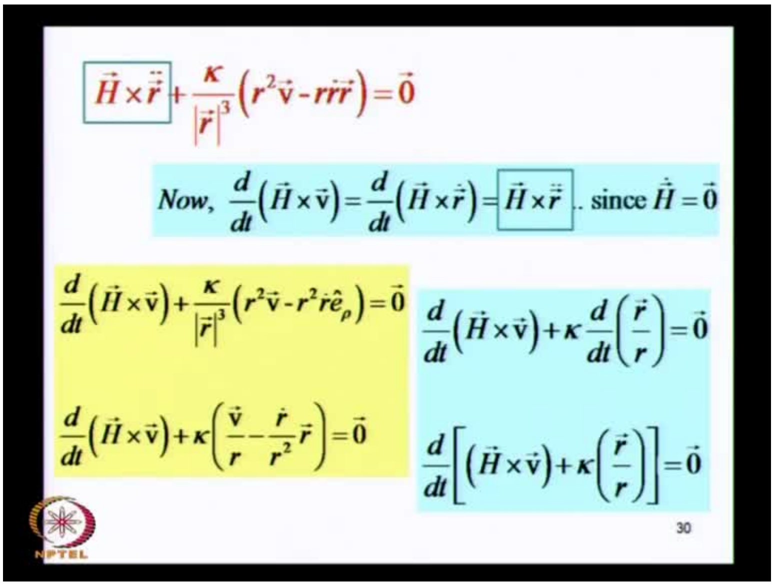

Let us play with this further. We recognize the fact that if you take the time derivative of this cross product of H with v, this will give you d H by dt cross v plus H cross d v by dt. Here, d by dt of H cross v will give you d H by dt cross v plus H cross d v by dt, but d H by dt is the rate of change of the angular momentum. We already know that the angular momentum is conserved. So, d H by dt will vanish and you have the second term, which should go to 0. So, you get the time derivative of H cross velocity, which must be equal to H cross r double dot. This is the quantity, which is appearing in the first term over here. So, for the first term, we can write the time derivative of H cross v. I written the time derivative of H cross v and the remaining terms. I have written as r and the whole thing equal to 0.

I now take this 1 over r cube inside this bracket. So, r square 1 over r cube gives me 1 over r. The first term become v over r and the second term gives me r dot over r square. This r square will cancel r square and will give me only 1 over r, but there is a unit vector over here. So, I write it as r over r square. Remember that all we have done is to use equation of motion played with it. You take the cross product with this specific angular momentum. Everything else is the consequence of the vector algebra and no new principles are inserted in our analysis.

Now, what we have, if you look at this quantity here, v over r minus r dot over r square. This is nothing but the derivative of this unit vector because this is the derivative of a ratio. So, it is 1 over r times d r by dt 1 over r times the rate of change of the position vector, which is velocity over r. Then, minus the position vector times the derivative of 1 over r, which is minus 1 over r square and that is what you see over here.

This second term can easily been seen as the derivative of this unit vector. Having recognized this, we noticed that the two terms involve time derivatives. You can write this as a common differential operator and time derivative of the first term. It is H cross v plus kappa times this unit vector. What you have here is a conserved quantity because the quantity in this bracket must be conserved, since its time derivative vanishes.

Do you see how we got a conserved quantity? We began with the equation of motion played with it. We did some algebra by taking the projection of the equation of motion on the velocity. We discovered the conservation of energy, when we constructed the cross product of the equation of motion with the position vector. We discovered the conservation of angular momentum.

Conservation of energy is associated with symmetry and with respect to time evolution. Conservation angular momentum or specific angular momentum is connected with the central field symmetry or the rotational symmetry. Now, we have discovered that by composing the cross product of the equation of motion with this specific angular momentum. We have discovered that d by dt of this quantity in this square bracket goes to 0, which means that the quantity in this square bracket is constant. Therefore, we have discovered a conserved quantity and it does not look over. Energy does not look like angular momentum. What is it? It is a new physical quantity, which is conserved.

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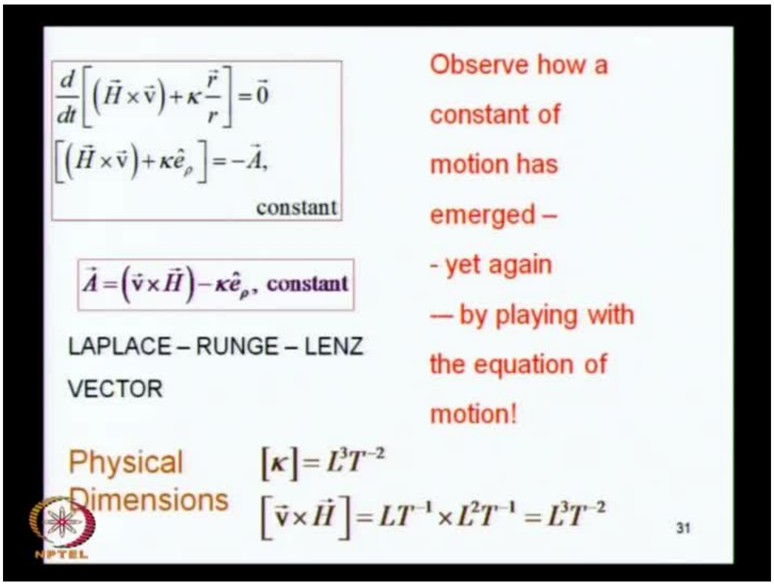

This is the conclusion that we get. The time derivative of this combination of vectors goes to 0, which means that the vector inside this square bracket is equal to some constant vector. I call this constant vector as minus A. It does not matter and minus sin has got no specific meaning, you can always write it as b and say that that is the constant vector that define negative of that. It does not matter minus A is as constant or A itself because minus 1 does not change in physics.

Here, we discover a conservation principle and it is not one of the usual conserved quantities. It is not energy, it is not angular momentum and this is a new conserved quantity, which we have arrived. We have arrived at this by playing with the equation of motion and this quantity is called as the Laplace Runge Lanze vector. It is got a very nice name; this is the Laplace Runge Lenze vector. Depending on what book you are reading, you might read this differently. Instead of v cross H, you might read it as p cross l, where p is the linear momentum and l is the angular momentum. I am developing it, in terms of this specific mechanical energy, this specific angular momentum. So, the mass does not appear in this, but it is the same vector and same physics.

This quantity call is the Laplace Runge Lenz vector and this is the definition of the Laplace Runge Lenz vector. It is v cross H in the units, which we are developing in algebra. We cross, H minus kappa time 0 and this is the constant of motion. It is the conserved quantity and we have discovered a constant of motion, yet again by just playing with the equation of motion. I want remind you that the dimensions of kappa are L cube T to the minus 2. The dimension of v cross H will be L cube T to the minus 2 because if you see p cross L, you will have different dimensions.

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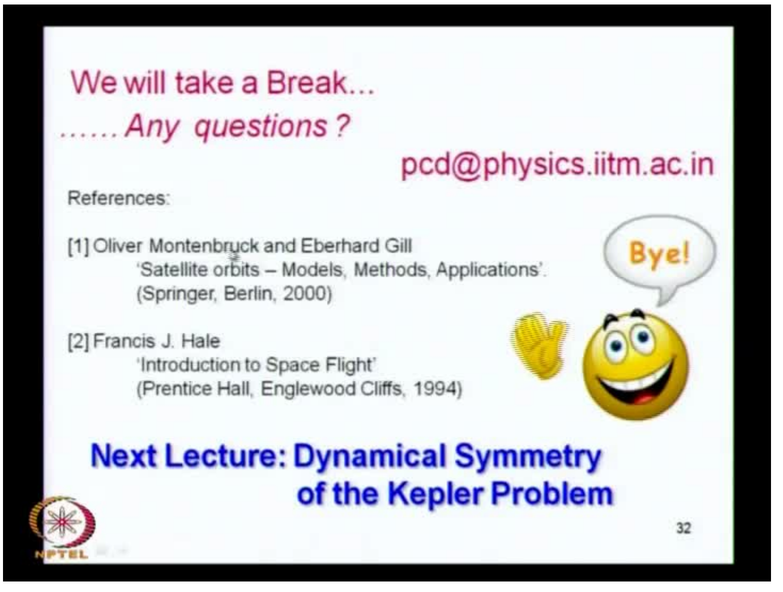

Now, we will take a break at this point. If there is any question, I will be happy to answer. We will continue from this point in the next class. So, these are some reference, which I would like to bring to your attention. This is very nice book by Montenbruck and Gill on satellite orbits – Models, Methods and Application that you might want to read, Francis hale has a book on Introduction to Space Flight and you might to want to read that book. We will take a break after this. In the next class, we will discuss the dynamical symmetry of the Kepler problem because we have discovered the conservation principle.

Now, you must ask what associated symmetry is. When we discover the conservation of energy, we knew that the symmetry involves a symmetry with respective temporal evolution changes in time translation along the time access. When we ran into conservation angular momentum, H is the specific angular momentum. We associated it with the central field symmetry with the rotational symmetry. Now, we have done into a new quantity, which is conserved; namely the conservation of the Laplace Runge Lanze vector, which is the constant quantity. We must ask what the associated symmetry is. So, we will discuss this question in the next class. Thank you.