Select/Special Topics in Classical Mechanics.

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Module No.# 08

Lecture No. # 26

Gauss Law Eq of Continuity (i)

In this unit, we will study the Gauss's laws. We will study the equation of continuity and these are very important topics. They have got crucial applications in fluid mechanics, electrodynamics and in many other branches of physics, very many other branches of physics. So, this is just the introduction. So, I will not even make an attempt to give the entire range of application.

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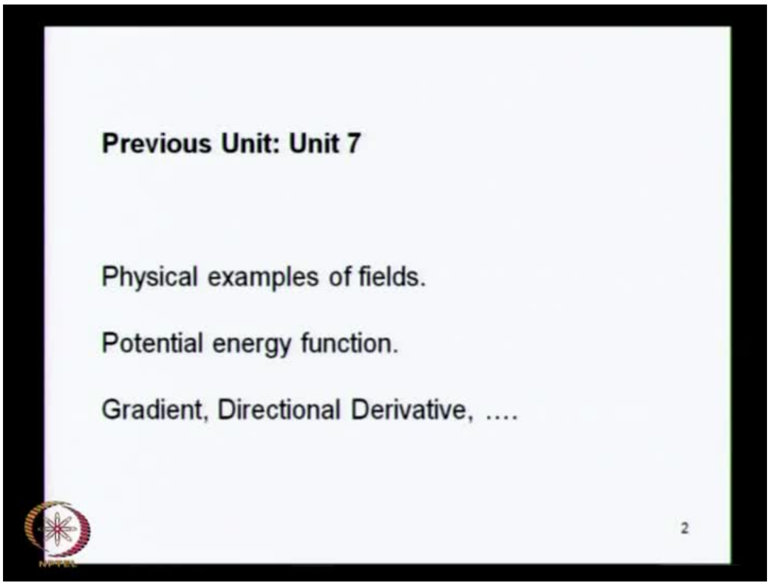

Let me quickly remind you that in our previous unit, we studied the directional derivative in particular, and we learned how to express the directional derivative using the gradient operator. We discovered that the gradient of a scalar function is in the direction in which a scalar function has its steepest descent and we saw that, this has important applications in the study of various fields.

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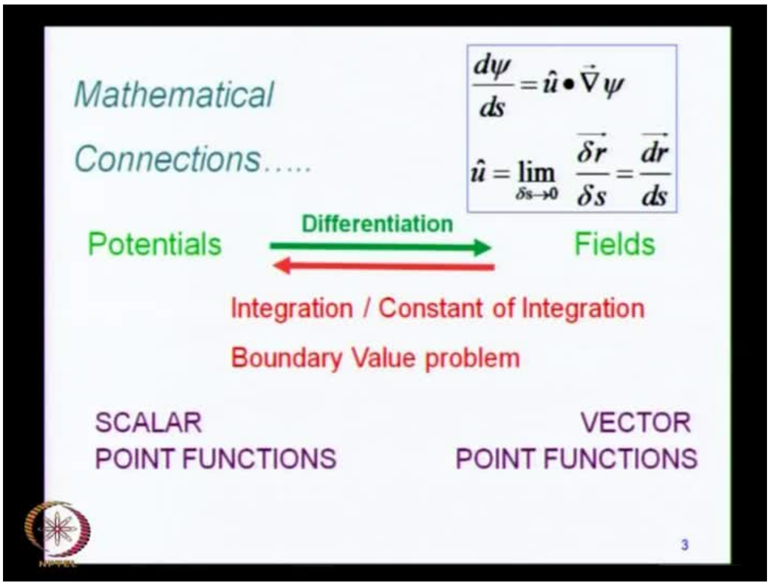

We learnt that there are important mathematical connections between potentials and fields and these connections also calculus. There are inverse connections from fields to potentials and they are through the inverse mathematical operations. So, if differentiation takes you from the potentials to the fields, integration takes you from the fields to the potential. Of course, you need to plug in the appropriate boundary conditions.

And then, we learned how the potentials generate scalar point functions, scalar fields and then, the gradients of the potential will give you the field intensities or vector point functions. So, this is the primary relationship that we invoked, which is to express the directional derivative which is the scalar, but it has got a directional attribute.

So, we already learned not to define a scalar as some quantity which has got just magnitude alone or a vector as the quantity which is defined in terms of just direction and magnitude that, these definitions are rather inadequate and, here was another example, in which we have a scalar quantity which does have a directional attribute. And, we have already learnt that the definition of a vector as a quantity having just magnitude and direction is not enough, and you have to really see how its components transform when you rotate a coordinate system, and that is the only way you can really define these quantities comprehensively.

We also learnt, when while we were discussing this issue that, there are you know, vectors are basically tensors of ranks one, rank one and then you know there are tensors and pseudo-tensors and they transform differently. So, all these things, we have, you know, exposed ourselves to.

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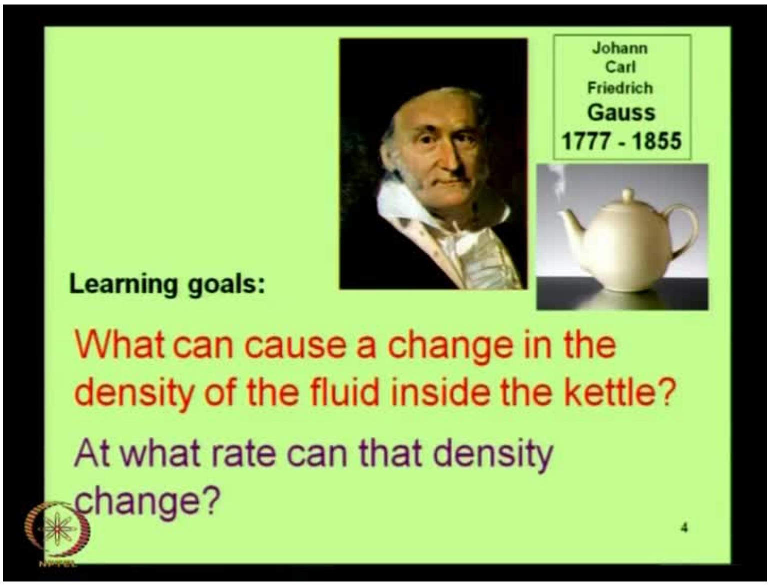

Having met the gradient operator, which is this nabla. In today's topic, we will see another very important application of this gradient operator and this will be expressed in terms of what is known as Gauss's law. So, this is named after Johann Carl Friedrich Gauss who lived in the 19th century and what we will learn, just to illustrate the point as a situation like this, that what is it that can cause a change in the density of a fluid inside a kettle, for example. So, this is a physical problem. You are directly related to Physics. What is it that can cause a change in the density of the fluid?

The other related question is that, if this density is changing, then what would be the rate of change - the time derivative of the density? What will be the rate at which this density changes? So, if there is a change it will have to be at a, at a certain rate, assuming that it is changing continuously.

So, these are some of the questions, which are very primary questions and we will try to see, how these questions are tackled and using, you know, appropriate techniques and the Gauss's law gives us the basic tool to develop a formalism which will eventually lead us to the answers to these questions.

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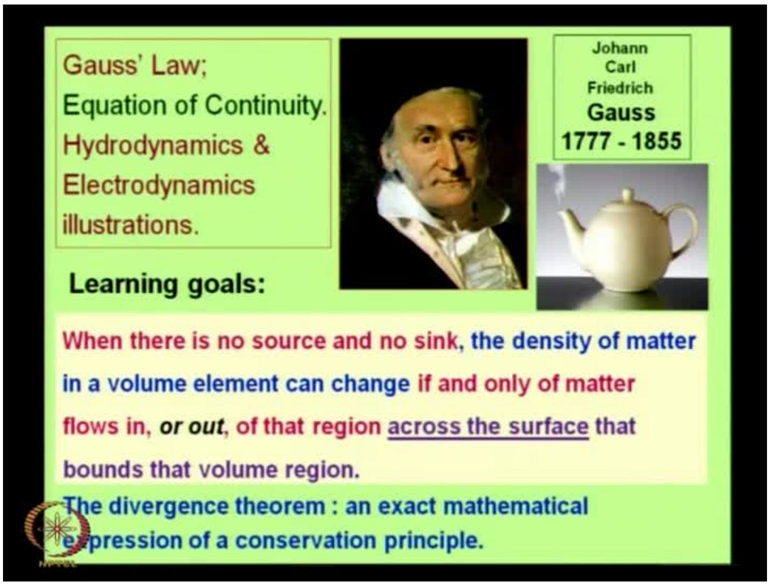

Now, we do know that if there is no source and no sink, then the density of matter in a given volume element can change, if and only if, matter either flows in or flows out of that region and where can it flow in or out from, it will have to do so, across the surface that encloses that region.

So, this is essentially a conservation principle, because that is the only way you know the density can change, if there is no source or sink, right. Then, the matter will have to either flow in or flow out of that region, and if matter is to flow in, it will have to flow in from the surface, like, as long as this bottle is closed, no water can get in, but then I open it, right, and I can pour water in, or if I am thirsty, I can consume some of it. And, the water will have to flow, across the surface which encloses the volume element, which is encapsulated in this bottle. No matter what the shape is, it does not have to be regular shape, it can be a kettle, it can be a bottle, whatever.

Now, what we are going to learn is what is called as the divergence theorem or sometimes called as Gauss's theorem or more completely as the Gauss's divergence theorem. Now, this is going to give us an exact mathematical expression for this conservation principle.

So, this will be our learning goal, we will learn to apply the Gauss' law to develop what is called the as equation of continuity, which is a conservation statement and I have already illustrated the idea. And, we will see that it has applications in hydrodynamics electrodynamics and a number of different branches in Physics.

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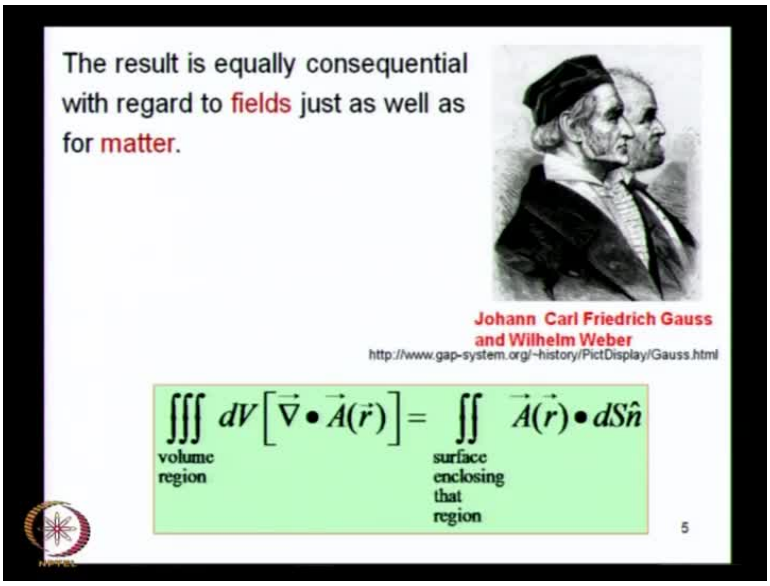

Gauss is very well known for the work he did with Wilhelm Weber on magnetism, especially terrestrial magnetism, but we are going to study a particular piece of work by this brilliant physicist and mathematician, which is known as the Gauss's divergence theorem and you see this theorem stated in this rectangular box. It uses, you know certain quantities, which you have not perhaps met before, at least in this course, like you see, in the integrand, you have got a volume integral over here. So, this is a triple integral because, it is a volume integral. So, in the Cartesian system, it could be integration over x, y and z if you like, or in spherical polar it could be over coordinate system, it could integration over r theta phi. d v is the volume integral, volume integration element and the integrand over here, which is in this rectangular box, is the mathematical quantity which we are going to introduce now. That is called as the divergence of a vector field A.

So, A is the vector field. It is a vector point function, which is defined at every point in space. So, A is a function of r and what you see is this del dot A, which is the notation used to express the divergence of a vector A. So, this is the quantity we are going to introduce here and then on the right hand side, there is a surface integral, but we will study this as the discussion progresses.

Now, the conservation principle in fact, works for fields as well as for matter. May this not just a question of, you know water flowing in or out or electrical charges flowing in or out. When charges flow in or out, they constitute a current, which is how you find applications in electrodynamics, but then there are also applications to fields. But we will come to that, especially in the context of applications and electrodynamics, when we will apply it to the electric fields and to the magnetic fields.

So, this requires us to develop the machinery of vector calculus further, and as we have now come to accept, we find that mathematics keeps lending itself as a very beautiful and very exciting tool to quantitatively express the laws of Physics.

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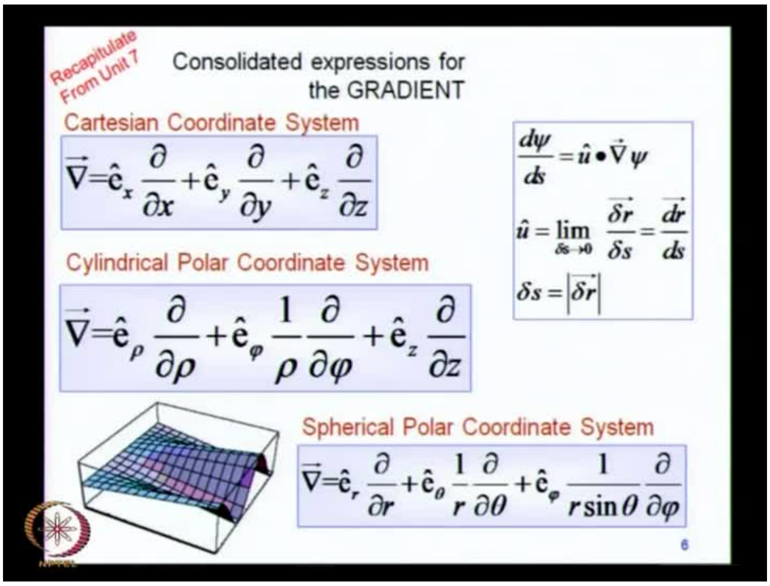

So, we will continue to take advantage of this relationship between physics and mathematics, which is very intimate. Let me quickly recapitulate the expression for gradient, which we have discussed in the previous unit, in unit 7.

We arrived at the expressions for gradient in different coordinate systems, with reference to, you know, analysis of some function which changes in different directions, and then by looking at the directional derivative of that function. This was the primary relationship that we exploited, that the directional derivative in a particular direction u is the component of a vector, which is called as the gradient of that scalar function, in that direction.

So, this was our defining criterion. And from this, we learned how easy it was to develop an explicit mathematical expression for the gradient in any coordinate system. In Cartesian, it is very simple, but then it is just as simple in the cylindrical polar coordinate system or in the spherical polar coordinate system, because this is the criterion and then there is only one way, you can meet this criterion in any coordinate system.

So, these were our results which we have analyzed in some detail in the previous unit, unit number 7. This is the expression for the gradient in the Cartesian coordinate system and then, we had a similar expression in the Cylindrical polar coordinate system and we also obtained the expression for the gradient in the Spherical polar coordinate system.

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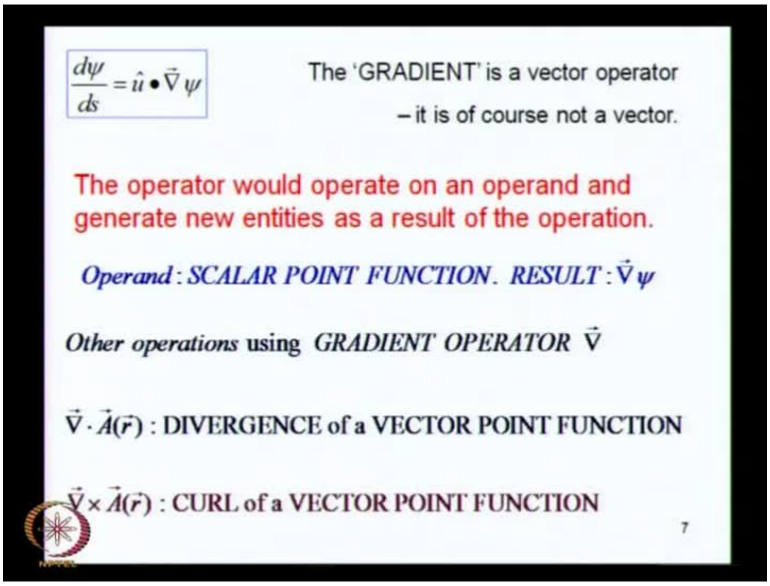

Now, let me remind you that, we recognize a gradient as a vector operator. It is of course, not a vector. We have defined a vector very carefully. Being an operator, it would need an operand. And as a result of this operation, it will generate new entities, which will be amenable for physical interpretation.

So, the operand that we considered so far, was the scalar function, which gave us the result, which is the gradient of the scalar function del psi. So, this is the operation that we defined, quite extensively in our previous unit.

Now, there are other operations, using the gradient operator. One is written as del dot A. Notice that, there is a dot over here or a point. This is the kind of notation we use, when we write the scalar product of two vectors.

So, it is a, a similar kind of notation, but being similar, let us not say it is the same. So, I only point out the similarity at this stage and then I will underscore the differences. So, there is another operation using the gradient operator, written as del dot A. This is called as the divergence of the vector point function A and there is in fact, another operation called as the curl of the vector point function, which is written as del cross A, which also looks, you know on the face of it, like a cross product of two vectors, which it is not, but it looks similar.

So, let me only highlight the similarity at this stage, but then I will underscore the differences. So, it is only a notation and this del dot A is, what is called as the divergence of a vector and the del cross A is called as the curl of a vector and what we are going to meet in this unit, is the divergence of a vector point function.

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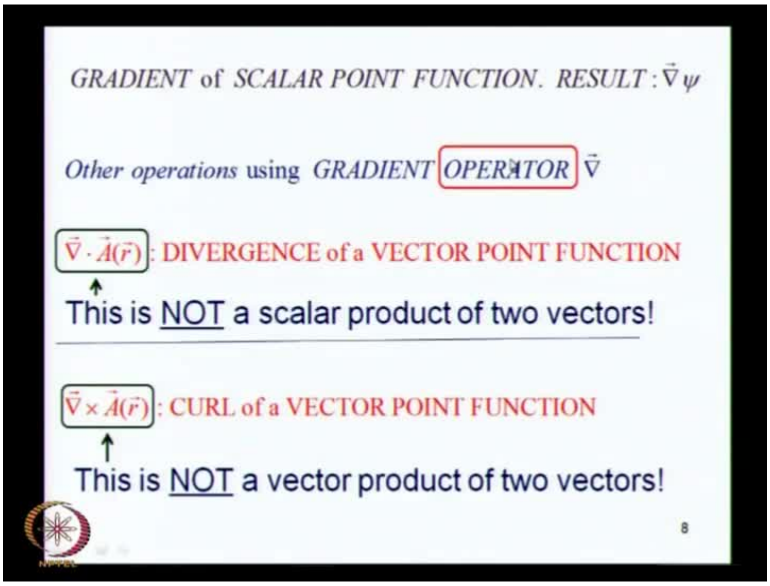

So, these are the 2 new operations involving the gradient operator, which is this nabla or del operator as it is called. I want to underscore the fact that, this is not a scalar product of 2 vectors. It has similar appearance. Likewise, del cross A is not a vector product of 2 vectors. A scalar product of 2 vectors has its own definition. A vector product of 2 vectors also has its own definition.

But here, the divergence of A and the curl of A will have to be defined in a very precise way, which we are now about to do. The reason for the difference is the fact, that, the gradient is an operator; it is not a vector.

So, this is not a dot product of 2 vectors. One is an operator, the other is a vector. These are different creatures. Same thing over here, because, del is an operator. You do not talk about the cross product of an operator with a vector. A cross product is defined for 2 vectors. So, this is a different creature altogether. So, the reason these are not just scalar products or vector products is because of this fact, that the gradient is an operator and not just a vector. It is a vector operator.

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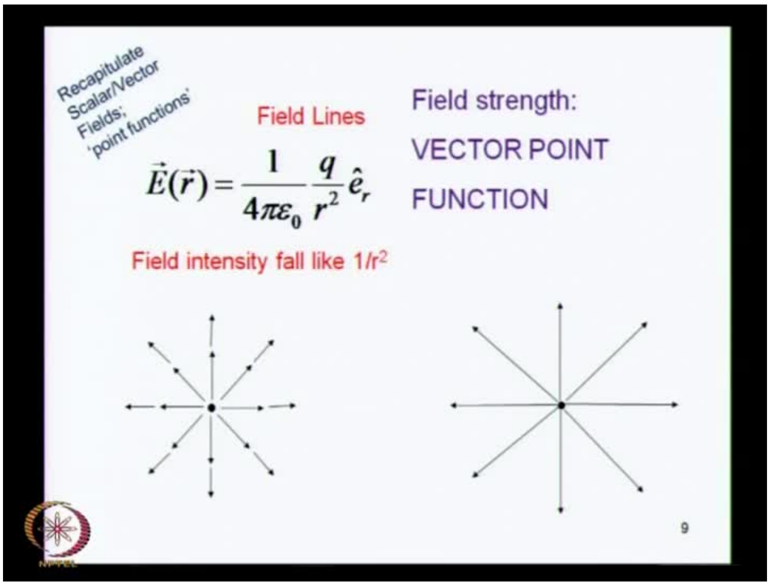

Let me also remind you, of what we know very well about scalar and vector functions, because, these properties are fundamental to our understanding of what a divergence of a vector is and what the curl of a vector is.

So, we will very quickly recapitulate what we mean by these scalar and vector fields, because then, we can proceed to introduce the divergence of a vector precisely. So, we know that the electric field lines and that is an example which is well known. So, we can discuss it easily. The electric field lines go as one over r square and these are radials, so indicated by the unit radial vector. This is the field created by a point charge that is a simplest case that one can think of.

We know that the field falls, that is one over r square and the field strength at any point, is a function of the, where that point is located. And therefore, it is what is called as a point function. The nature of this function being a vector, it is called as a vector point function.

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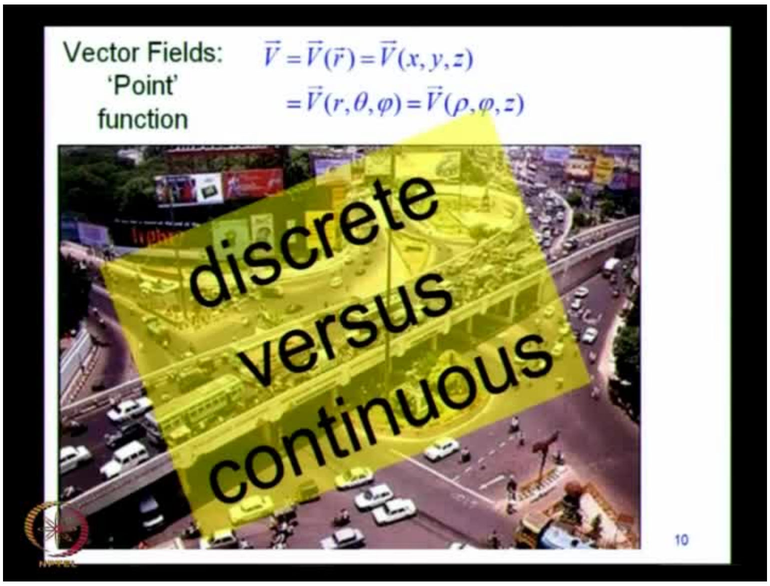

Now, we discuss various vector point functions like the velocity field and here, you have got a velocity of various objects and the velocity of any vehicle at a given point, depends on which point you are talking about. So, it depends on all the coordinates x, y and z. You have got a flyover. So, there is a z coordinate that you must worry about. So, all the 3 coordinates come into the picture.

But we also have to remember, that an example of this kind does generate a velocity field of a certain kind, but this is not the kind of field that we are dealing with. The reason is that, the velocity at a given point in such a case is a discrete function, like between this car and this car, there is no vehicle having any velocity at all, there is no vehicle at all. So, this is the discrete kind of quantity and it is, it will not lend itself for analytical operations, like a taking the derivative of its components and so on. So, this is not the kind of, you know, field that we are talking about. We are talking about analytical fields, which are differentiable and then, we also talk about these quantities being functions, not just of position, but also of time, because the velocity of any object could change from time to time, at any given point.

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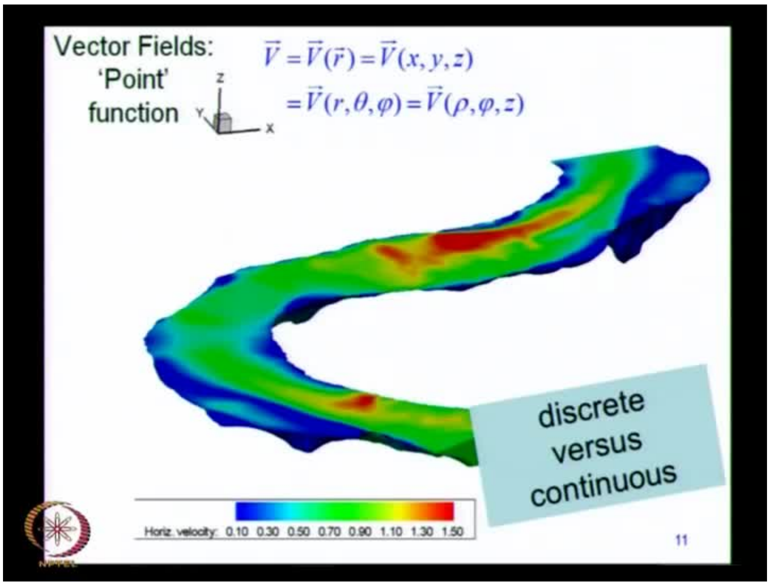

So, this is an example of what can offer itself as a function having continuous derivatives, because if you have a water flow in a region of this space and then you have got a horizontal velocity, which is different at different points, and then, this is the colour coded map, but at every point, there is a certain velocity that you can define and here we have got continuity and the function then becomes differentiable.

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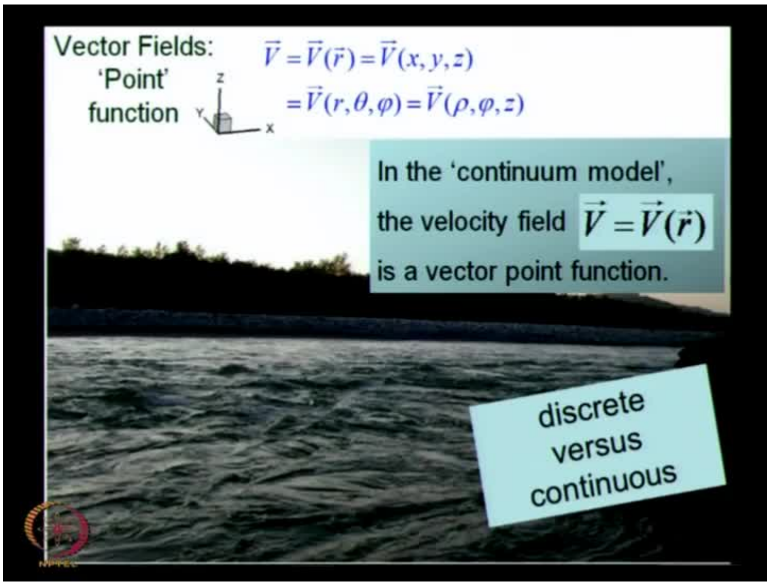

So, this is the kind of continuity that, we require, for the analysis that we have undertaken. And, the velocity of water for example, over here, is such an example, but, then again, we work within the realm of what is called as the continuum limit, because if you go to the molecular level, then of course, the water is made up of, you know, each molecule of water is made up of, you know, two atoms of hydrogen and one atom of oxygen and we really do not go into the sub molecular structure and what happens at that level.

So, this is the continuum limit that we shall have at the back of our mind, when we talk about the fluid velocity. So, the velocity will then be a function of the position and this generates a vector point function, whose divergence and curl we will be talking about. Besides, it could be a function of time as well.

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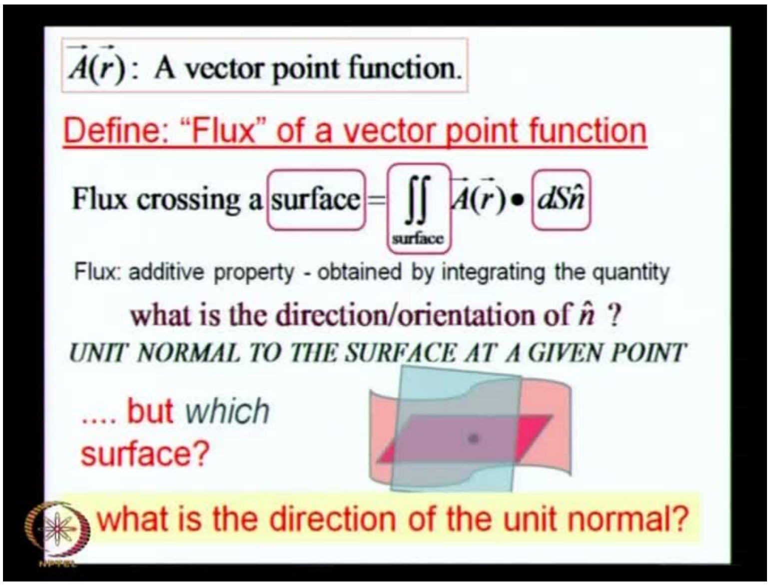

So, that is an additional complexity that we can introduce, when it is appropriate to do so. Now, we know what a vector point function is and we will talk about vector point function, which is defined for every point in space and if it is a velocity field, it will be considered in the continuum limit.

So, at every point in space, such as the one that you see on your screen, there is a certain, there is a certain vector which is defined at each point in space. What we will do is, is, we will now define, a quantity which is known as a flux of a vector point function. So, first we have to introduce ourselves to what a vector point function is. Now we will define, what is called as the flux of a vector point function. Now, this is the definition of a flux. It is defined for a vector point function. It is a property of the vector point function, described in terms of a feature, which crosses a surface.

So, these are the additional attributes which are coming into our consideration. And this is defined, in terms of the right hand side, which is a surface integral of a dot product of the vector point function, with an infinitesimal surface element dS, but this is an orientated surface. This surface has got a certain orientation, a certain directional attribute, which is shown by the unit vector n.

So, we have to understand what is meant by the direction of a surface? What is the directional attribute of a surface? So, we have to discuss this and then in terms of, once we understand how this direction n is assigned to an infinitesimal surface element dS, then we can certainly talk about the dot product of these two vectors and go ahead and add it up to construct this surface integral, which is a double integral, because the surface, of course, has 2 degrees of freedom

So, x and y if you like, or if you can express it in any other coordinate system and once we defined this quantity, the directional attribute of a surface, which has to be done very carefully. It is not something that we have discussed earlier in this course, so, we have to introduce this idea.

Then, we construct this dot product, then add it up, take the limit of the sum, we get a surface integral. Now, we have the right hand side, which defines the left hand side, without any ambiguity, right. In any equation, the left hand side is defined in terms of the right hand side. So, we have to introduce everything that is there on the right hand side and we will be in business.

So, this is an additive property and we have to discuss now, what is the directional attribute of the surface element. Now, the first property, but not the only property is that, this direction must be normal to the surface. It means by orthogonal to the surface at the given point. That is a necessary requirement. It is not a sufficient requirement because, if you take any surface of an arbitrary shape like this bottle, it has got a flat, more or less flat base. Then at any point on the, at the base, you can define the normal, either going out of the bottle or into the bottle. Likewise, no matter what the curvature of the bottle is, at every point, if you take a sufficiently tiny region and take the limit that the tiny region of the surface becomes infinitesimally small, you can define a direction which is orthogonal to it, normal to it, at 90 degrees to it, but again it will be a direction which is pointing outward or into the bottle.

So, we will have to figure out how to define this outward and inward and what is our convention, if any, because, there are 2 directions which immediately come to our mind. It turns out to be a little more complicated than that, because, we have to resolve not just between in and out, but there are some additional complexities that we must worry about.

First of all, let us ask this question which surface, because, if it is a surface going through a point, then for this bottle, we have already defined the surface, but if you just have a point like this, then which surface are you talking about? Is it this surface? Is it another one which is orthogonal to this, right.

So, now, you have a surface passing through essentially the same point, but a different surface and the direction which is at 90 degrees to the first surface is obviously different to that of the second one and there could be n number of different surfaces and different shapes that you can think about.

So, the directions will keep changing. So, we really have to figure out, what is it that we are doing and all of these things need to be resolved unambiguously, and then we have to ask ourselves what will be the direction of the unit normal.

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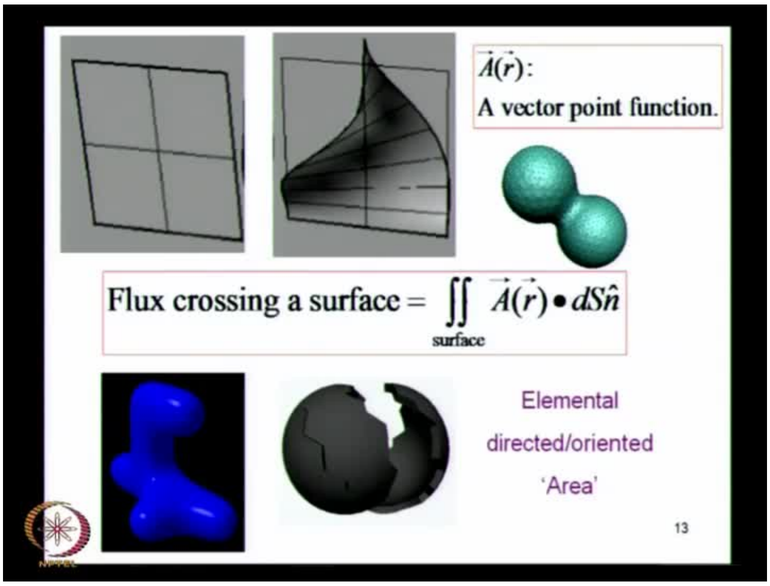

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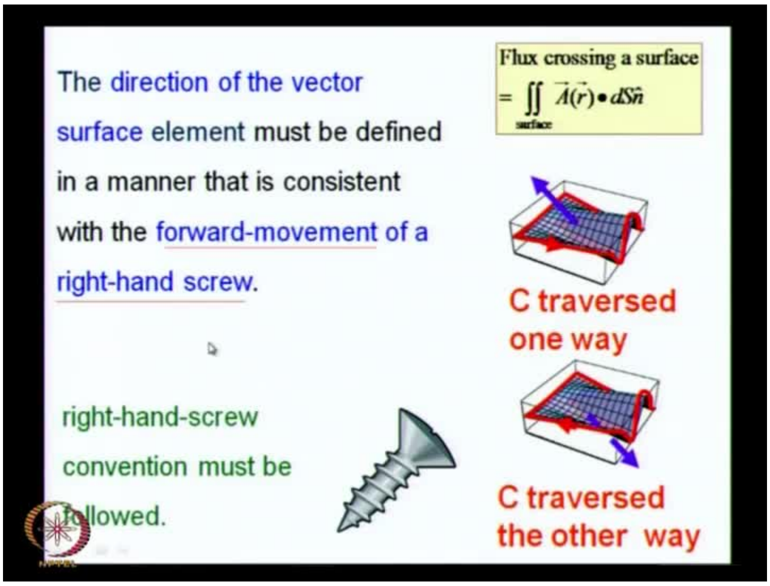

So, first of all, we have to understand what is meant by this oriented area or a directed area. Now, what kind of surface elements come to your mind? Here is a simple one. It is a flat piece of surface. The surface may be warped, right. Here is another shape and you have a different surface here. Here is one more. It can get more and more complicated or it may even be fractured. And in each case, how we define the orientation of that surface or the normal to the surface has to be done very carefully. Now, what we do is, follow certain conventions. We are going to define this flux in terms of the surface integral.

So, the first thing we do is, if you have any surface element, you consider its boundary and this boundary you can traverse which is the perimeter of the surface, no matter what its shape is. It could be a handkerchief that you have pinched and the surface may be warped. It does not have to have any regular shape.

But once you have a perimeter, you can traverse. You can take a walk along the perimeter in one way or the other. So, first you decide which way you are walking on that perimeter. So, let us say this direction is indicated by this arrow over here. Now, once you do that, you think of a right hand screw which you would have used and use a right hand screw rule to define the unit normal.

So, what would be the forward direction of the right hand screw, if you were driving it into this surface, if the screw were turning along the direction you have chosen to traverse the perimeter. So, now, obviously it would be this, right. And this blue arrow tells us the direction of the unit normal without any ambiguity, but it is done so, with reference to the right hand screw rule. So, that is the basic requirement. If you traverse it in the opposite way, the same perimeter, but now you traverse it in the opposite way. Then the right hand screw would move forward, if you were to turn it along the perimeter in which you would traverse the perimeter, would be the opposite one, right.

Now, mind you, do not forget that you must mention the forward movement of the right hand screw, because, if you turn the screw anticlockwise, if you, right, the screw of course, comes backward and it gives you exactly the opposite direction.

So, stating the right hand screw rule is never enough. You should also indicate that, it is defining the forward movement of a right hand screw rule, if you were to traverse it, if you were to rotate it in the direction in which you think of traversing the perimeter.

So, this now, this gives us a convention which unequivocally defines the normal. So, it cannot be both ways. It is only one way or the other and which way it is, depends on how you choose to apply the right hand screw rule.

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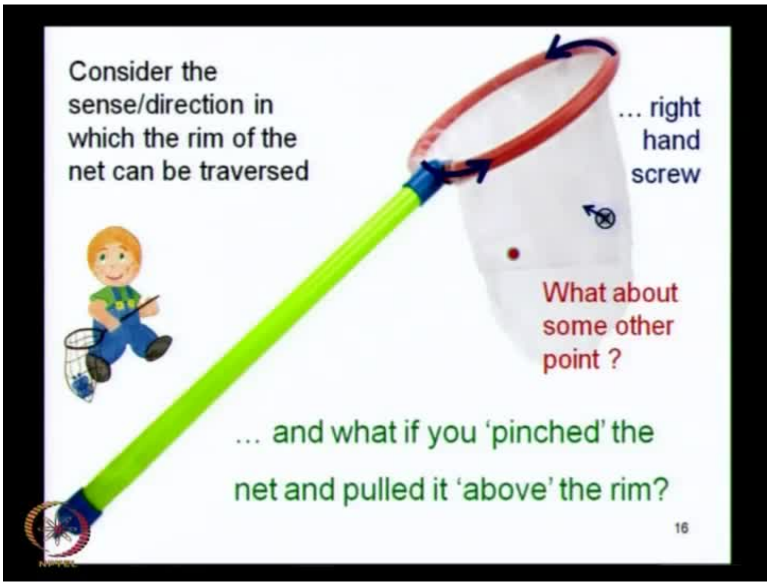

Let me also tell you, how different shapes can be addressed. You would have seen the butterfly net. As children you would have used one to catch butterflies in it, right. Now, if you look at this butterfly net, then you can think of the perimeter of this butterfly, being traversed in this direction. That is one sense, in which you can think of traversing it, like this, but, you can also think of traversing it in the opposite direction.

So, now, if you have chosen this as your direction of traversing the butterfly net, and now, you think of a point on this net and ask what would be the direction of the unit normal at this point, then you can see that it will be given by the right hand screw convention and it will have to be given by this arrow and you would see the tail of this arrow from outside the net, right.

Now, ask yourself what would happen if you were to pinch the net and pull it above the rim, but you are not changing the direction in which you think of traversing the perimeter. So, the directions, this arrow will change its orientation, but, mind you, that all you are doing is to flex the shape of the net without changing the perimeter. So, the direction of the arrow is completely defined by the direction in which you are thinking of the perimeter of the net being traversed.

So, if you take any other point, you should be able to now define the unit normal to the surface without any ambiguity. So, quite independent of the fact that the net really does not have to hang loose below the rim, even if you were to pinch it, pull it up, distort it any which way, the direction can be unambiguously specified for the unit normal to the surface, as long as the perimeter and it is determined completely by only one thing as to how you think of the perimeter being traversed.

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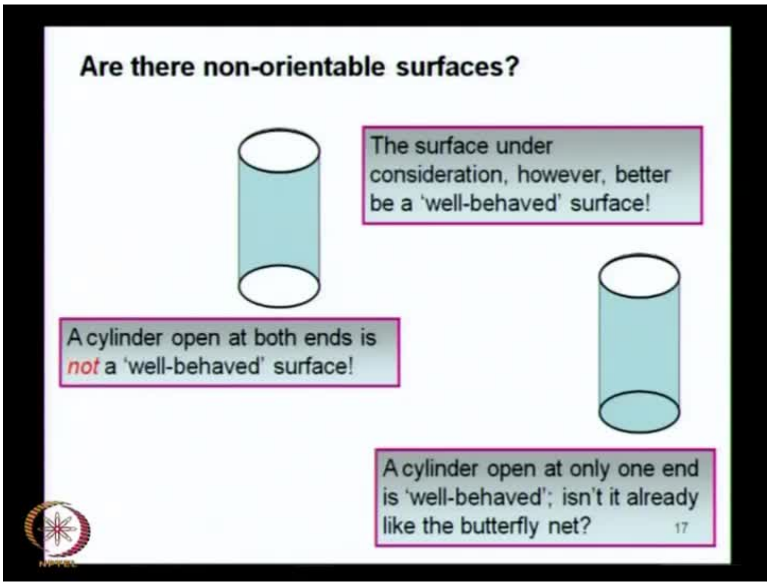

So, now we know how to orient a surface. Let us ask, if there is any surface which cannot be oriented? Is there a surface, which you will say is not an orientable surface? So, orientable surfaces are usually called as well behaved surfaces, but then there are other surfaces, such as a cylinder open at both ends and like some of us, it is not a well-behaved surface. There is no unique way of defining the unit normal to the surface, because the perimeter you can define, there are, there is not a single unique perimeter there are two of these. So, here is a surface which of course, you cannot orient.

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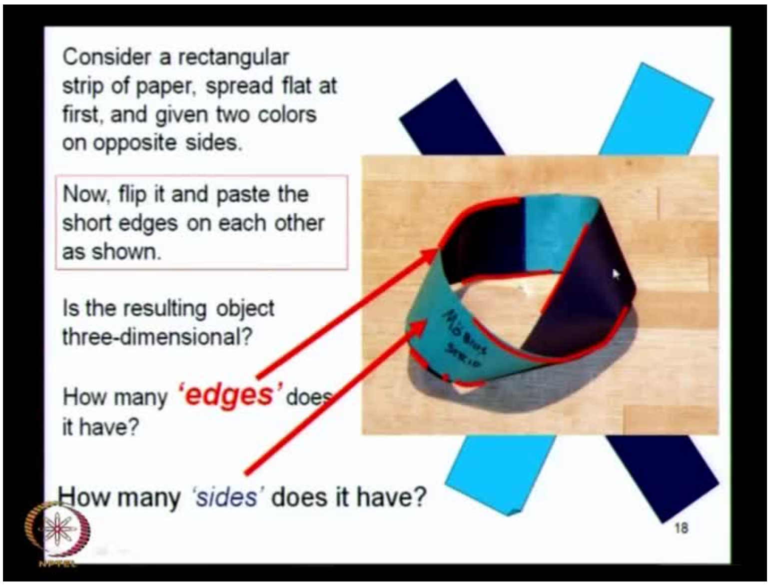

What about a cylinder which is open only at one end? This is not a problem. This is almost like the butterfly net, really. So, this is not a problem, but a cylinder open at both ends is not a well behaved surface. You cannot orient this. Then, there are some other surfaces and let me discuss one particular, exciting kind of surface by asking you to think of a rectangular strip of paper, spread it flat and give it two colours. Something of, something like this, I have it in my pocket, here.

This is a piece of cloth, a ribbon and you can see that it has got a white colour on this side and a orange color on the other. So, when I hold it like this, it clearly has got 2 sides, a white side and an orange side. A side 1 and a side 2. Side a and a side b. A side up and a side at the bottom, right. So, we think of two sides.

However, let us draw this in the diagram. So, here is a blue side. Here is a deeper blue or indigo, or whatever is the correct name for it. May I call it indigo, more or less. So, there is a blue side and an indigo side. What you do is, flip it, take this, bring it back and I can join it like this, or else, before joining I just give it a twist and join it like this, ok.

Now, this is the two ways in which I can make the ends meet. One is like this, the other is, after one twist, and then you know I, I bring it here, but give it a twist and I join it like this. So, when I do that, what I get is, what is called as a mobius strip, ok.

Now, let us count the number of edges, means if I just hold it over here, you can see that there is a top edge and a bottom edge. So, there is an edge 1 and an edge 2, an edge x and an edge y. So, there are two edges that we talk about. Now, let us count the number of edges of this strip, which is called as the mobius strip.

So, this object resides in 3 dimensions and if you count the number of edges, let us trace the edge. So, if you start tracking the edge, it goes over here, then it continues along this and over here, it is going to go underneath and come at the bottom side and come underneath this and come on this side and then flip over here, flip over here and then rise over here and join back on the top. So, you started out with what you thought was the top edge and without ever changing, without ever lifting your finger, you go through the entire perimeter and find yourself at the same place again.

So, you have only one edge. What about the number of sides. Here you are in the blue side, you run your finger on the blue side, along over here, go underneath it, and then you find yourself behind this indigo side, and then find yourself back again on the blue side and then you find yourself running on the indigo side, and then behind this blue side, and then back again on the indigo side. So, you are, find that there is only one side. So, surfaces of this kind cannot really be oriented. So, these will be excluded from our analysis.

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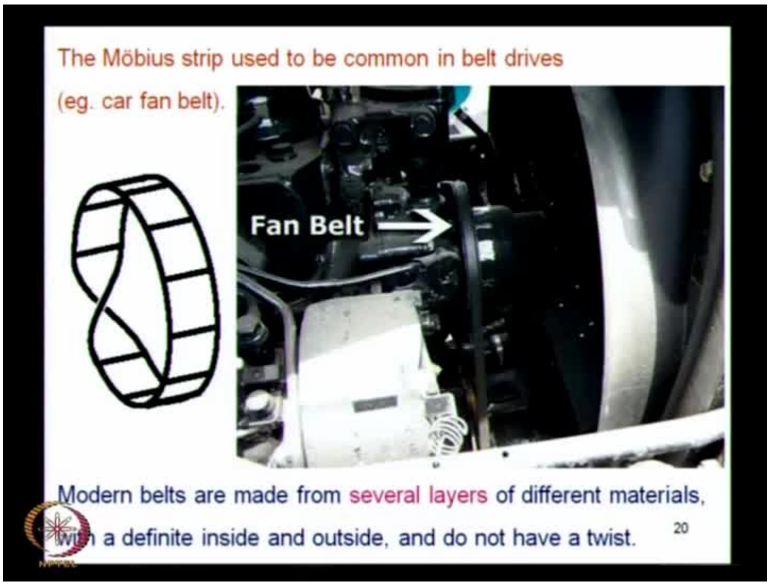

Now, look at this earring this girl is wearing. What she is wearing is in fact, a mobius strip and this is a mobius strip. So, you, you just turn it over and then instead of joining it like this, you flip it and then join it, right. So, this is the mobius strip. And, this has got some interesting applications. And, in the earlier days of machines, these fan belts were given a twist like this, because then, the wear and tear on just one side would be reduced. Because, part of the run would be on what we would have otherwise called as the opposite side, but because of the twist, it is no longer an opposite side. It is really the same side, just because it happens to be a mobius strip.

So, this is not however, modern fan belts are made because, these are one uses advanced technology in material science and you actually make them from different layers and these are different material properties. So, these are not given a twist anymore, most of the modern fan belts, but this was the technology which was adopted in the earlier days of, you know, fan belts.

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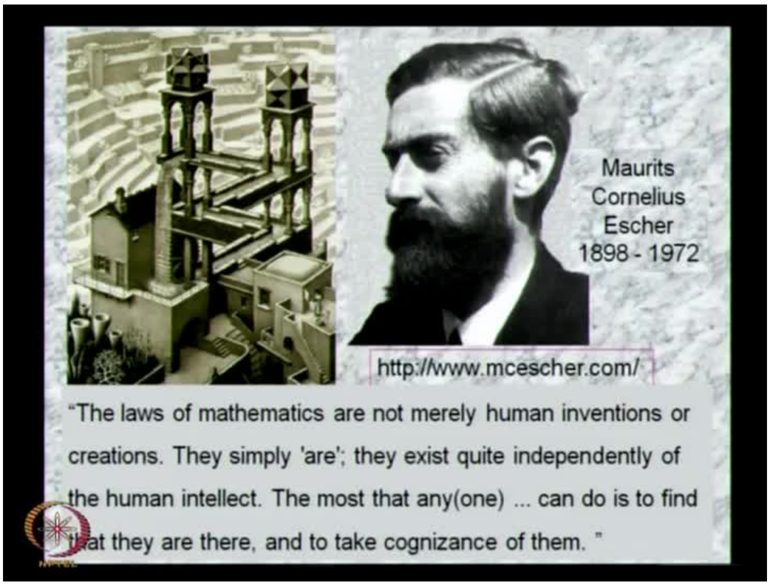

There are these interesting shapes and I think, I must introduce you to the wonderful shapes which were generated by this brilliant artist, who had a very intricate mathematical mind, Maurits Escher, and look at this drawing that he made, means. These are Escher diagrams and you can find many of these at this website and here you have a picture of water falling over here, it keeps falling, falls over here and keeps falling, goes to a lower level over here, keeps falling over here, and keeps falling, goes over here and then back again, it keeps falling. So, you really know, do not see while looking at these, as to where it is changing the levels.

So, these are very, these are exciting pictures which Escher has drawn. He is a very brilliant artist and I will like to quote him over here. He said that the laws of mathematics are not merely human inventions or creations. They simply ‘are’. They are just there. They exist quite independently of the human intellect and the most that anyone can really do, is to find that they are and to take cognizance of that.

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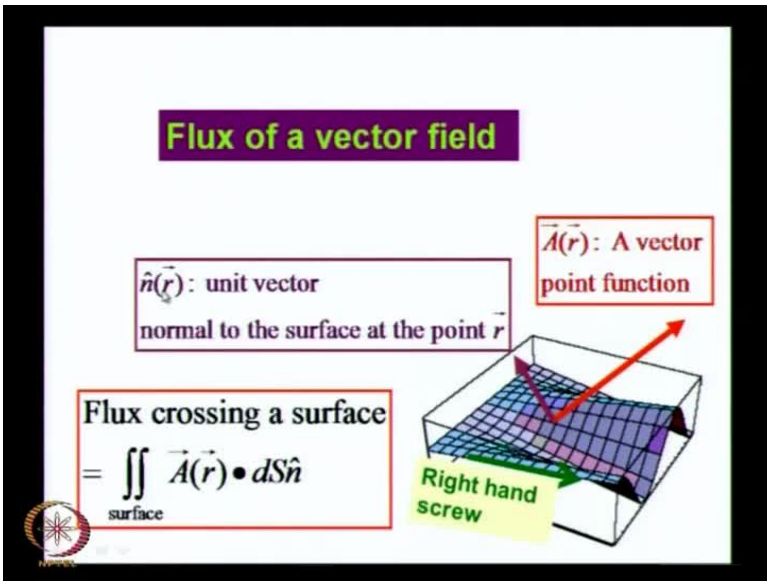

So, this is a, I am sure you will enjoy visiting these Escher diagrams. So, anyway, back to our definition of flux of a vector field. Now, we know how to construct an infinitesimal surface element. We know how to orient this. So, we can construct this surface element. We can orient it. We can now construct the dot product and we can add it up over a surface and what we have is a flux of the vector crossing that field. So, now the left hand side is completely defined in terms of what we have defined on the right hand side.

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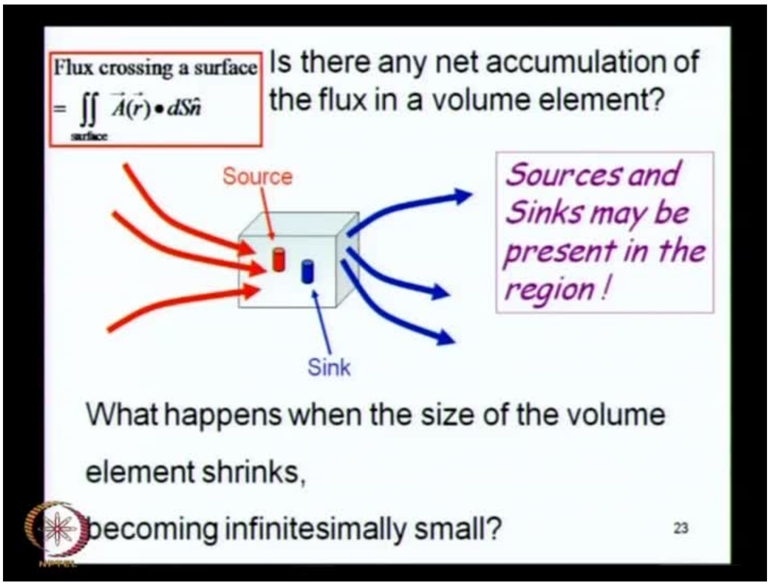

So, you have got a vector point function. You have got the right hand screw rule. You get a sense of direction for the surface. This is the unit normal, it is also a point function because it will change from point to point. The unit normal at this point is obviously, different for the unit normal at some other point, for the same sense of traversing the perimeter, and now what we will do, is to find how to determine the flux crossing a surface element.

So, we consider a rectangular parallelepiped and suppose you have got a velocity field which is crossing this, this could be the velocity field of a fluid which is passing through that region. So, you have got this region and then you have got water flowing through this, coming along different streamlines passing through this, and then it, it exits this region also along different streamlines.

Remember, there may be sources and sinks inside this region. So, there may be a source, there may be a sink. You do not know what is inside and there both the possibilities, that we must include and our question is, is there any net accumulation of the flux in a volume element. That is a question, that is a physical question that we are dealing with.

Is there a net accumulation of the flux in a volume element, and what would happen if the size of the volume element shrinks, and in the limit it becomes infinitesimally small, falling into a point, so that, we can define some property, which is a property of that particular point. As this volume region becomes smaller and smaller, smaller and smaller, smaller and smaller, reducing it in the limit, to a point, then can we define a property in terms of the net accumulation of the flux in that volume region, in the limit that the volume becomes infinitesimally small. This idea will lead us to the definition of a divergence.

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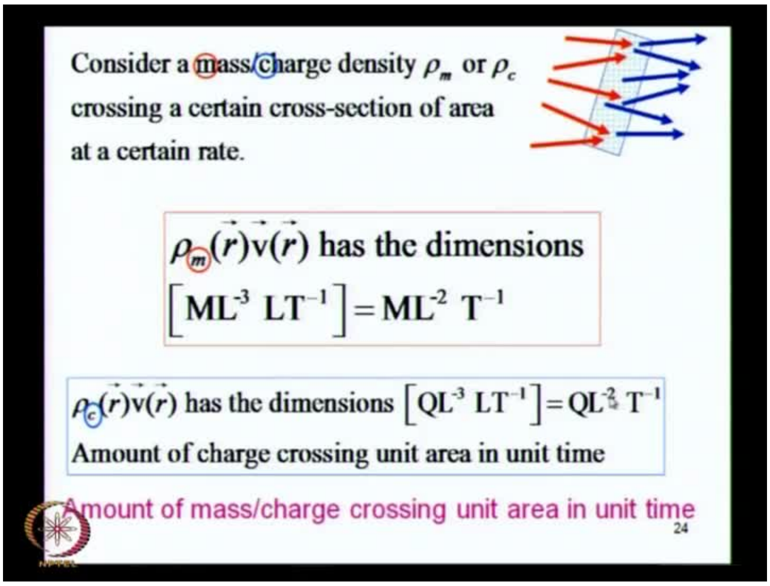

So, now, let us consider the flux passing a certain surface element and what is really passing is, either some mass of the water or it could be, you know, some other form of matter it could be electrical charges for example, and when these electrical charges are crossing a surface element, they are constituting a current.

So, you have got, let us say, a certain mass density rho m or a charge density rho c. And, we will develop the discussion in parallel, for both the mass density and the charge density, because the formalism is essentially the same.

So, you have a certain mass density or a certain charge density crossing a certain surface element, a certain cross-sectional area at a certain rate, whatever be it and we introduce a quantity which is defined by the product of this density and the velocity at that point r.

So, it is a point function. What will be the dimensions of this quantity? The dimension of the density is mass per unit volume. So, it is M L to the minus 3. The dimensions of the velocity is length over time. So, the dimension of the product of density and velocity is M L to the minus 2, T to the minus 1. So, it is mass per unit area per unit time. So, that is the quantity we are dealing with.

If we were talking about charges, the charge density is charge per unit volume and the dimension of the quantity density times velocity, will be charge per unit area per unit time. So, it will be Q L to the minus 2, T to the minus 1. So, these are the quantities we are now introducing. And, what it represents is the amount of either mass or charge crossing unit area in unit time.

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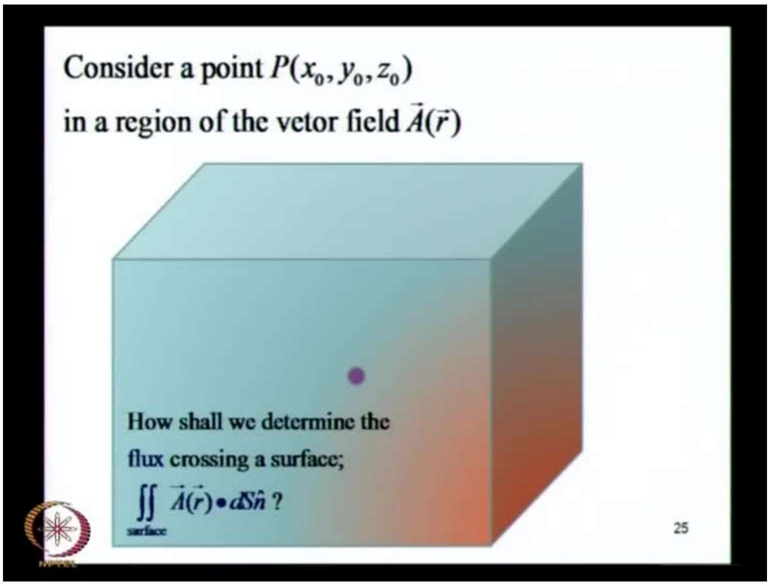

So, now, let us think of this parallelepiped region and you consider a point inside this, where a vector field A of r is defined and here is a point inside this, and now let us ask ourselves, how dowe precisely determine the flux crossing in area now.

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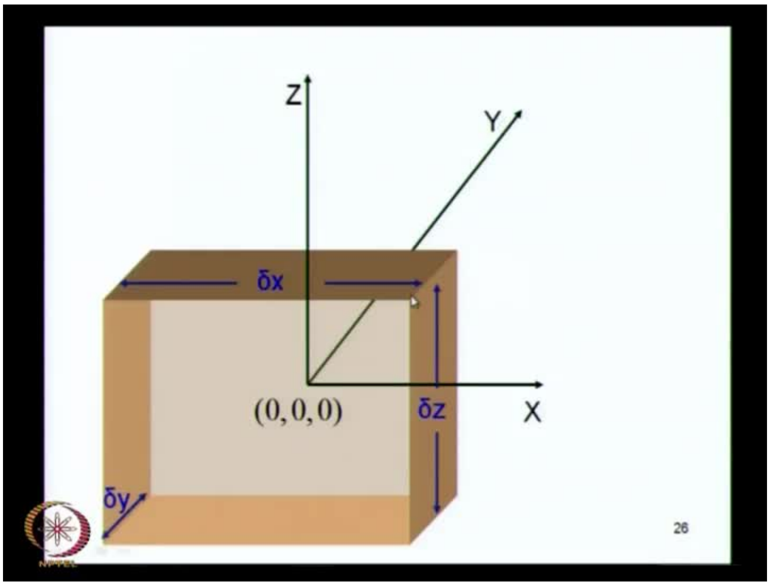

So, here is the parallelepiped we, we set up a coordinate system and it is convenient to set up a Cartesian coordinate system with the axis parallel to the respective edges of the parallelepiped.

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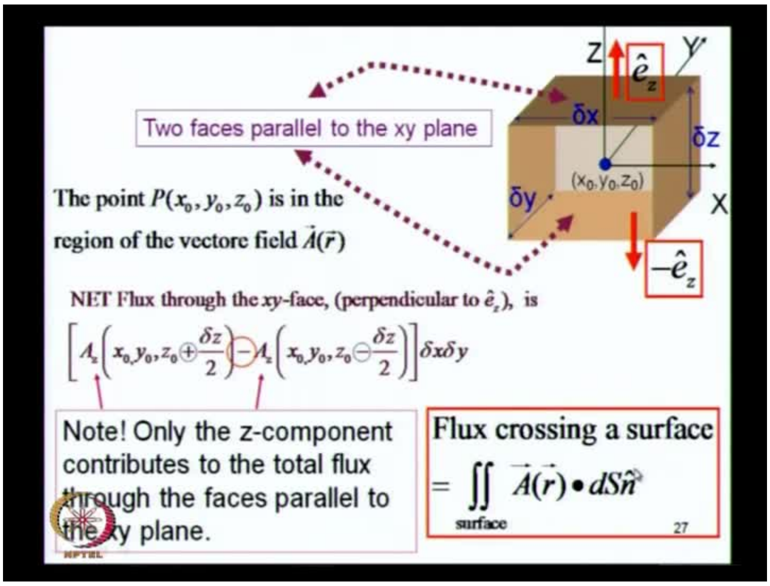

So, this diagram is self explanatory. I will not explain it any further. And, in this diagram, you have got the unit normal to this flat plane on the top, which is parallel to the x y plane and the normal to this is the e z, the normal to the bottom plane is minus e z and we now have oriented areas.

So, this is the point at x 0 y 0 z 0 and you consider two faces parallel to the x y plane, one at the top, one at the bottom. Then, the net flux which is the addition of these quantities A dot dS. So, you must determine A dot dS on this surface and add to it the A dot dS at the bottom surface. The two together will give you the flux through the faces, which are parallel to the x y planes.

Then, there are two other sets of parallel planes, the y z and the z x, when you do the addition, over all the 3 pairs of parallel faces, you get the net flux through this volume element. It is as simple as that.

So, now to get the flux through the top face, which is… So, you need to construct this A dot dS, which is the magnitude of the function A at the top face, but the top face is not at x 0 y 0 z 0. It is at x 0 y 0 z 0 plus delta z by 2, where delta z is this increment, because the top face is not at z 0. It is at half delta z above this.

So, the value of the function would have changed. It is a point function, which changes from point to point. So, it is a value of the function at the top face times the area of the top face, which is delta x times delta y, but you must subtract from it, the corresponding quantity at the bottom face, subtraction because, the unit normal to the bottom face, which we will define to be the normal which is outside, which is pointed outside to the region.

So, this is the closed region and for a close region, the direction of the unit normal to a surface is always defined as the normal to the surface which is pointed outside, not inside. And, now at the upper face, this unit normal is plus e z, but at the lower face the unit normal is obviously minus e z, because it is pointed downward. It is the unit normal pointed outward to the box, which will be along minus e z on the bottom face.

So, you must subtract from this, because this is, this one now comes with the minus sign, so that, the A dot dS, this projection along the unit normal will pick up a minus sign. But then the value of the function at the bottom face, is not the value of the function at the point z 0, but it is the value of the function at the point z 0 minus delta z by 2, because the bottom face is at a distance delta z by 2 below the x y plane, which is passing through z 0.

So, now we have got a plus sign over here, because this is the top face. We have got a minus sign over here, because this is the bottom face and we have got a minus sign over here, because this bottom face has got a unit normal which is pointed downward as opposed to the unit normal at the top face, which is pointed upward. So, now, we have got all the signs, understood?

And now, all we have to do is, to add up these pieces from the remaining 2 pairs. So, we have considered only the z component over her, because the x and y component, when you take the dot product with e z or minus e z, will give you zero contribution. The dot product between two orthogonal vectors vanishes.

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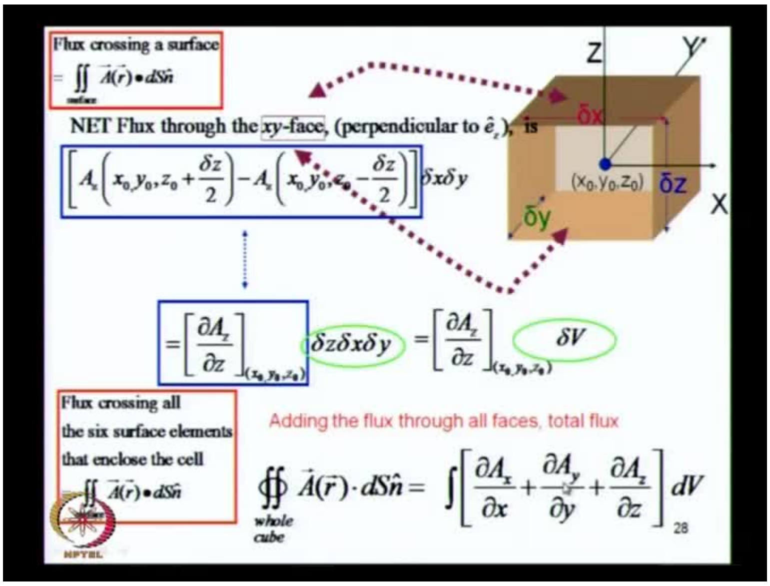

So, we have got this contribution from the planes which are there parallel to the x y plane and this difference, between the value of the function at these two neighbouring points and we are going to consider that, neighbours get infinitesimally close to each other, because our interest is in looking at, shrinking that volume to a point. So, in the limit, these surfaces are going to get closer and closer to each other, so that, you can apply a simple differential calculus kind of application, and you immediately see, that the difference between the value of the function at the top face and the value of the function at the bottom face is nothing, but, the rate of change of A z with z multiplied by delta z itself, delta z being the difference between the bottom edge and the top edge. It is half delta z below z 0 and half delta z above z 0. So, the two together give you delta z.

So, this rate of change of A z with z, del A z by del z times delta z gives you this difference and this delta x, delta y is the area which sticks in over here, and this product delta z delta x delta y is obviously nothing, but the volume of the parallelepiped.

So, this is the volume of the parallelepiped and you can get a similar expression from the other two faces, which means that if you integrate this, over all these 6 sides, which enclose this parallelepiped region, you will have a volume integral, a triple integral in which delta V can be factored out as a common, and then you have got a sum of these 3 terms del A x by del x, because similar to del A z by del z that you have over here, you will have a term in del A y by del y and del A x by del x. So, you are integrating this quantity in the rectangular box and you are carrying out its volume integral. It is a triple integral.

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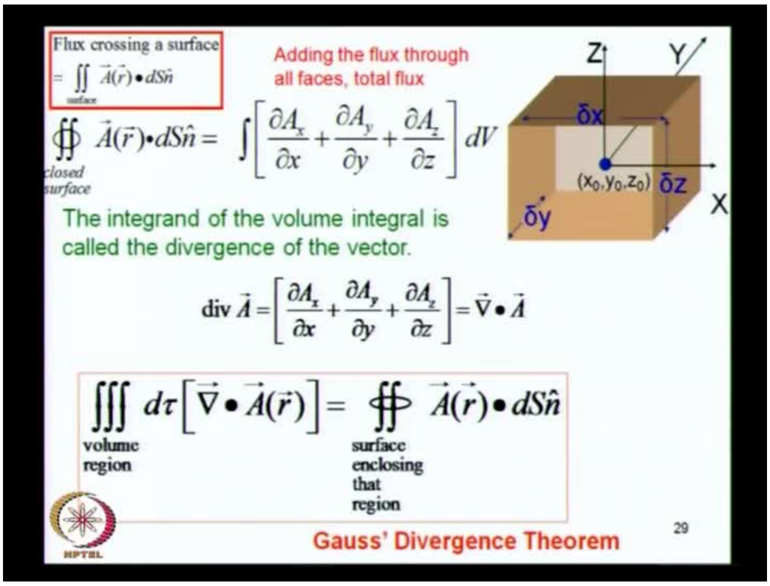

So, here you go. This is the triple integral and the quantity you have in this box, which is the sum of these 3 partial derivatives, is what gives us what we are after. This is the divergence of the vector field. This is the divergence of the vector at that point. Point, because you are considering the limiting value of this volume, as it shrinks to the point. So, the separation between parallel surfaces delta z between these faces and this face it reduces, delta x would shrink to 0, delta y limit going to 0.

So, those are the limits you are taking. And this quantity which is being integrated, the integrand of this volume integral, is what is called as divergence of A. It is written as d i v of A or d i v being an abbreviation for divergence and this is the notation. It is not a scalar product of del and A. It is a notation to express the divergence of A. It is just a notation, a notation is a notation is a notation, it is not a dot product of two vectors.

So, now if you add up the flux through all the surfaces, you are essentially determining the volume integral of this integrand, over the whole space and the result pops out automatically and this result, which is just come out of our understanding of what the divergence of a vector is, this result is really the Gauss's divergence theorem.

We have actually established it. And this Gauss's divergence theorem, it expresses a volume integral of the divergence of A and expresses it as a surface integral of A dot dS, the surface integration being over a closed region, because we have determined the surface integral over all the surfaces, all the 6 faces of this rectangular parallelepiped and added up all of that.

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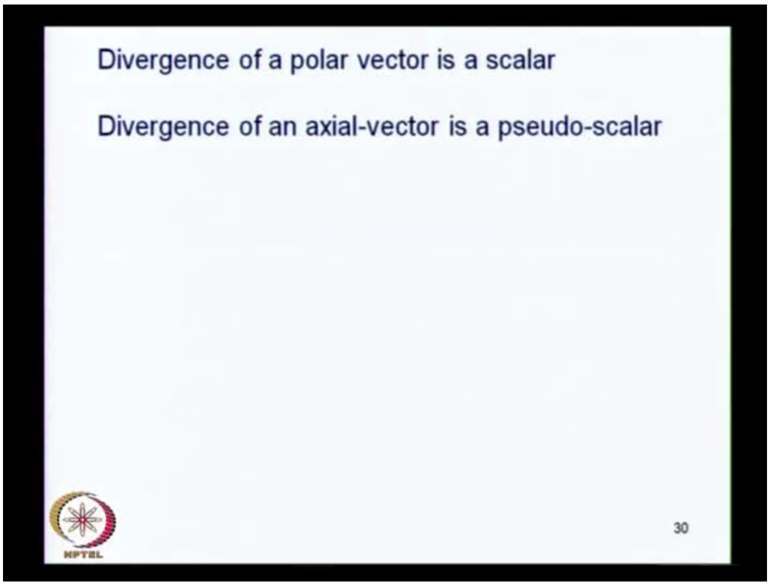

So, it has to be over a closed surface, which bounds a finite volume of space. So, this is known as the Gauss's divergence theorem and I should mention over here that, the divergence of a polar vector…