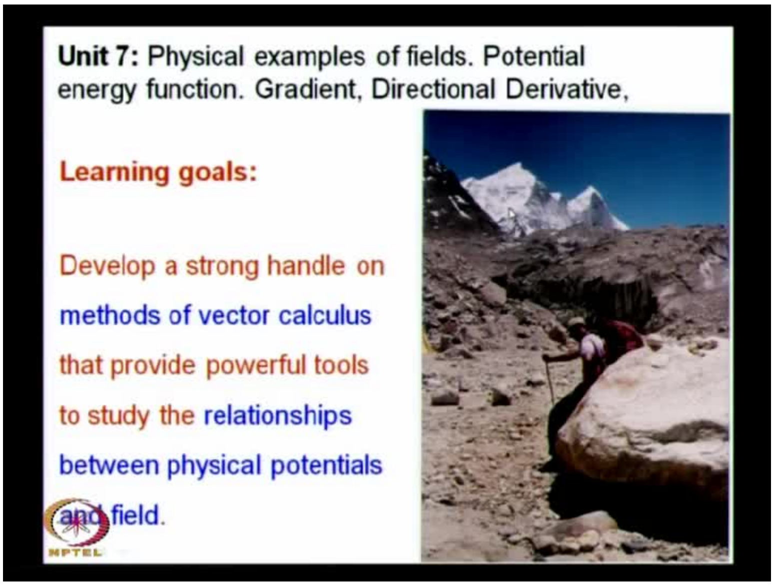
Select / Special Topics in Classical Mechanics

Prof. P. C. Deshmukh
Department of Physics
Indian Institute of Technology, Madras

Module No. # 07
Lecture No. # 23
Potentials Gradients Fields (i)

(Refer Slide Time: 00:32)



Greetings, welcome to the 7th unit of this course, and we are now into the second half of this course, and we shall discuss in this unit potentials, gradients, fields, and some related ideas. Our learning goals are to get us very strong handle on methods of vector calculus, because you know there is a very intimate relationship between mathematics and physics which we exploit for various applications in solving problems in physics.

(Refer Slide Time: 01:16)

So, we will find rigorous connections between potentials and fields. That, what you have over here is a picture of this glacier which is Gomukh, and behind this is the Bhagirath peak, and the river Ganga, it originates at the Gomukh, and the question is once this glacier starts melting and, you know, Ganga gets form drop by drop [fl]

(Refer Slide Time: 01:39)

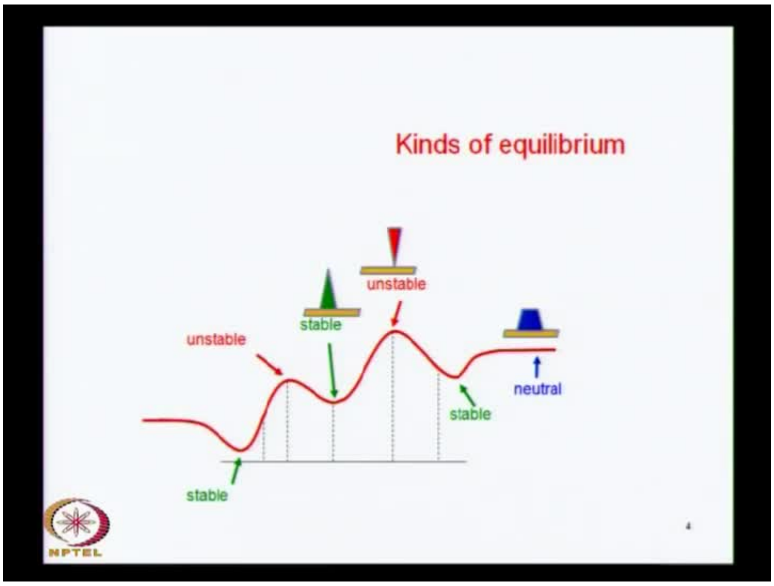
What path does the river take? That is a question we are going to address. There it is starts from Gomukh over here, and then, goes through, you know, Gangotri, and then, Rishikesh, Haridwar, then runs through parts of, you know, Uttar Pradesh, Kanpur and then Prayag, and goes through, you know, Bihar, and then, into the bay of Bengal.

So, beginning with the melting of the glacier Gomukh, at Gomukh, how does the Ganga flow? Why does it take the path that it actually does? When water is flowing and it goes in a certain way, why does it go this way and not the other way?

There is a certain path that the water takes, and it all happens naturally, thank you. So, once the river starts flowing, it takes a particular path, and we all know that water goes along the path of what we call as the steepest descent.

If the slope is steeper in this direction compare to any other, this is the path along which the water would flow. Now, this is the idea whose rigorous form we shall discuss in this unit, because it has to do with slopes; it has to do with tangents; it has to do with something changing which makes the water flow from one point to another, what is it that changes? And how does this change result in the water flowing in a particular direction? So, all of these ideas are related to the notion of potentials, gradient, steepest descent, and so on.

(Refer Slide Time: 04:09)

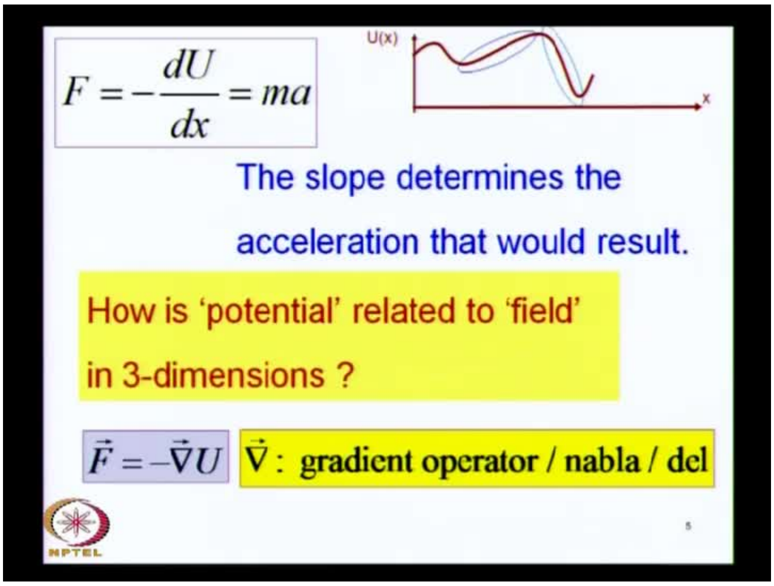


So, we do know that when an object in a state is in a state of equilibrium, it will maintain its uniform motion, and along a region of flat space, as you see over here, an object at rest will remain at rest; if it is moving at a constant speed, it will keep moving at a constant speed; its state of equilibrium will remain undisturbed, determine completely by the initial condition, but if it is rolling, if you have got a marble which is rolling from right to left over here, and it comes over here with where the potential is dropping, then the marble is going to pick up speed; it is going to get accelerated.

So, this is what is going to happen when the potential changes, and at what rate this potential changes will determine how much acceleration will result in the marble.

So, we have some sort of intuitive idea about these issues and we will try to develop it quantitative estimate of these connections, because as we can see, they have clear applications in various branches of mechanics, and in fact, all branch of physics and engineering.

(Refer Slide Time: 05:38)



So, what needs for acceleration to result, this is acceleration, is that the potential must change as long as the potential is constant? There will be no acceleration, potential must change with the x; this is the simple one dimensional illustration. So, unless there is a rate of change of potential dU by dx, there will be no acceleration, and in the Newtonian idea of F equal to ma, this acceleration will result whenever the potential changes with respect to x, and the quantitative relationship is given by this F equal to minus dU by dx. So, here is the slope which is going to determine how acceleration will result.

Now, the slope is obviously fundamental and how steep the slope is will determine the actual quantitative estimate of the acceleration. So, for example, over here, you see that there is a certain slope. On the other side of this, unstable equilibrium, in on this side, this slope is steeper.

So, if there is a marble at the top, at the point of equilibrium, whether you move it to the left or right is going to determine whether it moves into a region where the slope changes less rapidly or more rapidly, and it will gain that much more speed, the acceleration will be higher, if the slope is higher.

So, these are the physical ideas which go into our consideration, and we are familiar with this relationship between the force and the potential in 1 dimension, and now, we ask the question, what is this relationship in 3 dimensions? In 3 dimensions, you will need to deal with vectors, because any physical quantity which has got directional attributes could have components along any one of three linearly independent directions.

So, you can have a three dimensional space, and in this three dimensional space, we must generalize this expression, and this generalization is achieved through an idea which is very similar to this. You have the primary idea is that of derivative; this is derivative with respect to a space coordinate.

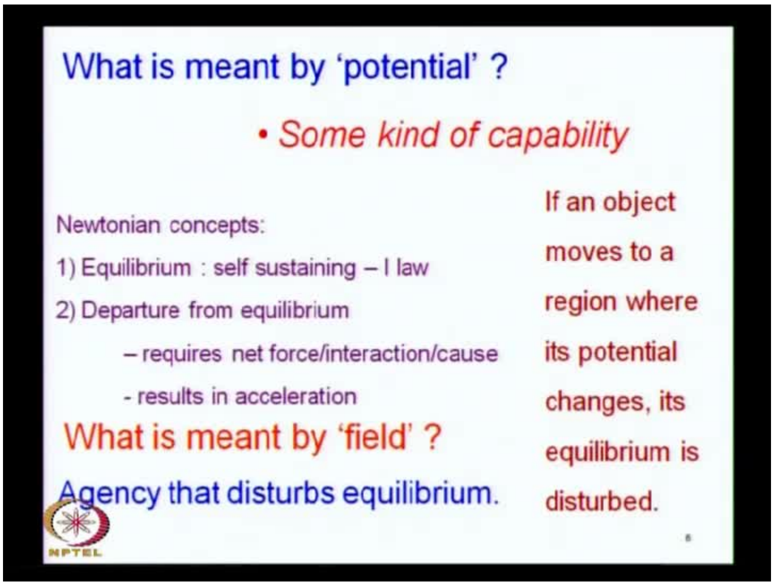
So, this idea of derivative with respect to space coordinate d by dx is what we must extent to 3 dimensions, and when we do so, we come into an operator which is known as a gradient. So, the 3 dimensional analog of this expression F equal to minus dU by dx is this force which is now a vector which is equal to the negative gradient of U.

So, here you have an operator this triangle with an arrow on top; this is an operator just the way d by dx is the differential operator. So, this operator, this triangle, let have which is read as a gradient; this is an operator, this operates on the potential which is a scalar function, and the result of this operation to get with this negative sign will give you the force. So, this is our definition of a gradient.

Mind you, it is an operator, and it is a vector operator which is why you put an arrow on top of it. It is not however a vector, it is a vector operator and operator must seek an operand on which it would act, and once it completes that operation, you get a result which is amenable for interpretation.

So, this is an operator, this is a vector operator, it is also called as nabla or sometimes just as Del operator, and all of these are used synonymously in this context.

(Refer Slide Time: 10:18)



So, we are going to discuss the gradients of a potential, and let us discuss the idea for potential further. What is the potential? Potential is, you know, when your dad says that you have tremendous potential, they talking about a capability. A certain ability which out of their love, they hope you have, maybe there are other reasons also which are equally good or even better, but certainly the love is perhaps one of the strong reasons.

And it is some kind of a capability, and this capability would manifest when you perform, until then it is just a property that you have, and only when you perform, the result of this capability becomes manifest.

So, potential in the context of physics is a somewhat similar idea, and what it tells us is that as long as a body is in a state of uniform motion, when it is in a state of inertial motion, this is what Galileo recognized as the law of inertia that in an inertial frame of reference motion of an object is self-sustaining.

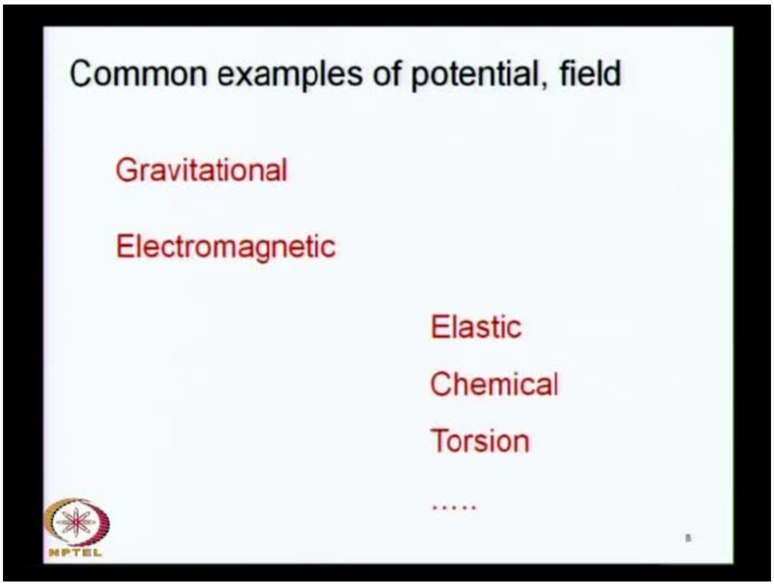
And this is what happens when you have to look for no cause to explain this, because it is completely determined by initial conditions, and this is what happens as long as an object has its motion beat one of rest all of uniform motion. In a region where there is some property which remains constant, this property is a potential.

As long as the potential is constant, its equilibrium will not be disturbed, and the equilibrium will be disturbed, if either you apply an external force on this object, or during this motion, the object finds that it has reached a point where the potential certainly changes. For example, if this bottle were rolling and on a frictionless table, then, can you get this on the camera? If this bottle is rolling and it keeps rolling on a frictionless table right up to the point added meets the edge, but here the potential changes and it would drop off. So, that is when you will see that the uniform motion is disturbed.

So, as long as the object moves in a region of constant potential, there will be no acceleration, and what causes the equilibrium to be disturbed is an interaction, or what in physics, we call as a force or a field.

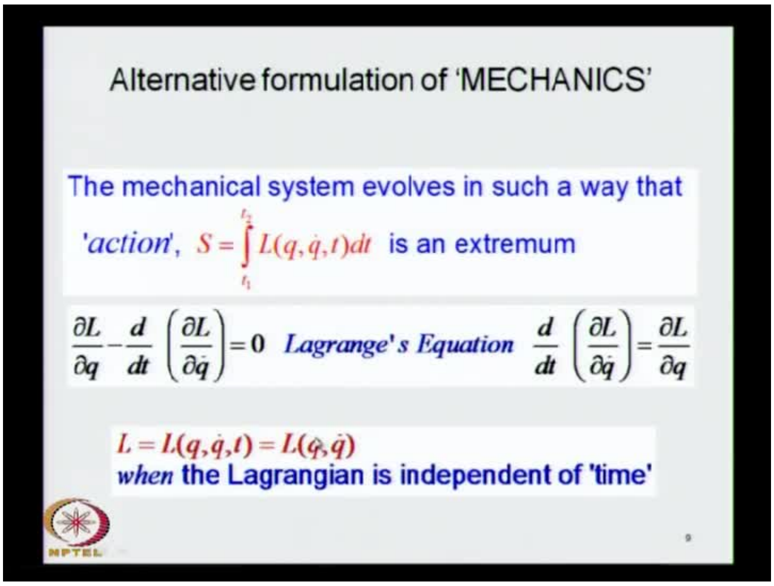
So, these are all related ideas. In a particular context, they acquire more precise meaning and the context will reveal this meaning to us, but these are all related ideas, that these are the ideas of force and interaction of field will be necessary to cause acceleration; in the absence of this, it is only uniform motion which will be sustained.

(Refer Slide Time: 14:50)



So, this is field is then the agency that disturbs equilibrium. Now, the interesting thing is that there is a very rigorous quantitative relationship between force and a field. So, let see some of the common examples which you are certainly familiar with. The gravitational potential, for example, the electromagnetic potential, then there is something that you might call as the elastic potential, chemical potential, torsion potential, these are various forms of potentials that one comes with across.

(Refer Slide Time: 15:12)

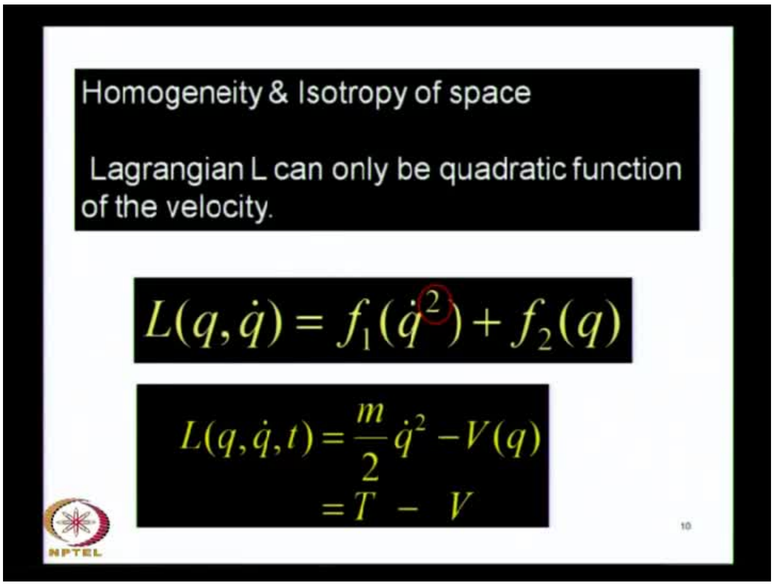


We learning to this idea even in the alternative formulation of mechanics which is not the Newtonian formulation but one that is based on the principle of variation, and in the principle of variation, the evolution of a mechanical system is not explained in terms of the principle of causality and determinism, then Newtonian idea of force is not used, but when explains the evolution of a mechanical system by saying that the system evolves in such a way that the action which is this integral of a quantity that we call as Lagrangian is an extremum, and we have discussed this in unit 1, so, I will not spend any time discussing it in details but just remind you that this is an alternative formulation of mechanics which also leads us automatically to the notion of potential.

So, not just from the point of view of potential whose negative gradient gives you the force, the Newtonian force, but also the potential which appears in the Lagrangian. Now, the Lagrangian is one whose time integral which is an action give is an extremum between the initial state and the final state, and the principle of extremum action leads us to the equation of motion which is Lagrange's equation or one can also go further and get the Hamilton's equations.

And we shall deal with those situations in which the Lagrangian is independent time, and in this case, the Lagrangian is a function of position q and velocity q dot. Position is what I denote by q and the velocity is q dot.

(Refer Slide Time: 17:07)

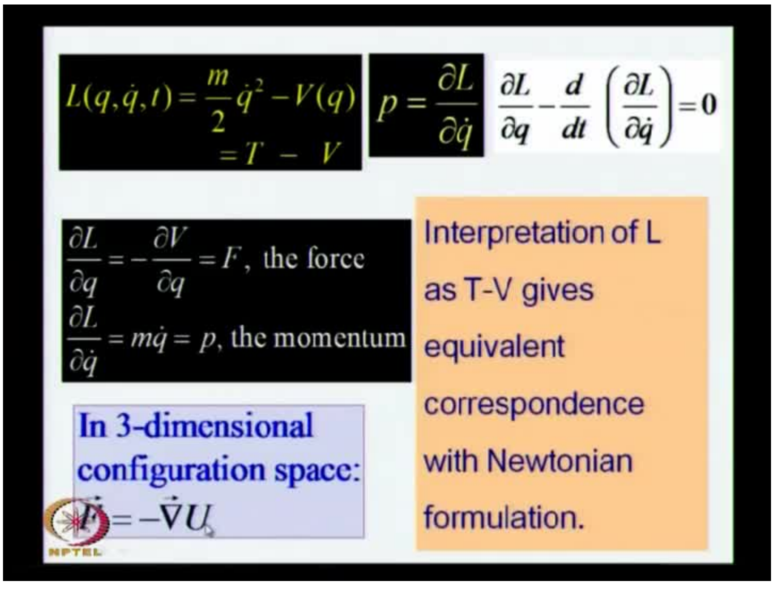


And you will remember that in unit 1 we discuss that in a space which is homogenous and isotropic, the Lagrangian can only be a quadratic function of the velocity; its must depend on the velocity; L must be a function of q and q dot, but in isotropic space, it cannot depend on any direction, now, velocity is a vector.

So, you need a quantity which depends on the velocity but not on its direction. So, it must depend on V dot V rather than on V itself, because V dot V will depend on V, but V dot V is a scalar and it is independent of the direction.

So, the Lagrangian must be only a quadratic function. So, it must be a function of q dot square, and when you write this Lagrangian as the simplest combination of a function of q dot square and q, then a direct interpretation of this function as the kinetic energy and f 2 q as the negative of the potential energy so that the Lagrangian turns out to be T minus V. This interpretation gives direct connection between the Lagrangian formulation of mechanics and the Newtonian formulation of mechanics.

(Refer Slide Time: 18:39)

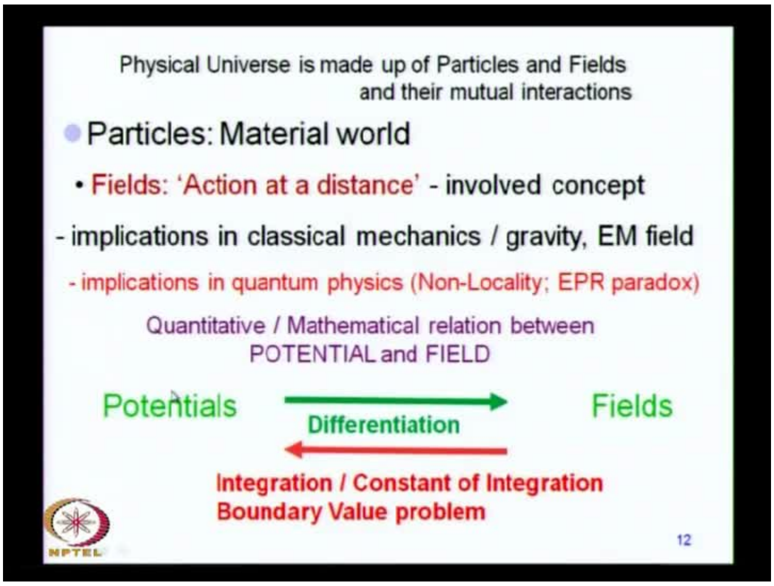


Because these cannot be independent of each other; they cannot give you two different results, they must give you results which completely correspond with each other, and that correspondence is easily seen, because the moment you recognize your generalized momentum as the partial derivative of the Lagrangian with respect to the velocity.

Then you find that the partial derivative of the Lagrangian with respect to q del L by del q. From this relation del L by del q will be minus del V by del q, this is the Newtonian idea of the force, and del L by del q dot which is here, which is the momentum itself, which is mass times velocity, and its time derivative d p by dt will be equal to the force. So, this comes from the Langrage's equation because del L by del q is minus del V by del q, so, this is the force, and this becomes mass times acceleration, because this is the momentum and the rate of change of momentum gives you the force.

So, this gives you the direct correspondence between the Langrage's formulation of mechanics and the Newtonian formulation of mechanics, and the potential appears in the Lagrangian as T minus V, so, L equal to T minus V gives you the correct form of the Lagrangian which you can relate to Newtonian formulation of mechanics.

(Refer Slide Time: 20:10)



Now in three dimensions I have pointed out that the force turns out to be the negative gradient of the potential, and we analyze the physical universe around us as made up of particles and fields. The particles are constitute the material world around us.

And the fields provide us some mechanism to deal with action at a distance. Now, this is really a very involved concept, it is used very easily in physics, but it is a very involved concept. It has very intricate applications in classical mechanics, and one there by talks about the gravitational field, the electromagnetic field in classical mechanics, and essentially what it does is the presence of a field enables an object which generates the field to perform action at a distance. So, this is the idea in classical mechanics, like gravity for example.

So, the earth generates a gravitational field and this field is able to influence other objects which have got masses wherever they are, whether it is over here or here, so, the gravitational field of the earth operates at different points in space. So, it is the point function we changes from point to point.

And there are similar ideas in the in electromagnetic fields which are generated by charges and currents, and the idea has even more certain applications and quantum mechanics, because there is this idea of non-locality which was contained in the EPR paradox and so on, and I will certainly not go into these issues at this point, but the action at a distance is a very settle idea and it has got many connotations in physics, in classical mechanics, and quantum theory, and so on.

Now, what is interesting is that there is a very rigorous mathematical quantitative relationship between potential and field.

Now why mathematics play such a strong role is a question one might ask and I am not sure that there are easy answers, but you can go from the potentials to the fields through mathematical operations by carrying out differential calculus.

So, you take the derivatives of the potential. If, once you use the derivative operator, now this operation in the middle differentiation is a mathematical process which connects two physical entities. This is a physical entity field and potential is a physical entity and these two physical ideas are connected to each other by mathematics. Others are part of the region; Galileo said that mathematics is the language of physics.

So, these are two completely physical ideas and they are rigorously quantitatively exactly connected by a mathematical operation which is differentiation, which is to carry out the derivative of a function.

(Refer Slide Time: 23:56)

Likewise, if you carry the inverse process of integration, solve it is a boundary value problem, plug in the constant of integration; from the field you can get the potential. So, these two are rigorously related by mathematical formulations and its seems almost unreasonable that mathematics is so effective in natural sciences, and I will draw your attention to a very exciting article by Eugene Wigner which was published in communications in pure and applied mathematics in 1960.

The title is the unreasonable effectiveness of mathematics in natural sciences, or whether it is unreasonable or not matter of, you know, semantics and you know that is what Wigner called it in his title.

But it is not exactly unreasonable; perhaps, there are good reasons for it. There are, there was another article by Maxborn, in which, he points out why mathematics and physics are really interrelated and it is really not separable in some sense.

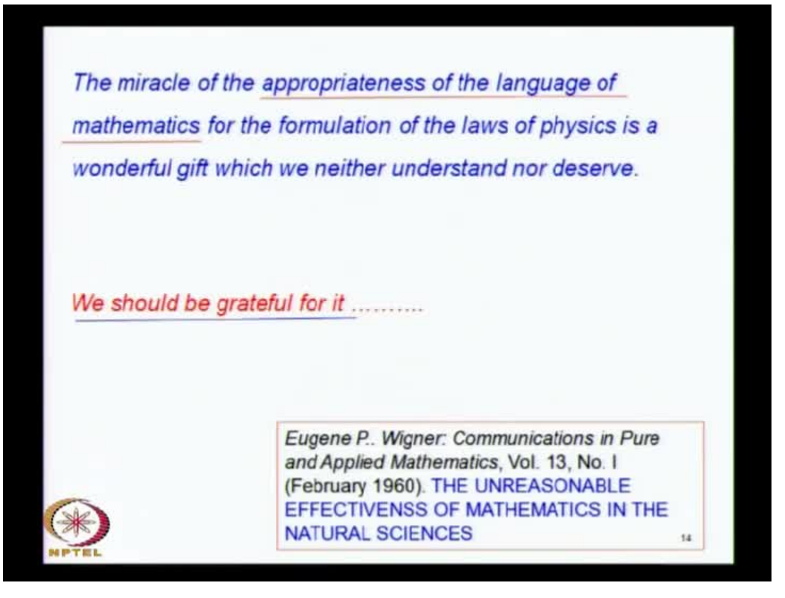
And in this article, what Wigner says and he quote Schrodinger in this article. He says that it is a miracle that in spite of the baffling complexity of the world, certain regularities in the events could be discovered.

That when you observe, (( )) almost everything is unpredictable means, you cannot predict really the future, and I think that is good. But the fact is that you cannot predict the future, and one knows it, one accepts it. Despite that, there are certain things you can predict.

For example, I can hold this bottle in my hand and let go, and before I let it go, I can predict that it is going to fall; I can also predict that its speed will increase, that it will get accelerated, I can also tell you how much the acceleration will be it will be 9.8 meters per second per second.

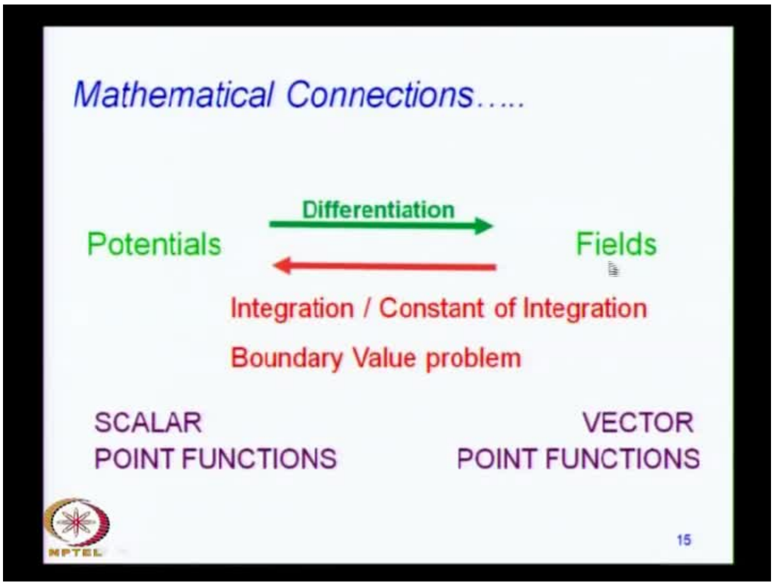
That there are certain things which can be predicted, these are what you call as the laws of nature; this is what one means by laws of nature that, there are, these are certain regularities which come out and these regularities which come out which are the laws of nature can be predicted not just qualitatively but quantitatively using precise and rigorous mathematics and that is what this unit is about.

(Refer Slide Time: 26:35)



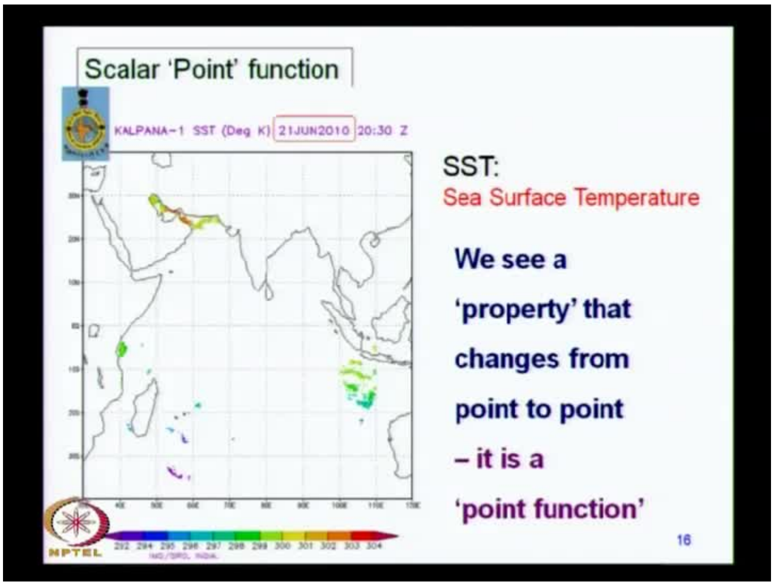
Because, it will tell us how the potential is connected rigorously, quantitatively with fields and how do you determine these quantities. In the same article, Wigner goes on to say that the miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve, but the good thing is that we should be grateful for it, because we can use it constructively and that is what we will go about to do now.

(Refer Slide Time: 27:02)



So, this is the connection. That from the potential by carrying out differentiation, you can get fields by carrying out the inverse process of integration; you can get potential from the fields; you have to set up the boundary values, and these are certain functions which are called as point function. So, the potential is the scalar quantity. We know what a scalar is? It is a scalar point function and a field is a vector point function, so, we need to introduce these ideas now rigorously.

(Refer Slide Time: 27:38)



Now, what is a scalar point function? Let us consider what is called as SST? The sea surface temperature and you can takes the satellite pictures and map the surface temperature at various points on the sea, and here is a satellite map which was taken two days ago on twenty first June by the Indian satellite known after Kalpana Chawla. So, these are images taken by Kalpana satellite.

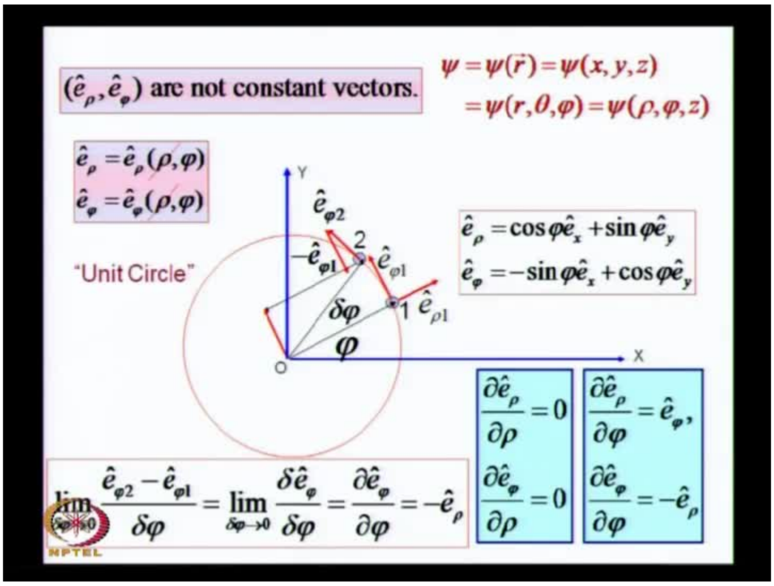
And what it does is it maps this sea temperature, you see the Indian Ocean here, and there is a temperature gradient, so, these are you know regions which are at a slightly warmer temperatures like 304 kelvin; this is the temperature color code.

So, this is about 303, 304 degrees Kelvin. If you go to the south and this being summer in India, it is winter in the southern hemisphere. So, you go south of equator on the southern hemisphere, the temperatures are somewhat lower, and these are called a temperatures.

Essentially what you have is a property, namely the temperature which changes from point to point; it is one thing over here, another over here, another over here, and as you go further down it changes.

It may remain in variant over a certain extended region of space that is a different issue, but it has got a certain quality, a certain property, which changes from point to point. When it is dependent on the particular point that you are talking about, it becomes what is called as the point function. The quantity itself is an invariant with respect to rotation of any coordinate system and it is therefore a scalar. So, this is what you call as a scalar point function.

(Refer Slide Time: 29:52)



Let us consider some other examples and we are going to describe these points in various coordinate systems. We could describe a point on a flat space by its Cartesian coordinates x and y; we could do it in terms of the polar coordinates like cylindrical polar or plane polar coordinates rho and phi, and when we use this coordinate system, the unit vectors are not constant vectors, so, we have to remember that. We discuss these things when we did unit 3, so, again I will not spend any time, you know, deriving these relations but I will use these results.

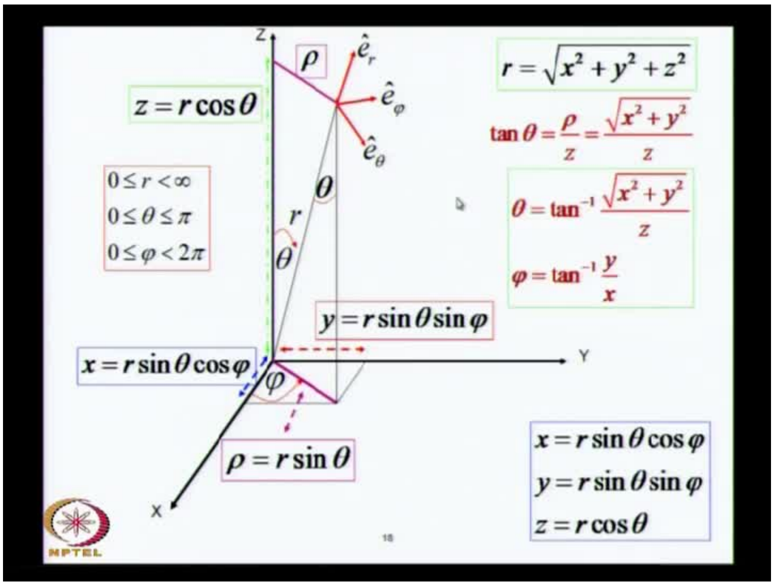
So, I will quickly remind you that you can describe a point in space either in Cartesian coordinates or in polar coordinates, cylindrical polar, spherical polar, or any coordinate system, it say irrelevant.

When we talk about scalar point functions, the coordinate system is not of important. So, what is of importance is the fact that you are talking about a physical property which dependence on a particular point.

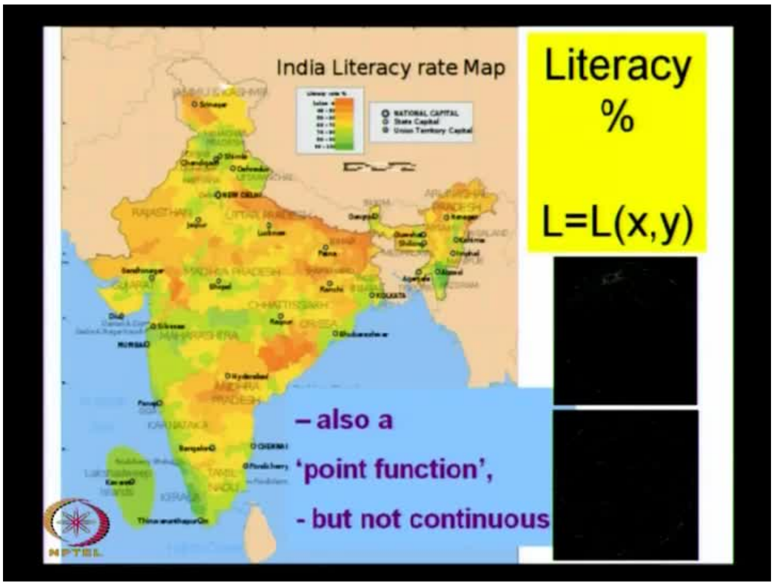
So, these unit vectors of the coordinate system of the plane polar or cylindrical polar coordinate system, these unit vectors change from point to point, and you can determine at what rate these unit vectors change with angles. Now we have derived these relationships in some detail in unit 3. Are we know how the unit vectors change with the angles? They do not change with distance from the origin we know that as well.

And the upshot of this that a scalar point function which is a scalar has a particular point whose position vector is r as its argument, but this position vector can be expressed either in Cartesian coordinates x, y, z, or in spherical polar coordinates r theta phi, or in cylindrical polar coordinates r phi and z, and we will use all of these coordinate systems, because depending on the convenience, that is offered in a given situation. We can then exploit the convenience of any coordinate system and work out our relationships accordingly. So, we need to develop the expressions for the gradient in all of these coordinate systems. So, that is the idea.

(Refer Slide Time: 32:23)



(Refer Slide Time: 32:52)



This is to remind you the spherical polar coordinate system. So, here the unit vectors are e r, e theta, and e phi, and the spherical polar coordinates are r theta and this azimuthal angle phi and this also we have done in some details in unit 3. So, I will quickly only remind you of the primary relationships which we shall use.

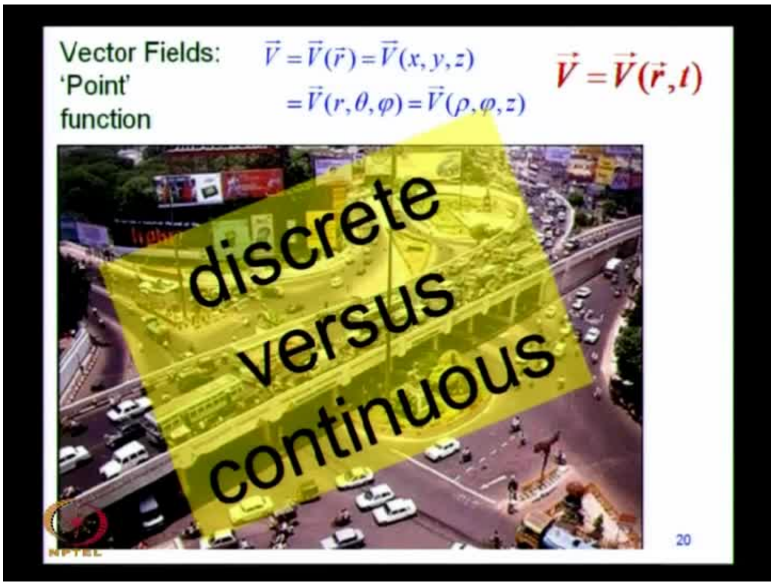
And we have agreed that we can describe a point in space, in any coordinate system, and we can talk about any kind of physical property. Here, we talk about the literacy rate in India, which is different in different parts of the country. So, it is very high here in Kerala and it is rather low in some other parts of the country; there is a color code also associated with this.

What is significant over here is here again we are talking about a property; we changes from point to point; we changes from one region to another. So, it is some sort of a point function. Nevertheless, this point function is really not continuous, because there will be region in between like forest and mountains, where there is no habitation at all. So, the question of having any literacy over there does not really arise.

So, this is a point function in a certain sense but it is not a continuous point function and we have to, we can describe it in terms of coordinates, and we can write the coordinates either as x and y or we can give it in terms of the longitudes and the latitude, it does not matter. Anyway, which is convenient, we can describe a point in space and we have these point function.

So, now, I believe, I hope that the idea of a point function is going home, and these point functions, you know, properties of this kind are not continuous, whereas, some other properties are continuous.

(Refer Slide Time: 34:40)



Look at this this is a picture of the Gemini flyover in here in Chennai and some of you would have met the traffic chaos over here, so, you are familiar with this picture very much. This is obviously a photograph, so, it is taken at a particular instant of time when the photograph is taken.

So, if you talk about the velocity at a given point in space, and we can talk about x, y, z, because there is a flyover. So, at the same x y coordinate, you could have a vehicle moving at a certain velocity on the lower plane and another vehicle moving at another velocity at the higher plane on the flyover.

Now the velocity is a vector quantity and it can depend on x and y and z, all of that, or you can describe the position vector in spherical polar coordinates r theta phi or in cylindrical polar coordinates r phi and z.

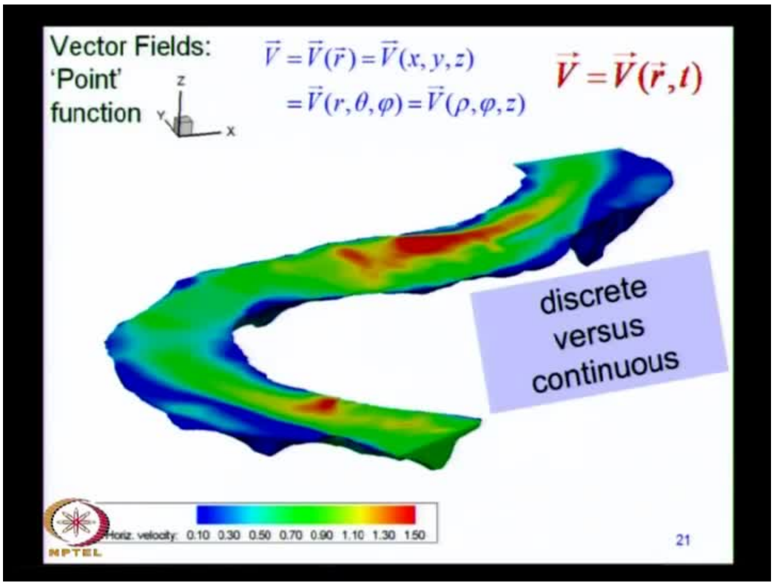
Now, here is an example of a velocity vector point function which changes from point to point. The velocity of a pedestrian here is obviously not as much as the velocity of this curve over here. The motorcycle is perhaps want to be at the highest speed, and furthermore, the velocity may depend not just on the position but also on time, because this is a snapshot taken at a particular time. So, if you took a movie it certainly going to change from time to time.

Here, again we must ask the question about discrete versus continuous functions, because we talk about the velocity of a vehicle or of a person or of a car or a motorbike, if and when, the vehicle is there at that point.

But then there are regions of the road. Here, for example, there is no vehicle at all in this part, at this particular instant of time, which is a rare thing to happen in Chennai that there is any part of the road where there is no vehicle that usually does not happen.

But here, you have a region of the road whether there is no vehicle at all, and obviously this is not a continuous function, and in the absence of continuity, the function will not be analytical and you cannot take its derivatives.

(Refer Slide Time: 37:25)



So, these are involved ideas. If you talk however about the velocity of water, and here you have got water which is flowing along a certain path, and you can actually map the horizontal velocity which in certain units is mapped and given a certain color code in this picture and you can map it at every point in this space.

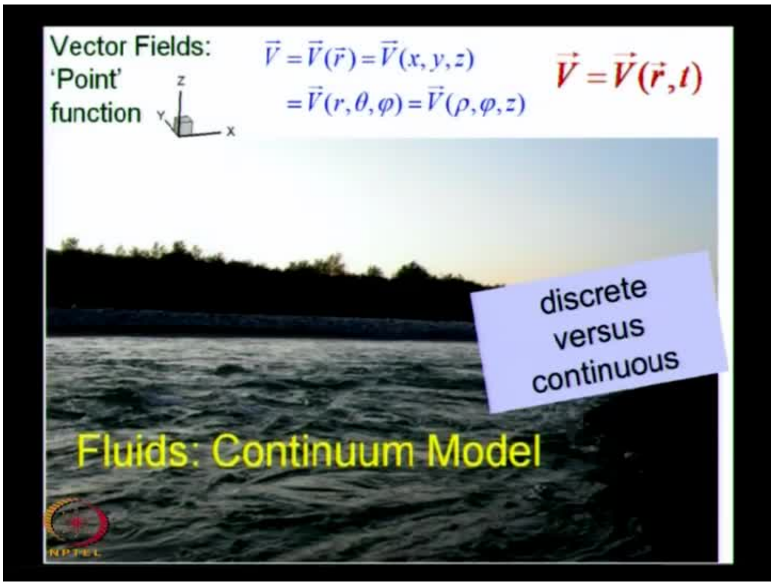
So, this is not like the rare incidence of a road in Chennai where you do not have a vehicle at all, because in this region through which water is flowing, you always have water everywhere. Here, again, this will be a function not only of the point but also of time, because this is the snapshot at a different instant of time, this map will change.

We must again ask this question because if we have to take derivatives, we know that the function must be analytical and it must be continuous, and at what level is it continuous is an interesting question because after all water is made of these water molecules H 2 O and they do not really are single molecules, they existence states of dimmers, may be trimmers, they are tumbling, they are bonded, these molecules are bonded by the hydrogen bond.

And this may be breaking, this may, there may be a dynamic process, or what happens if you, a, because whenever you take derivatives, you take, you know, limits of quantities when they become infinitesimally small like dy by dx is the ratio delta y by delta x in the limit delta x is going to 0.

So, when delta x is goes to 0 you need to make that quantity infinitesimally small, and what happens when this distance becomes smaller than the size of a molecule. A molecule has got two atoms of hydrogen and an atom of oxygen and as delta x goes to 0, this can become smaller than the size of a molecule.

(Refer Slide Time: 39:58)



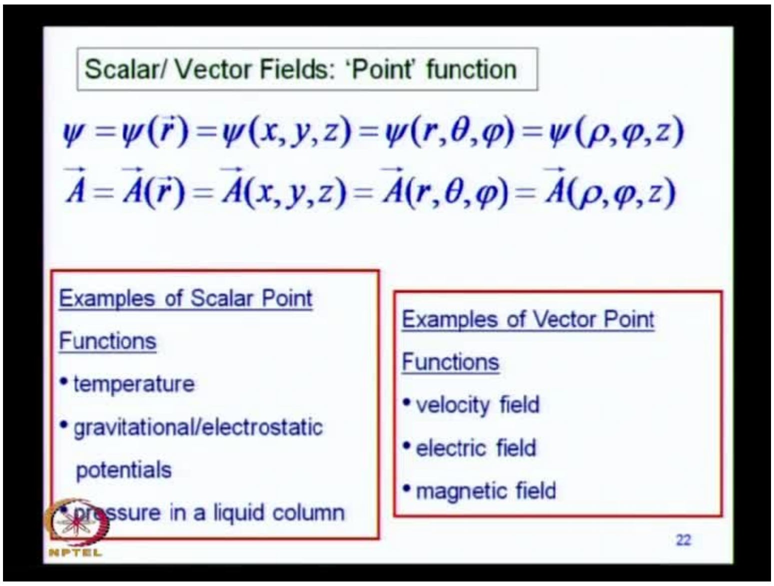
Do you really still have a continuous model over there, and what we, in the context, in which, we will be doing this analysis, we use what is called is the continuum model of fluids, and in this limit, we do not worry about the delta x going to 0 limit.

So, we take small distances which go to 0 which become infinitesimally small, and this is the continuum limit, but it will be large enough so that we do not have to look at the internal structure of molecules and atoms.

So, regions will be considered to be small enough so that we can use the continuum model of fluids, but which are not even so small that we really have to look at the internal structure of molecules and atoms. So, that is not the scale on which we shall discuss these situations. So, this is the continuum model of fluids.

And in this continuum model, you can have the velocity of fluid at any given point in space, in the region through which water is flowing, or any liquid is flowing, and you can define the velocity as a point function. So, this gives us the idea of a vector point function. So, now we know both what a scalar point function is and what a vector point function is. So, they must have continuous derivatives and it is in the continuum limit that we shall regard these functions.

(Refer Slide Time: 41:37)



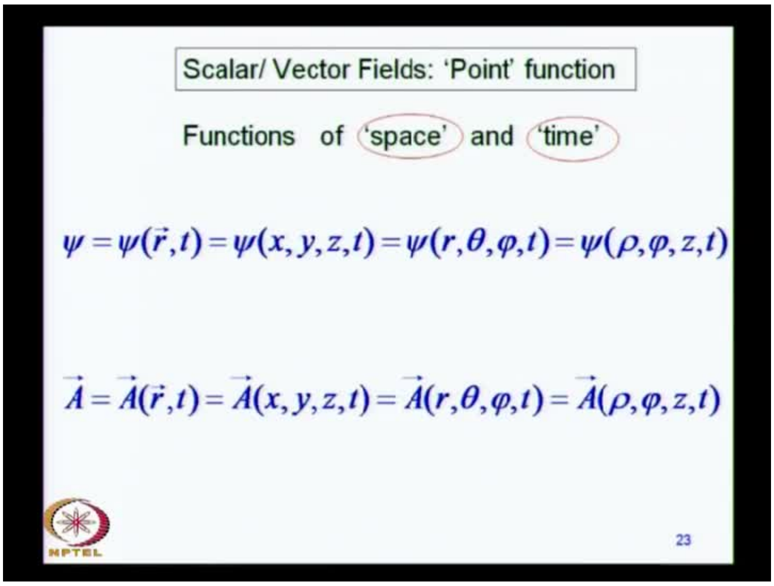
So, both, there are two kinds of point functions: the scalar point function and the vector point function. So, psi is an example of a scalar point function; the vector A is an example of a vector point function. Both are functions of specific points, so, the argument of the position vector of that point appears as an argument of these functions the scalar and the vector point function.

But the coordinates can be expressed either in Cartesian coordinates or spherical polar coordinates or the cylindrical polar coordinates, it does not matter, and our formulation must be able to exploit the convenience of using either the Cartesian geometry. If that is appropriate or the spherical polar, if that is more convenient and we must develop a formulation which is independent of all the coordinate systems.

So, familiar examples of the scalar point functions are temperature that we considered, the gravitational and the electrostatic potentials. The pressure in a liquid, now pressure is force for unit volume. So, force, of course, is a vector, but when you talk about pressure, you normally refer to it as a scalar because, you know, through Pascal's laws the same no matter in which direction it is considered.

So, it is in the limits of these approximations that we talk about these quantities and various examples of vector point functions of the velocity fields that we just discussed, and then, there can be electromagnetic fields, right. So, these are other examples of the vector point functions.

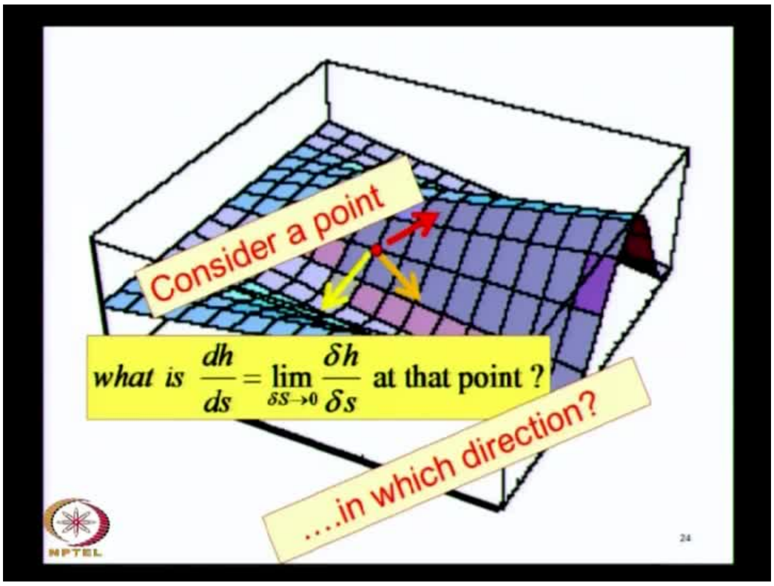
(Refer Slide Time: 43:16)



These we have noticed our functions of space which is what make them as point functions, but they may also depend on time or they may not depend on time. So, the gravitational field generated by the earth is not going to change from today to tomorrow unless the earth loses some of its mass which vanishes into the outer space.

So, by enlarge, these will not be functions of time but there may be functions which change with time. For example, if you have an electrostatic charges distribution which generates an electrostatic potential, if you either add or remove the charges in that region or change the charge density in that region, then, of course, it will change from time to time.

(Refer Slide Time: 44:21)



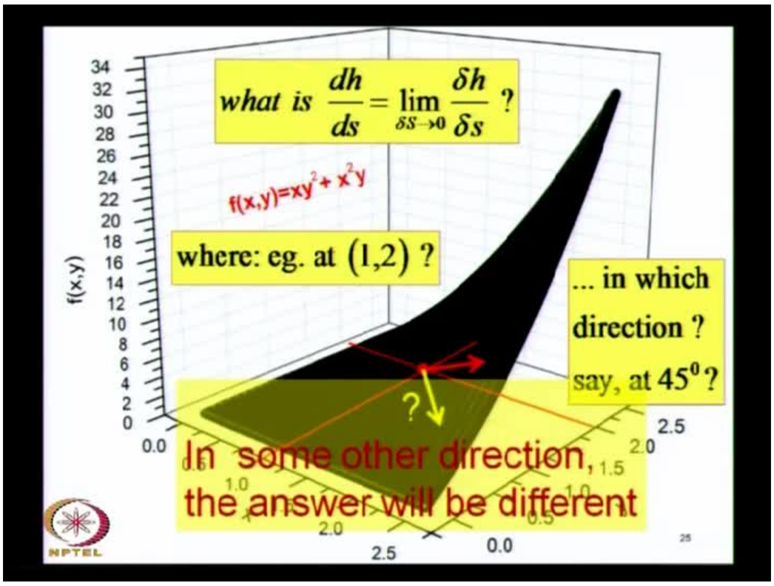
So, these functions by enlarge may or may not depend on time but they must depend on space which is what qualifies them to become point functions, and let us consider such a function here, a here is a function which is a function of x and y, so, as x changes along this line and y changes along this line, here is a certain property. Let us say this is the height of a certain sheet which is spread over this flat surface and this height changes from point to point in space and we ask this question that you consider certain point, let us take this one which is here at this red dot and we ask the following question: what is the rate of change of this height dh by ds? It is obviously the limit delta h by delta s, as delta s goes to 0.

The question we ask is what is dh by ds at this point. We must ask this question for a particular point, because this rate is obviously different for this point and it is different over here; I see that from this figure, it is mostly flat.

So, dh by ds over here is really not changing at all its 0. So, this question must be asked in the context of a particular point in space. So, what is this rate and at which point? We should also ask and then which direction, because the rate at which this height changes, depends on the direction in which you consider this change.

So, if you consider a change in this direction, this rate of change dh by ds is obviously different from what it is in some other direction. The rate at which the height changes, depends not only on the particular point that you are talking about but also the direction in which you are examining this change. If you consider some third direction, again this rate will be different, or it may be the same. That it depends on the details of the surface.

(Refer Slide Time: 46:42)



So, these questions become important and we take a particular example which (( )) plotted for may last night. Here is a surface which is mapped as xy square plus x square y, so, x changes from 0 to 2.5 and y changes from 0 to 2.5. We evaluate f of x y as xy square plus x square y and we ask a related question: what is this rate at which this height of the surface changes with distance? What is this rate dh by ds? We must ask at which point. So, we say, consider a point such as x equal to 1 and y equal to 2 in the Cartesian coordinate system. So, we take this point at x equal to 1 we draw a line and y equal to 2 we drawn another line and the intersection gives us this point, right, and now we must ask, in which direction are we examining this flow?

So, just to take a particular example, we will say how about the direction of 45 degrees. You can take any angle, it does not matter but just for the sake of discussion and illustration, I have chosen this angle to be 45 degrees. So, in this direction, what is the rate at which this height of the surface changes with distance?

Now, obviously you can see that this depends on the direction, in which, you are considering this change; it will be different in different directions. If you consider this change in different directions, the answer will be different.

And if you have a drop of water over here, it is going to move along the line of what you called as the steepest descent. So, what are we talking about we are talking about the derivative. A derivative here which is dh over ds or ratio delta h over delta s. This is a ratio of two scalar quantities which is obviously a scalar whose value however depends on the direction in which you are measuring it.

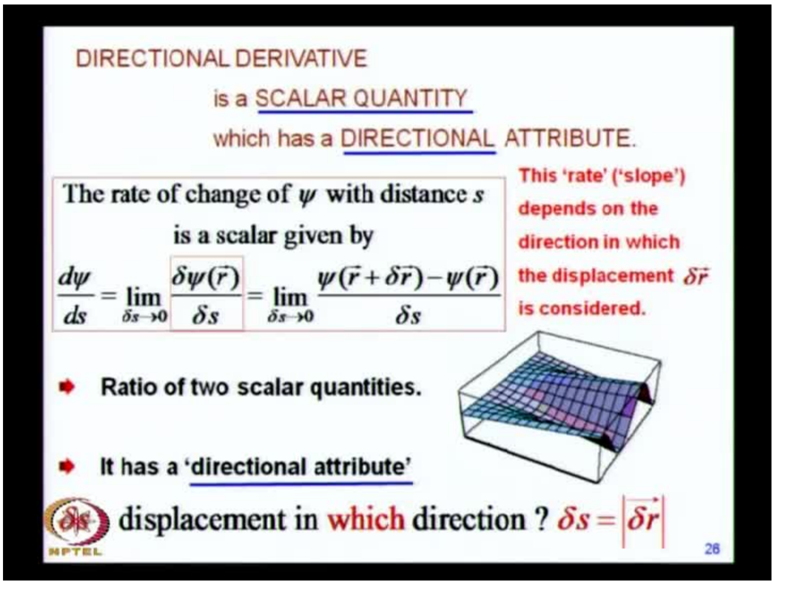
So, a ratio of 2 scalars which must give us a scalar which however has something to do with direction, because the direction in which you measure this derivative, whether you take consider this derivative along this red arrow or the yellow arrow is going to change its value.

Which is why when we introduce scalars and vectors, we persuaded you, not to define a scalar as a quantity which is got magnitude alone, and we persuaded, you to, not to define a vector as a quantity which is got a magnitude and direction.

Here you have a scalar quantity which does have a directional attribute. There is some directional property. It is a scalar, but there is some connection that sense of direction is not irrelevant to this, because this ratio has will have one value along the red line, along this line along 45 degrees and it will have another value along this yellow line. So, there is a certain directional attribute.

So, you must always define scalar as a quantity which remains invariant regardless of the rotation of a coordinate system in which you describe it, and a vector as a quantity whose components transform according to a given law and that law is different for polar vectors, it is different for axial vectors, and these details must go into the definition of a scalar and a vector.

(Refer Slide Time: 51:04)



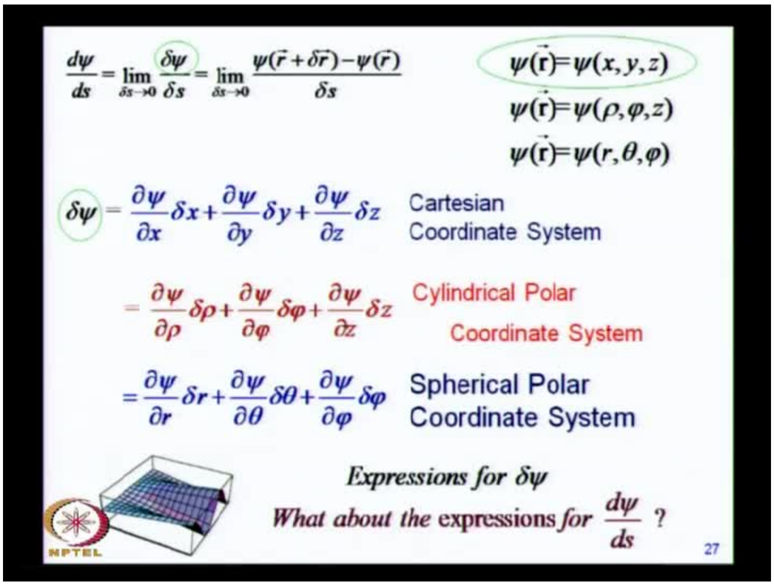
So, here you are meeting a quantity which is a scalar but which is got at directional attribute. This is called as a directional derivative; this is the definition of a directional derivative: it is a scalar quantity; it has got a directional attribute; it is a derivative; it is delta h by delta s in the limit delta is going to 0 which is dh by ds. So, it is the derivative.

But its value depends on the direction in which you measure it. This rate which is a slope, the derivative is the slope as we know. This depends on the direction in which the displacement delta r is considered, and if you have got a scalar function psi, then you define its directional derivative as a ratio delta psi by delta s in the limit delta s going to 0; delta psi is the difference between the value of the scalar at a neighboring point from what it has at a given point. So, that is delta psi; that is a numerator. Denominator is the distance between them, but the distance, whether it is considered along one direction or the other is going to change the value of this ratio.

Which is why it is called as a directional derivative, and this directional derivative is a scalar, it is a ratio of two scalar quantities, it has got a directional attribute, and the reason it has a directional attribute comes from fact that the denominator delta s which is a scalar. However, the direction of this displacement delta s must be considered.

So, this delta s is just the modulus of the displacement vector which is a scalar but the direction in which it is taken is going to play an important role in determining the directional derivative.

(Refer Slide Time: 53:14)



So, we will have to develop this formalism independent of the coordinate system whether Cartesian or a cylindrical polar or spherical polar. This is our primary definition of a directional derivative.

What it needs is the determination of delta psi in the numerator and this we can determine easily by using what is called as the chain rule, because psi is a function of x y and z, so, changes in delta psi come because psi changes with respect to x and that will not matter unless there is a corresponding change in delta x.

So, delta psi equal to the rate of change of psi with respect to x multiplied by the change in x will be your one dimensional model, and in three dimensions, you will have two other similar terms which is del psi by del y delta y plus del psi by del z delta z. So, this is the Cartesian picture, and because there are three independent degrees of freedom I make use of partial derivatives rather than total derivatives.

So, when you consider the derivative with respect to x, you must do so, you must determine this derivative while keeping the other two variables fixed at a given point at which the derivative is being taken.

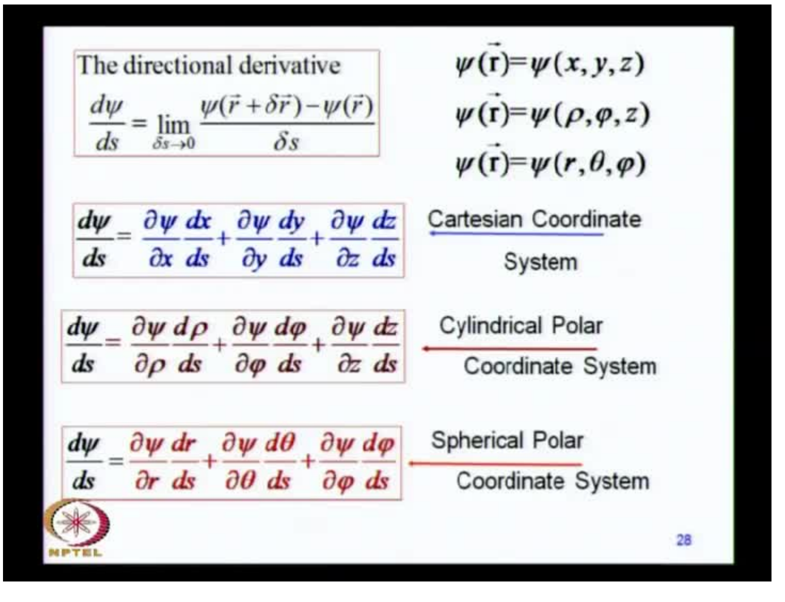
So, this is the chain rule; there is no big mystery over here. In cylindrical polar coordinates, the position vector is described in terms of rho phi and z, so, changes in delta psi which come in this numerator here. These changes are because of changes in psi with respect to rho multiplied by the change in rho itself.

So, you get del psi by del rho times delta rho which is the first term, and then, there are two other similar terms from phi and z. Once again we make use of partial derivatives and then you can also write the same in the spherical polar coordinate system. Here, you have got the position vector expressed in terms of three coordinates: the radial distance, the polar angle, and the azimuthal angle phi. So, changes in psi are because of change in psi with respect to r times the change in r itself.

So, it is del psi by del r times delta r plus del psi by del theta times delta theta plus the last component of a, here which is the azimuthal angle because psi can change also with respect to the azimuthal angle times the change in the azimuthal angle itself and this sum of these three terms will give you the net change which must go in the numerator which will determine this directional derivative.

We have got the expressions for delta psi. We should now get the expression for the directional derivative itself. So, what we are going to have to do is to divide this by delta s and take the limit, delta s going to 0. Now, that will be a fairly straightforward process.

(Refer Slide Time: 56:41)



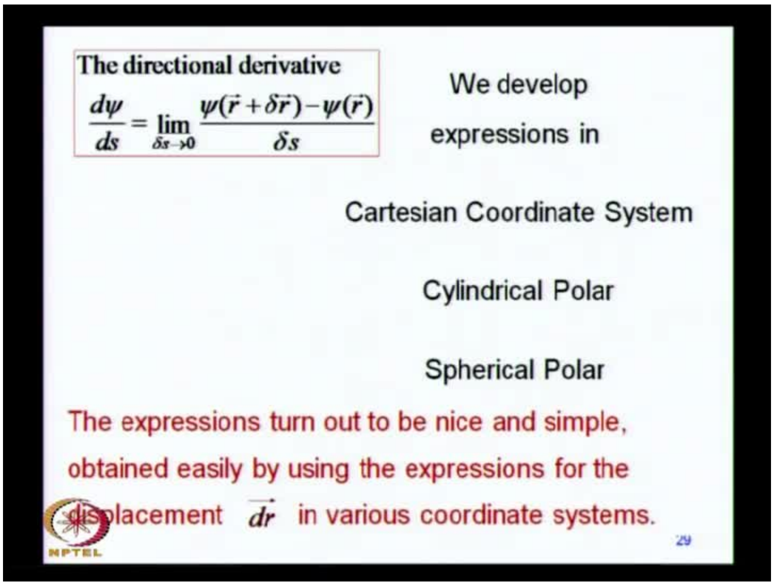
You divide it by delta s and take the limit, delta s going to 0. So, you get derivatives, so, delta x by delta s in the limit, delta s going to 0, gives you this derivatives dx by ds. Delta y by delta s in the limit delta s going to 0 gives you the derivative dy by ds, and likewise, you get dz by ds, and this is the expression for the directional derivative in the Cartesian coordinate system.

Now you do not have to mug up these relations because they are so easily obtainable from first principle. So, if you just keep track of the chain rule that what depends on what, what is changing with respect to what, why is it changing, and take all the causes which contribute to that change. The change we are looking at is the change in the value of psi. What is contributing to this change is the consideration that you measure the value of psi at one point and compare it with the value of psi at another point.

And when you go from one point to the other, what is it the changes? x y z changes, or r theta phi changes, or rho phi z changes, it depends on what coordinate system you are using. So, accordingly, if the changes are because of rho phi and z in the cylindrical polar coordinates, then you must take the rate of change of psi with respect to rho times how this rho changes with this distance s, but mind you somewhere the directional attribute will chase us, that is important. We have to keep that at the back of our mind.

In spherical polar coordinate system, we have got r theta and phi and you have got an exactly similar relation, so, you have got del psi by del r times dr by ds, and then, del psi by del theta. This is coming from the changes in psi because when you go from one point r to r plus delta r, the radial distance may have changed, but also the polar angle may have changed. So, that is the term which is contributing to this, so, no need to by heart these relations. Just remember, what are the changes taking place when you go from one point to another, which coordinates are changing, and then, use a chain rules with those particular coordinates in mind.

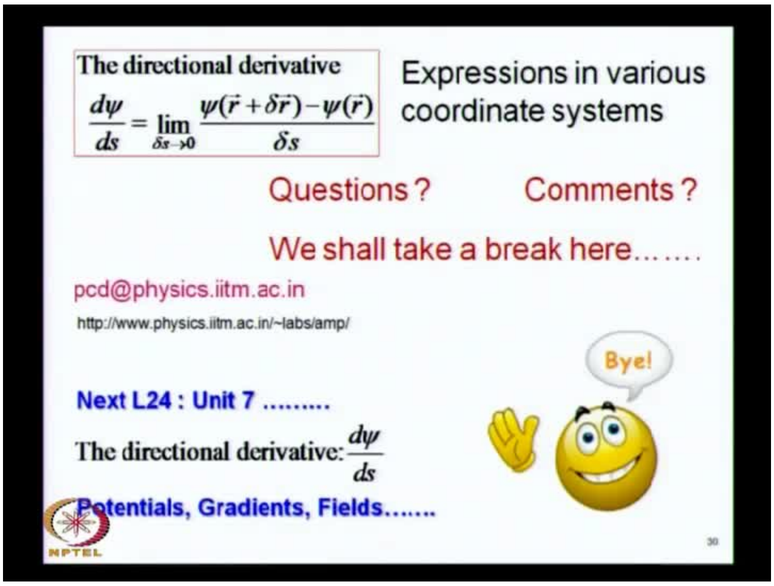
(Refer Slide Time: 59:18)



So, we will proceed to develop expressions in Cartesian cylindrical and spherical polar coordinate system, and what is important as we have recognized is this displacement delta r which is a vector. Now, this vector will of course have different expressions in different coordinate systems, because a position vector itself has got different expressions and different coordinate systems.

So, if you consider the expression for this displacement vector dr or delta r in different coordinate systems, it turns out that you can develop very simple expressions for the directional derivative, and this is where the expression, mathematical expression for the gradient will pop it.

(Refer Slide Time: 61:18)



So, we will take a break, and in the next class, we will obtain rigorous expressions for the directional derivative in which we will introduce the gradient. That will be for the next class. If there any questions, I will be happy to take, otherwise, we are ready to take a break.