Select/Special Topics in Classical Mechanics
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Module No. # 08
Lecture No. # 27
Gauss Law Equation of Continuity (ii)

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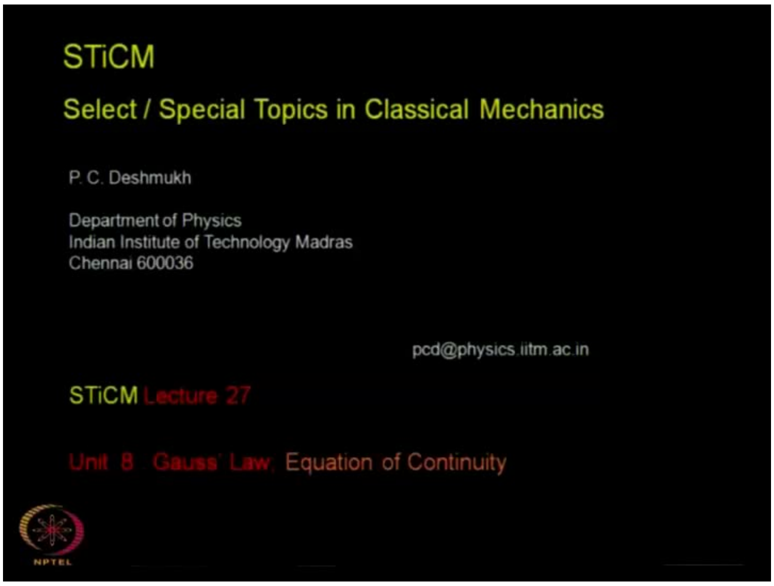

Very well let us resume our discussion on the Gauss's law and we will now apply it to fluid mechanics and develop what is a statement of conservation of matter expressed as what is known as equation of continuity

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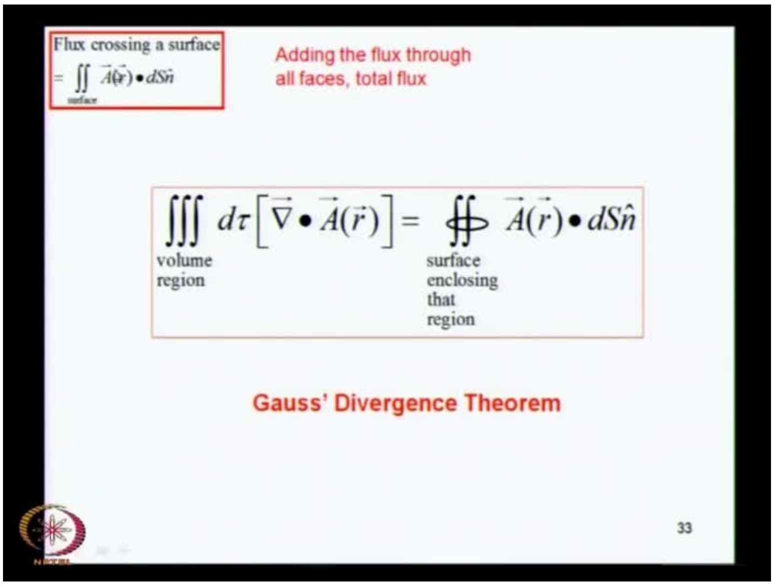

Now this is the Gauss's divergence theorem that we meet toward the end of the last class what it tells less that if you take a volume integral of this quantity which is the integrant this is what we defined as a divergence of the vector function A which is a vector point function we first define what are vector point function is

It needs to be continuously derivable so that we can define this so the continuity is an essential element then we defined what fluxes we have to define the flux in terms of the scalar product of the vector field with a directed area oriented area

So we defined how to orient an area we introduce that idea constructed this surface this this dot product this is a scalar product and then we added it up to get the surface integral and when we did this over a closed surface which encloses a certain finite volume of space we found that it is equal to the volume integral of a certain integrant which we called as the divergence and then mathematically quality that we established is what we call as the Gauss's divergence theorem

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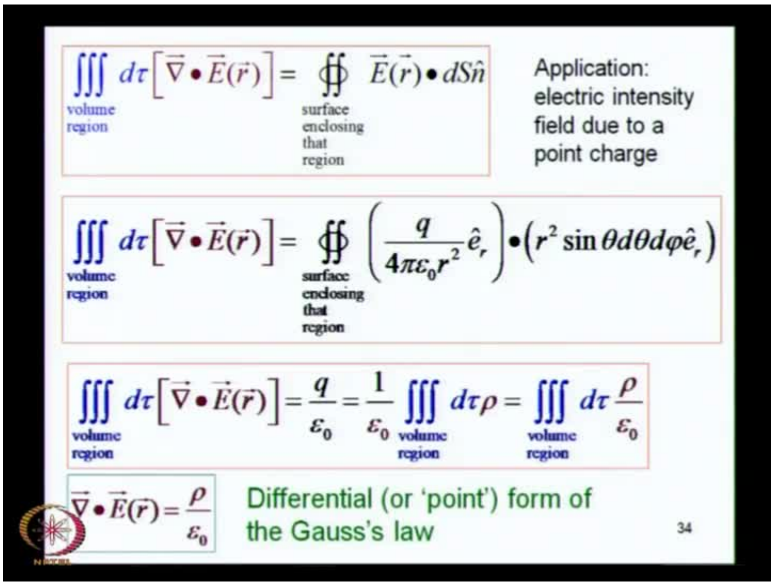

Now let us see that if this vector field A is specifically the electric intensity due to a point charge we let us take a very simple example just to see how it works this vector point function can be any vector field in particular we take it to be the electric intensity of a point charge

Then this is the divergence of E and on the right hand side you got the surface integral of E dot dS and there is this dS has got a direction indicated by the unit normal n which of course goes changes from point to point

So it is at the same point where you consider this E of r and about that point you construct an infinitesimal surface element which will then shrink to 0 and then you add it up integrated over the entire close surface

So now let us look at the right hand side electric intensity due to a point charge goes 1 over r square there is this usual 1 over 4 pi epsilon 0 factor there is this redial direction of the intensity vector then you have this dot product coming over here indicated by this dot over here

Now you have got a surface element which we know is r square times a solid angle which a subtended let say by a sphere at the center if the charge over at the center of the sphere and then the unit normal to this sphere is nothing but the radial normal vector which is plus E r in the spherical polar coordinate system so the right hand side is very easily spelled out

Now the evaluation of the surface integral is very simple because this r square cancels this r square in the denominator q over 4 pi is a constant which you can pull outside the surface integration what are you left with the 1 over epsilon 0 also comes out what do you are left with all you need to do is to integrate the solid angle over here all the angles which is nothing but 4 pi and that will cancel the 1 over 4 pi which is already there so the result is very simple it will be just this charge divided by epsilon 0 the 4 pi now gets canceled off so the volume integral of the divergence of A or divergence of E in this case is nothing but q over epsilon 0

But then you can also think of the charge itself as the volume integral of a charge density because it could come from a finite charge density which is sitting inside that regional space

So you can think of this as a volume integral of the charge density and then you can move this 1 over epsilon 0 once again inside the integral and make it a part of the integrant so you got a volume integral of rho over epsilon 0 and now you have these 2 volume integrals the one on the extreme left which is volume integral of the divergence of E and the extreme right which is a volume integral of rho over epsilon 0 and both of these are definite integrals

So they be in definite integrals over the same limits the corresponding integrands must be equal and you can equate the divergence of E to be equal to rho over epsilon 0

So this is also a result of the same theorem and this is an integral form it you need to carry out these integrations over in you know extended space so this is like a global form because you know it needs to consider all points in space whereas when you equate the corresponding integrands you get it in what is known as the point form so this is sometimes called as a integral form of the Gauss’s divergence theorem this is called as the deferential form or the point form

And this is the physical content or the mathematical content of these two is not different from each other they are completely equivalent

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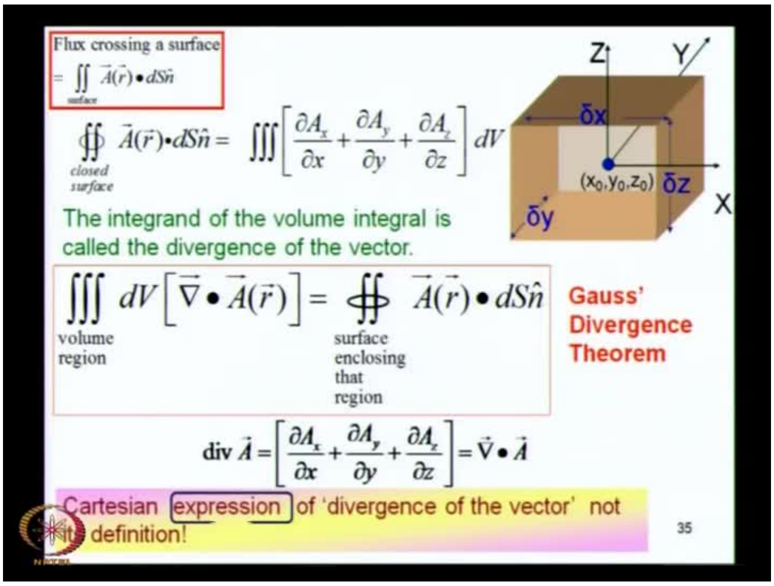

Now this is what we have got we recognized this integrant and decided to call it as divergence of a and offered it a notation written as del dot A I am choosing my words very carefully the integrant of the volume integral is what we have called as a divergence of a we have not defined divergence of a like this the reason is a definition of any quantity should not really depend on a coordinate system

So this integrant is what we have called as the divergence of a we recognize it it is of course an appropriate correct mathematical expression of the divergence of a in the cartesian coordinate system but this is not something that we will admit as the definition of the divergence of a because we will like the definition of quantity of any physical quantity to be independent of a coordinate system

So we have you know come a long way by exploiting the cartesian geometry but we must carry this mathematical formalism further in a manner which is independent of the coordinate system

So this is a cartesian expression but not the definition of the divergence so it is correct in the cartesian frame of reference nothing wrong with it but let us not give is give a this stratus of a definition it is not the signature

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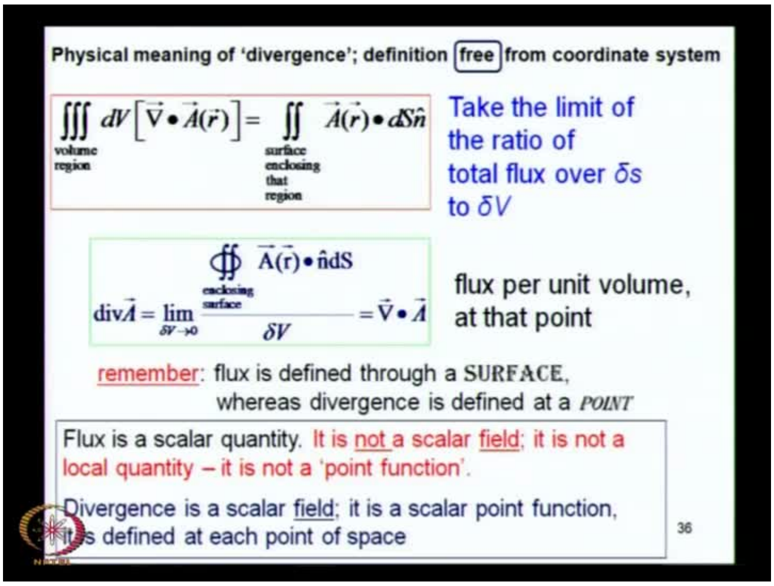

So let us now introduce a definition which is free from any coordinate system and if you look at this mathematical equality it already suggest to you what the definition should be and this mathematical equation which we now know is correct we have actually seen that it is correct it has been arrived at by introducing the idea flux very carefully then by adding up the flux crossing this surface which bounds a close region very carefully

And by extending this result to a region which does not depend on this rectangular parallelepiped shape but make it applicable to an arbitrary shape and then expressing the vector in any coordinate system the definition of the divergence of a automatically pops up because on both sides we have integrals and we know that these are limits of a sum

So if you take a sufficiently small region of space in a sufficiently tiny volume no matter what is shape is it does not have to a rectangular parallelepiped it does not have to be cylinder this is not even a cylinder it has got cylindrical symmetry but it is not a cylinder

You can take any close shape this is a close shape it is not even cylindrical it is not spherical it is got some obituary some different shape that of a rat so you take any regional space not quite a rat it does not even have a tail

So this is a anyhow you take any region of space and construct this flux and divided by the volume which is enclosed by that surface take the limit delta V going to 0 and now you have got a right hand quantity which has nothing to do with the cartesian geometry it has nothing to do with the cartesian coordinate system

Or we have exploited is the relationship between sums limits of the sum and integrals that is all we have done regular calculus and if you take this region to be sufficiently small take the limit delta V going to 0 we now have what we will call as a definition of the divergence of a vector quantity this is no longer just restricted to a cartesian geometry it is applicable to any coordinate system but then we must find how would we express it in other coordinate systems we will learn that

So it is meaning really is total flux this is the total flux per unit volume in the limit volume going to 0 so the divergence of a vector quantity is the total flux of that vector field across the region which bounds a certain finite volume of space in the limit that volume shrinks to 0 so it becomes a point function flux must be defined through the surface no matter how small that surface is and therefore the flux is the property not just of one point

So the flux is not a point function but the divergence of a vector field is a point function because you define it in the limit when the volume shrinks to a certain point

So the flux is a scalar quantity it is not a scalar field a scalar field is described by a scalar point function so that it is defined for every point in space so you have to be a little careful about these details the divergence it will describe a scalar field it is a scalar point function defined for each point in space

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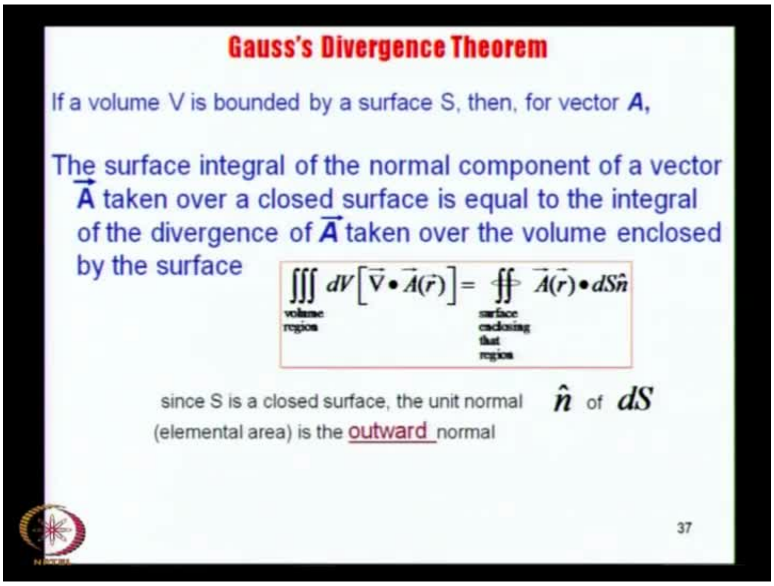

So this is now our divergence theorem that the volume element of the divergence of A is equal to the surface integral of the flux of a through the surface which encloses that volume so very simple statement

The direction of the oriented surface is always consider to be the normal to the surface it is orthogonal to the surface and it is always considered as the outward normal the definition of out and in there is no ambiguity in it because this is a close region of space so the one that is pointed out over to the space is the direction of n the unit normal

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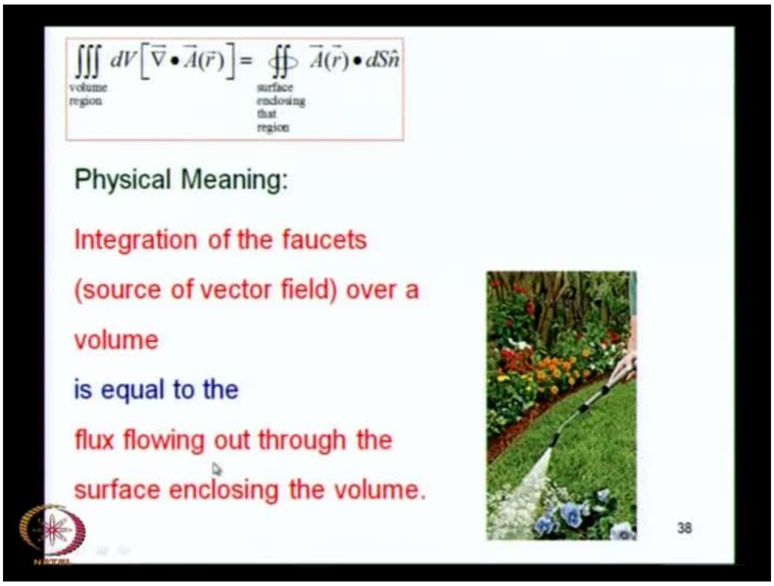

What the Gauss's divergence theorem does is it enables us to express a surface integral in terms of a volume integral and vice versa what it physically means is that if you have water coming out of a faucet and if you integrate over all the faucets there may be some other holes at different places if this is a rested one there may be some holes on the top as well that is not a one that you want use but who knows there may be some

And if you integrate over all the faucets over a certain volume region then you are going to get the net flux which is coming out through the surface which encloses the whole volume so it is essentially a conservation principle

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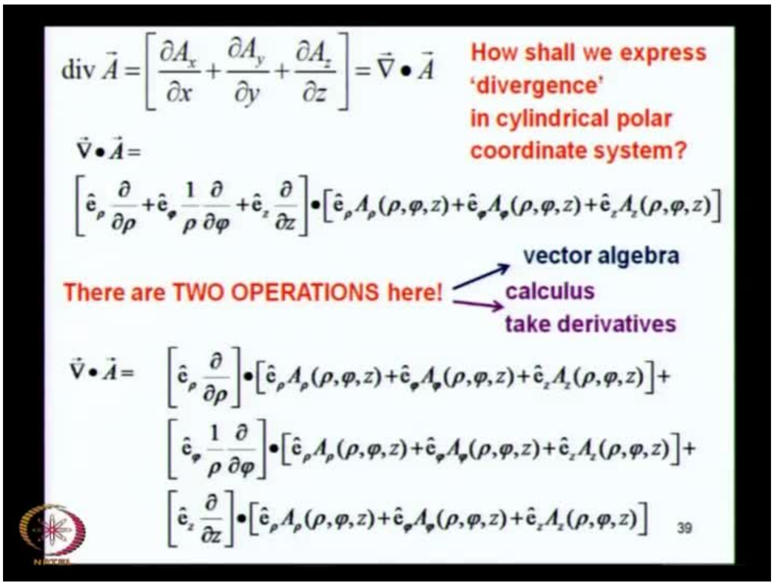

Now this is what we have in the cartesian coordinate system we must ask how shall we express the divergence of A in the cylindrical polar coordinate system some other coordinate system let us do it

So we already know how to write the gradient in the cylindrical polar coordinate system this is something that we spends quite some time discussing in unit 7 that was long time back once you know once up on a time but I hope you remember it

That the gradient has got a very simple expression in this cylindrical polar coordinate system it will include the unit vectors of the cylindrical polar coordinate systems which are e rho e phi and e z it will have partial derivatives with respect to the polar coordinates which are rho and phi and also z in the cylindrical polar coordinate system

So all you have done is to write the gradient over here in cylindrical polar coordinate system place the dot and written the vector A again in the cylindrical polar coordinate system so you resolve it in 3 components along e rho then along e phi and along e z that components being A rho A phi and A z but these are point function so each component of A is a point function and in general each component may depend not just on the subscript that it is referring to but all the 3 coordinates

So A rho will be a function not only of rho but also a phi and z likewise e phi will be a function not just a phi but also of rho and z and A z will be a function not just of z but also of rho and phi because these are point functions so they change from point to point

Now here I had learned how to get the divergence in the cylindrical polar coordinate system all we have done is to written it but what we have in front of us are two operations

In what we have written here there are two operations which are involved one is vector algebra because there is a dot sitting over here the other operation is calculus because we also have to take derivatives

So there are 2 different mathematical features that we must simultaneously respect and whatever we do must not be in conflict of vector algebra and it must not be in conflict with the calculus and our mechanisms mathematical mechanisms to take the differentials and the derivatives

And the reason we have to be careful with in the cylindrical polar coordinate system which we did not have to do in the cartesian coordinate system is because the unit vectors of the cartesian coordinate systems which are e x e y e z they are not changing from point to point in space

But the unit vectors of the polar coordinates they do change from point to point well not with respect to all the points because e z of the cylindrical polar coordinate systems is of course a constant vector it does not change with respect to anything and e rho and e phi change only with phi and not with rho we have done this quite carefully when we introduce these coordinate systems that was also once up on a time

So the unit vectors e rho and e phi change with the azimuthal angle phi but not with rho so whenever you take derivatives and this is where you take the derivatives derivative with respect to rho not just a derivative but this is the partial derivative which means that when you take derivative with respect to rho you hold the other 2 coordinates phi and z constant

You take the derivative with respect to phi over here holding rho and z constant so whenever you take the partial derivatives you must take partial derivatives not just of A rho which obviously depends on all of these but also of e rho which depends at least on phi if not on rho and z

So the partial derivative with respect to phi of e rho must be considered explicitly so whenever you take partial derivatives you have to treat this as a product of 2 functions one is a vector function e rho the second is a scalar function A rho

And treat it as a product of 2 functions so that you get the first function times of derivative of the second plus the derivative of the second function times the first function that usual rule of taking the derivative of a product of 2 functions will come into play

So let us do this carefully there are two operations so I do is carefully now all I have done here is to write it term by term the first term is e rho del by del rho then I have this dot and then I have the rest of the vector so I am still not carried out any of the two operations I haven't carried out either of the two operations I have only separated the 3 terms so that we do one step at a time

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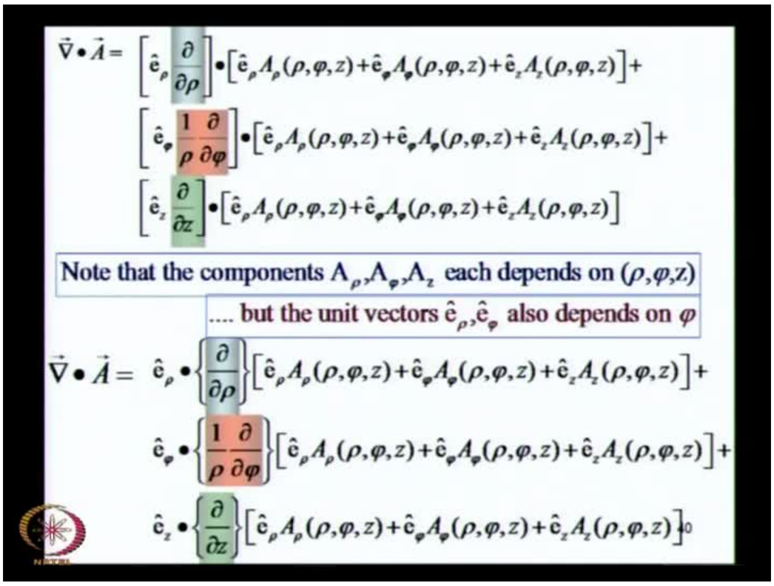

Very carefully one step at a time so we have separated the 3 terms one is this this is the second this is the third you have got the dot placed over here and then you have got the vector a expressed in cylindrical polar coordinates on the right of this dot

And now you carry you respect the fact that you have to take the derivatives not just of the components but also of the unit vectors so before you take the scalar product this derivative must be carried out this process of differential there are two operations the vector algebra and the differential calculus or the differential vector calculus if you like

the components a rho a phi a z of course depend on all the 3 coordinates but the unit vectors also depend at least on phi f not on others in spherical polar coordinates the dependence is a little larger but in this case there is no dependence on rho and z

So here we go so you must first take the derivative this operation which is del by del rho let just look at the first term look at the first term you have to take partial derivative with respect to rho of this entire thing so I move this derivative operator del by del rho to the right of this dot

So then I can carry out the derivation of the product of e rho and A rho with reference to rho with respect to rho holding phi and z I do the same with the second term which is 1 over rho del by del phi and then I operate I think of this as a deferential operator which is a derivative operator this must operate on the right hand side and this right hand side has got 3 terms each of which is to be seen as a product of a scalar function and a vector function of the point

So and keep a track of where the differential operators are coming so this deferential operator which were setting to the left of this dot has now moved to the right of this dot likewise this differential operator has move to the right of this dot it really does not matter where the 1 over rho comes because that is just a multiplying factor here so it does not matter and then same thing with the z the derivative with respect to z

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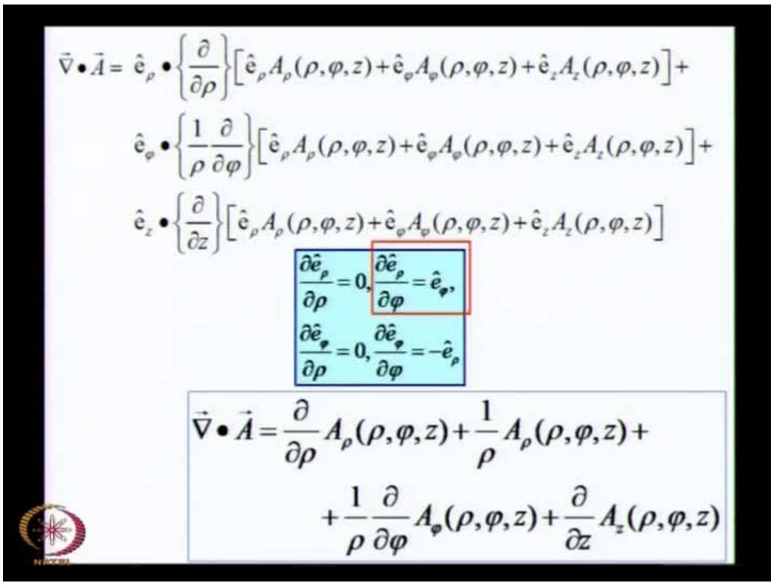

So now what we have to do is to carry out the deferential calculus first and then we will do the scalar product which is the vector algebra that is something that we will do in the end now when you do the calculus you have to remember that the derivative of the unit vector e rho with phi is e phi the derivative of e phi with respect to phi is minus e rho so this we have done we will use this results and we will need this when we take the derivatives of e rho and e phi with respect to this deferential operator del over del phi

So if you do this term by term and I will let you work it out as an exercise it is a simple one then the result for the divergence of A is del over del rho a rho 1 over rho A rho rho phi z 1 over rho del by del phi a phi and then del over del phi del z A z so there are really 4 terms 1 2 3 and 4 that you must take into count

So I hope that the procedure to obtain the explicit expression of the divergence in the cylindrical polar coordinate system is clear see here arrived that it in 2 different steps first we establish the Gauss's divergence theorem in the coordinate system then we suggested that the result must be independent of the coordinate system and when we expressed it in a manner which is independent of the coordinate system we had the left hand side of the Gauss's divergence theorem to be given by a volume integrand the integrand of which we defined as the divergence

We recognize a form of the divergence in the cartesian coordinate system and then we asked what must be its form in the cylindrical polar coordinate system and to get that form we carried out the 2 mathematical processes which are involved one is calculus the differential calculus the second is the vector algebra in the correct sequence so that whatever is derivable with respect to the independent degrees of freedom is properly accounted for by using a calculus after which we use the vector algebra so the result is rather simple

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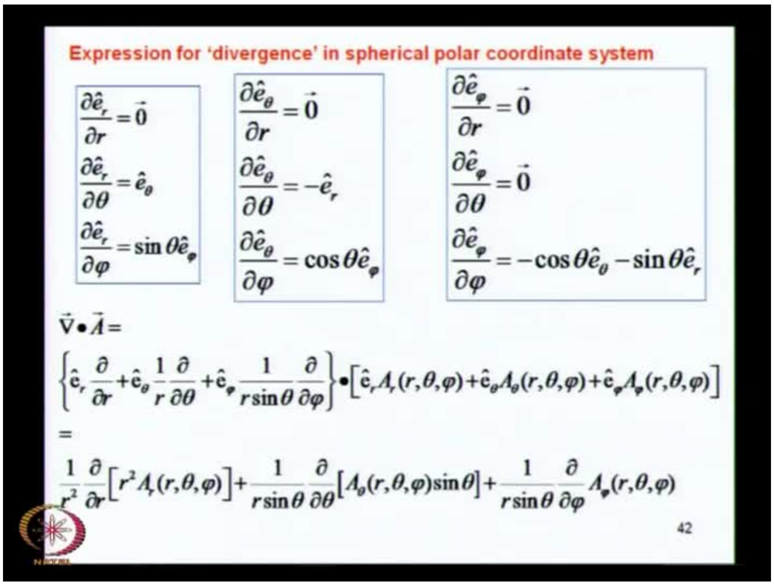

In this spherical polar coordinate system the same procedure has to be adopted details are for you to work out at home staying awake not watching a movie but enjoying calculus

And you have met these results earlier actually it is more fun than many movies which are so boring but some are good but only some anyhow so you have these relations which we have established already and using this you can get the expression for the divergence of A in this spherical polar coordinate system there is absolutely no need to by heart this expression because it is so easily derivable

So I am particularly alerting those of you who have a very sharp memory because you will mug it up very fast and you will remember the expression and then when you if you ever have to go for an exam you would have spent what you call as night out and then you will be so tired at the time of the exam then you will miss the 1 over r sin theta you will forget about it so no matter how sharp your memory is please do not by heart it it will take you only 2 or 3 minutes to derive it from first principles it is very easy

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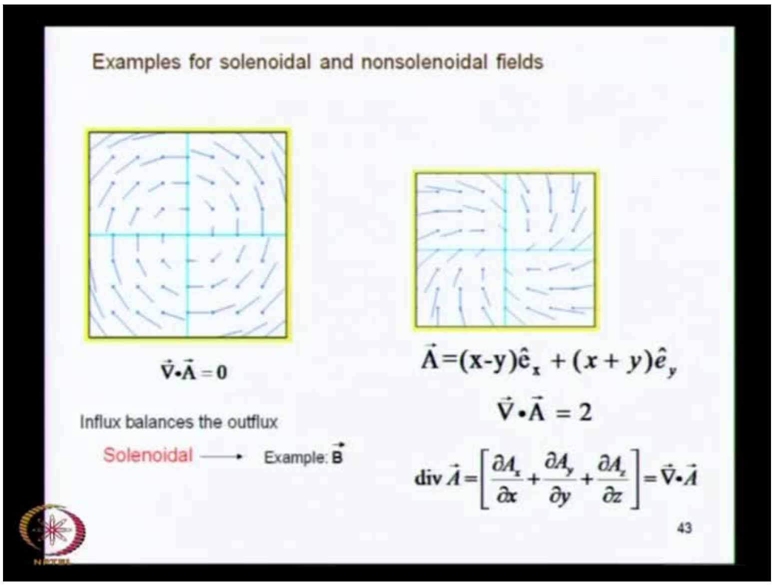

Let us take up some examples of the divergence so whenever the divergence of vector field is 0 that vector field is said to be solenoidal because that is the kind of thing you have for a magnetic field in a solenoidal so when the divergence of a vector field is 0 it is called a solenoidal so it is just a name of the divergence name of a vector field when its divergence is 0 the magnetic field is a good example of this

If you have a some other field here is an example of another a vector field and you can determine its divergence by using the cartesian expression and find it is divergence is constant it is a scalar it is a constant number it is a same no matter where you find it

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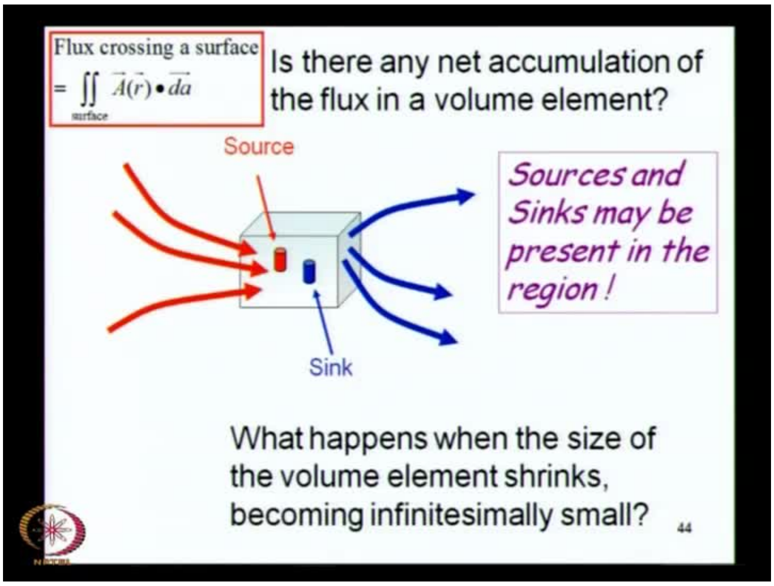

And now we can ask if there is any net accumulation of the flux in a volume element this is the question that we had raised earlier on but now we are ready with the answer even if sources and sinks were present

Our interest is in determining not just what will happen if there will be any net accumulation of the flux in a volume element but especially so when the volume element shrinks to a point so that we can talk about point functions we can talk about properties of a given point in space

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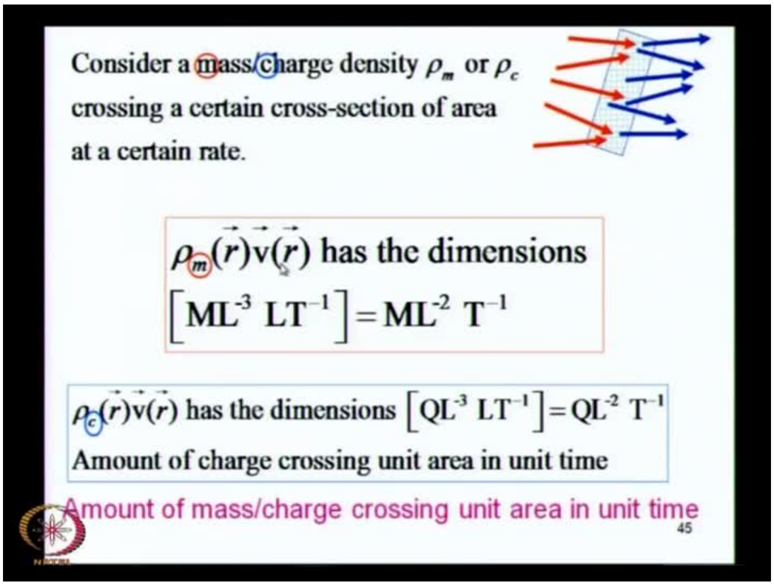

So we consider you know mass or charge density either rho m or rho c which is crossing a certain cross sectional area at a certain rate we have defined this quantity called density times velocity I introduce this quantity earlier or it could be a charge density so rho c which will have the dimension of charge per unit area per unit time so QL to the minus 2 T to the minus 1 and the quantity we have introduced is a product of density and velocity what it represents is the amount of mass or charge crossing unit area in unit time

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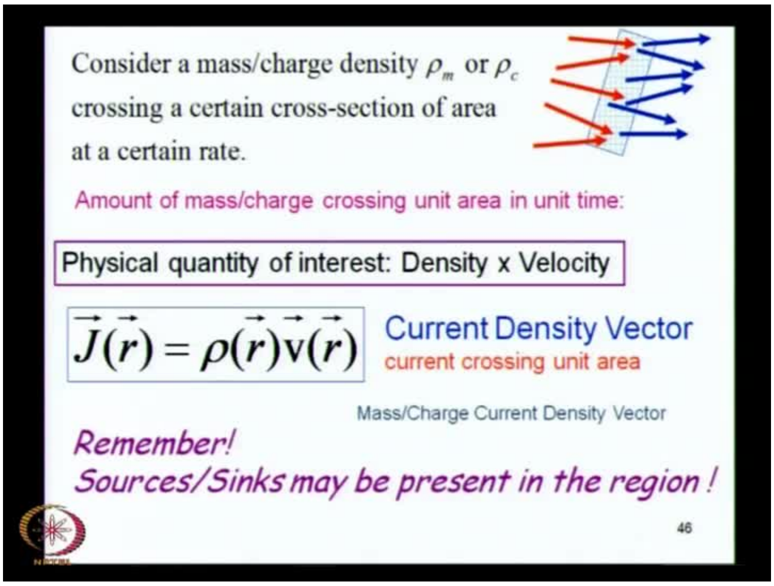

Essentially it is density time's velocity and it is called as current density vector that is what it is called whenever you consider the product of density time's velocity it could be the mass density it could be the charge density it does not matter

It is called as a current density vector which is a vector point function because for each point in space you have a density and for each point of space you have got a velocity defined and this is the velocity defined in the continuum limit of fluids

And accordingly this is the mass current density vector or a charge current density vector and in general sources and sinks may be present in the region

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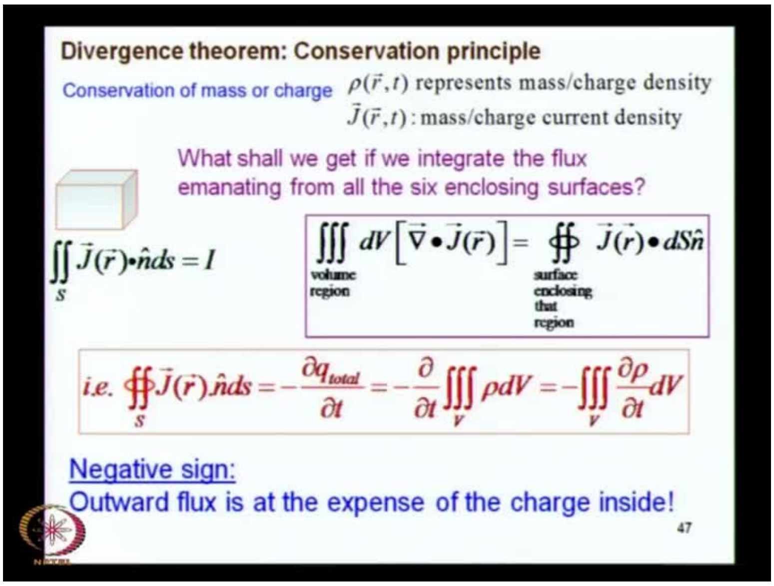

What we have is a conservation principle because if you look at the net charge current density which is flowing out of a close region of space charge per unit area integrated over area will give you the charge

And the charge flowing per unit time will give you the current right d q by d t is the current so when you are integrating the current density vector over the surface you will essentially get the current because of physical quantity which the current density vector is which we have seen is the quantity of either mass or charge crossing per unit area per unit time a quantity which is defined per unit area integrated over the whole area will give you the current itself will this is the charge per unit area so it will give you the net charge per unit time will give you the current

And therefore when you integrate this flux of the current density vector you are going get the current which is oozing out of that region we have already establish the Gauss's divergence theorem I have written it not for the obituary vector field a but now specifically for the current density vector we know are we know are talking about specific physical quantity which is either the mass current density or the charge current density and our formalism develops in parallel applicable to both

So this current which is coming out which is the rate of change of charge rate of flow of charge it will be del q by del t right it must come with the minus sign because there is a conservation principle it has to come at the cost of the charge which is inside the charge which is coming out has to be at the cost of the charge which is inside you are not creating any charge or you are not destroying any charge in the absence of any source of sink right

So whenever you are not creating any charge whenever you are not destroying any charge then the net flux which is coming out of a closed volume which is given by the surface integral of this flux must be exactly equal to minus del q by del t this is the simple conservation principle and this conservation principle now that you have you understand what this minus sign is telling us then you also see that this charge itself can be written as the volume integral of the charge density

And now the integration is over special coordinates the differentiation is over time so these are independent of each other so I can carry them out carry out these 2 process one is differentiation with respect to time and integration with respect to space independently of each other and in any order that are like because here as are completely independent so I move the derivative operator del by del t inside the integral sign

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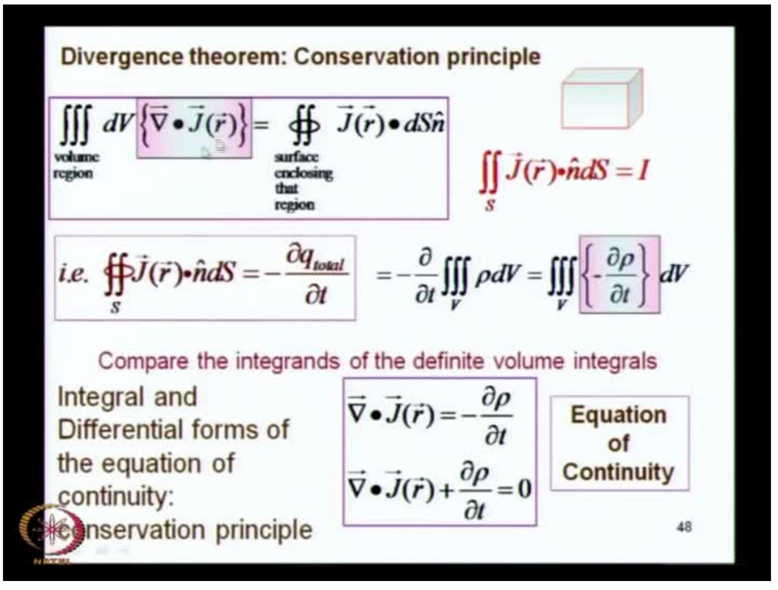

And now if you see this relation I have got a volume integral of the rate of change of density now what this allows me to do is to interpret the net current which is oozing out of their region in terms of this volume integral of the time derivative of the density so we can write this a little more clearly now

These results we have got but this charge now the total charge we write as the volume integral of the charge density move this time derivative operator inside the integration sign because these 2 processes can be carried out independently of each other and now you have got a volume integral over here which is equal to the surface integral but this surface integral is equal to this volume integral over here

And now if you look at these two volume integrals this one over here and these this volume integral these are definite integrals over essentially the same volume of space so the corresponding integrands must be equal in other words del dot J must be equal to minus del rho by del t the corresponding integrands must be equal

So we have got a mathematical equality now which is that the divergence of J is equal to the negative partial derivative partial time derivative of the density now you can write this term on the left hand side and you get an equation which is called as the equation of continuity the del dot J plus del rho by del t is equal to 0

And the entire derivation is based on a conservation principle so the equation of continuity expresses a conservation principle but you can express it either in this integral form or in this differential form this is a point form because it corresponds to properties at a given point here it includes integration over various elements surface elements on the right hand side volume elements on the left hand side so these required extended features whereas in this form when you equate the appropriate integrands of definite integrals you have expressed it as point functions

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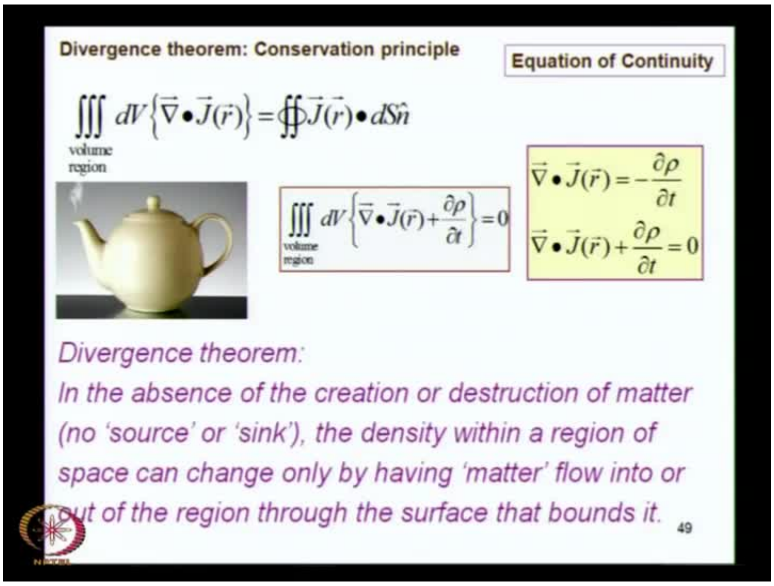

So this sums it up you have got the equation of continuity what it tells us is that if there is no creation or destruction of matter in the absence of any source or the sink that density within a region of space can change only by having matter flow into the region or out of it

And it can do so only across the surface so if you carry out the integration of the flux across the surface you get the net quantity which has oozed out or converged in so this is the divergence theorem this is the mathematically very regular statement of the conservation principle we will take a break here if there are any questions or comments I will be happy to take them

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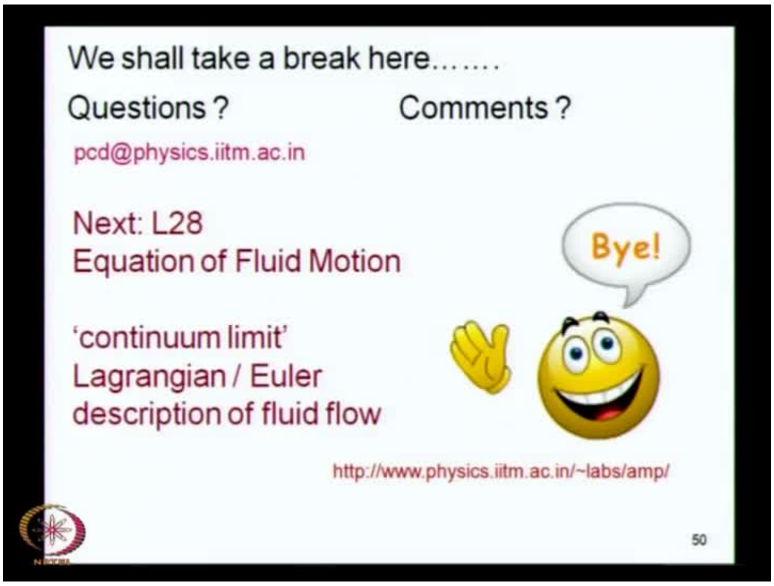

Otherwise we take a break here and in the next class we will establish the equation of fluid motion which is the equation of motion for a fluid this will require us to consider a very rigoures description of a fluid in what is called as the eulerian description or the lagrangian description of the fluid so we will discuss this in the next class

So that is the topic for the next class and at this point I will take a break if there is any question I will be happy to take