Select/Special Topics in Classical Mechanics

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Module No. # 09

Lecture No. # 29

Fluid Flow Bernoulli Principle (i)

Greetings. We will begin our discussion on unit 9 today. This is on fluid flow and in particular, we will discuss the Bernoulli's principle.

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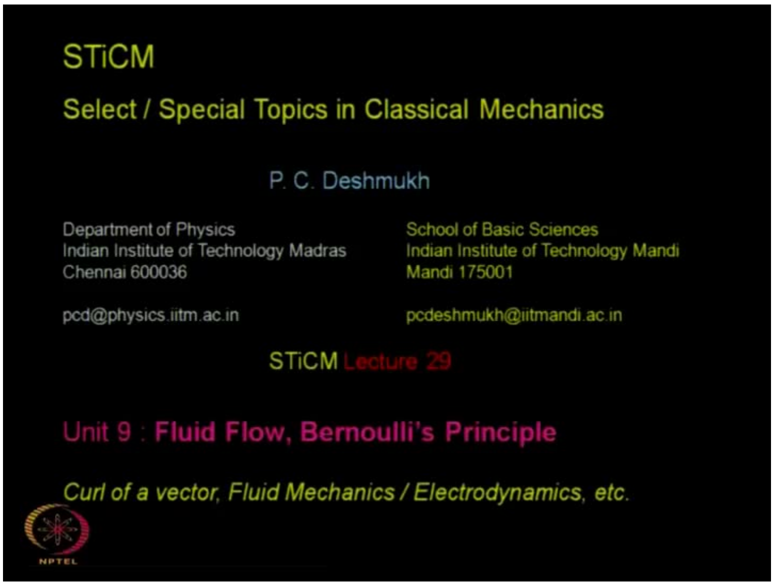

We will be talking about the curl of a vector and find applications in fluid mechanics, and prepare ourselves for applications in electrodynamics.

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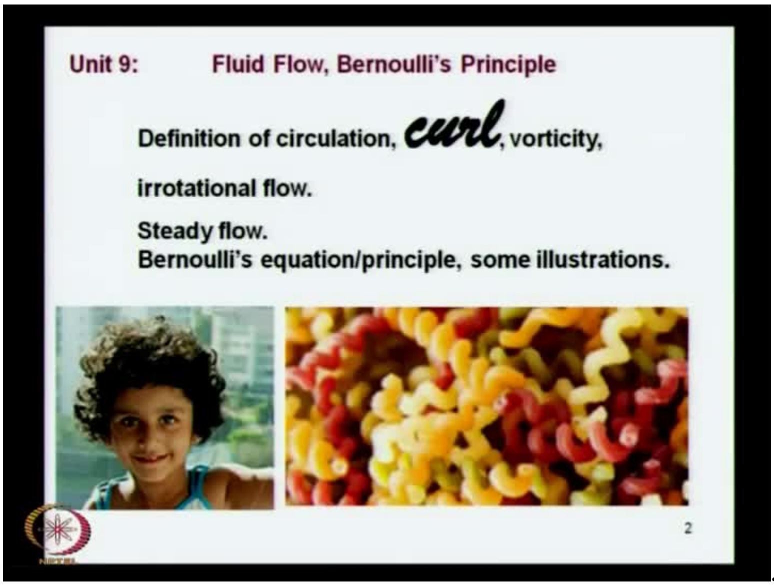

So, in the Bernoulli's equations, we will also be, you know, taking up a few illustrations. We will talk about what is meant by an irrotational flow, what is meant by a steady flow, and in particular, we will introduce the curl of a vector and the vorticity.

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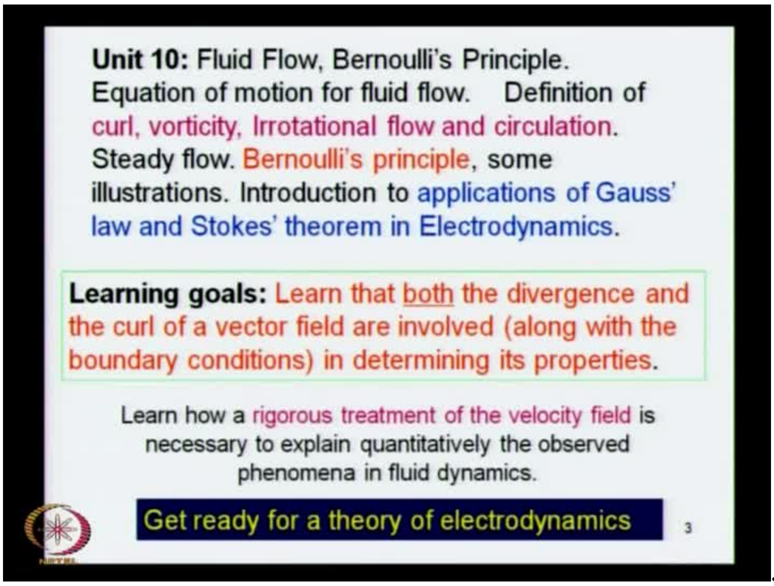

So, our learning goals are that having introduced the divergence in the previous unit and then once we introduce the curl of a vector in this unit, we will find that both the divergence and the curl of a vector are required to describe a vector field. One is not enough; you need both; in fact, you also need something more than the curl and the divergence of a vector you need the boundary conditions, but I will come into this to as we make some further progress in this unit.

We will learn to develop a rigorous treatment of the velocity field and then explain fluid dynamics in terms of the formulation that we develop, and most importantly, we will get ready for applications in electrodynamics, which will be our next unit in after this.

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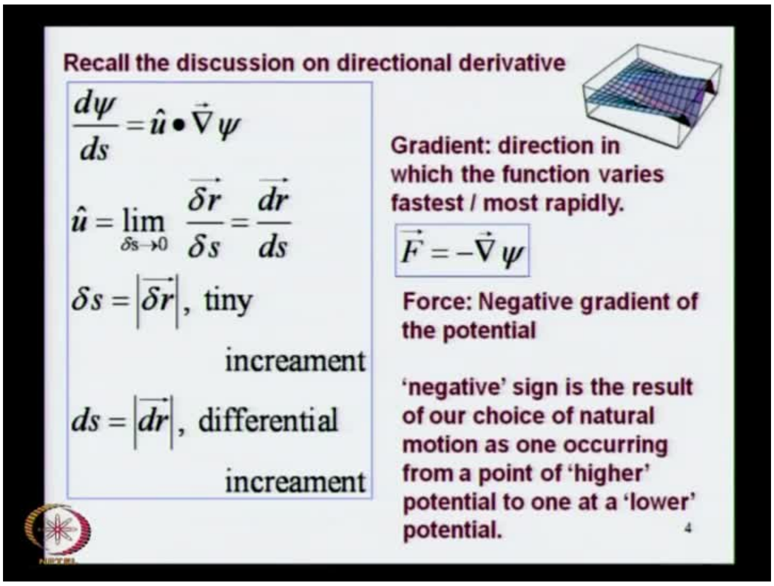

So, let me first recall a few things that we studied in the previous unit. So, we introduced the idea of a directional derivative and we know that it is very intimately connected with the idea of the gradient of a scalar function, and we know that the gradient is always in the direction in which the scalar function changes most rapidly.

We found that you can introduce a potential whose negative gradient is the force. The reason there is a negative sign, is a matter of, you know, a convention that we have adopted, because whenever we talk about natural motion, we talk about the motion - the natural motion being from point of higher potential to lower potential. So, that is the matter of choice. So, it is mostly semantics and it is for this reason that the negative sign gets in. And we have learned that the directional derivative is just the component of the gradient in a given direction.

So, the directional derivative is a scalar quantity and it has got a directional attribute, and the directional attribute comes from the fact, that, if you take a unit vector in a particular direction, then the component of the gradient in the direction will give you the corresponding directional derivative. So, this is just to recapitulate some of the ideas of gradient and the directional derivative that we had used in our previous unit.

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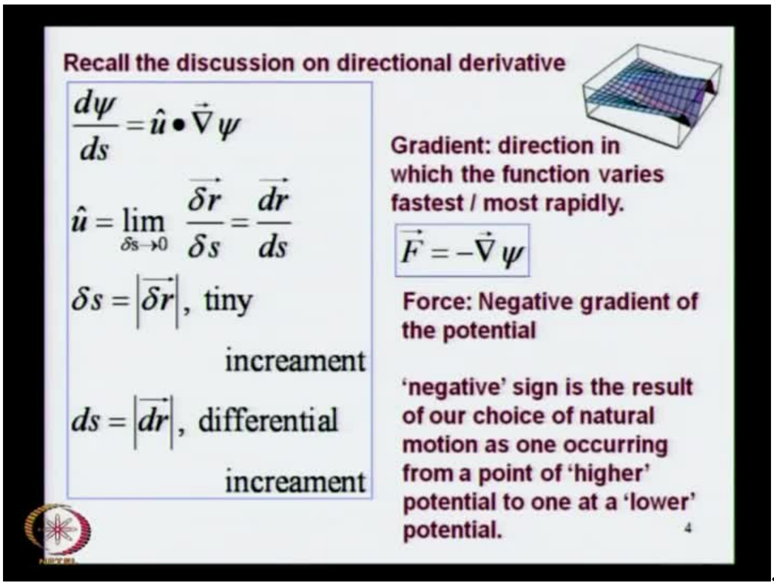

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Now, we have another definition of the force, which is in fact, the very first one that was introduced in Newtonian mechanics. This came from the cause effect relationship. This is the linear response, the stimulus response formalism in Newtonian dynamics and we have this definition that the acceleration is proportional to the force.

Now, we have another definition that the force is given by the negative gradient of the potential, and we should now ask - are these two conditions - we have two equations in front of us - do they necessarily correspond to each other? Are they consistent with each other? Are they compactable with each other? Or, are we defining different quantities by these definitions, because the negative gradient of a potential is one idea; inertia times acceleration is a different idea.

And what we are claiming is that both the ideas give you the force which is the same thing, and we therefore must enquire if we get consistency in these two relationships. So, it turns out that when the force is conservative, then these two ideas generate the same force. So, the criterion of a conservative force is that, if you take the line integral of that force over a closed path, it must vanish; or equivalently, if you take the line integral of this force between two arbitrary points a and b, then the path integral of the work done must be independent of the path; so, how you get to the point b from the point a does not matter.

You could go head on to the point b or you could take some curved path to the point b, you can take a path which crosses itself, which goes all over the globe and finally ends up at the point b, does not matter; finally, the net work done must be independent of those path. So, this is our idea of a conservative field.

And if you look at this relationship over here (Refer Slide Time: 06:40), you have a path integral over a closed path and this is what is called as a circulation. Because it is not that it has to be, this path has to be over a circle; it is over any closed path. It absolutely does not have to be circle; it could be any closed path, nevertheless it is called as circulation because it tells you that wherever you begin, finally, wherever else you go, you come back to the same point. So, this is called as circulation.

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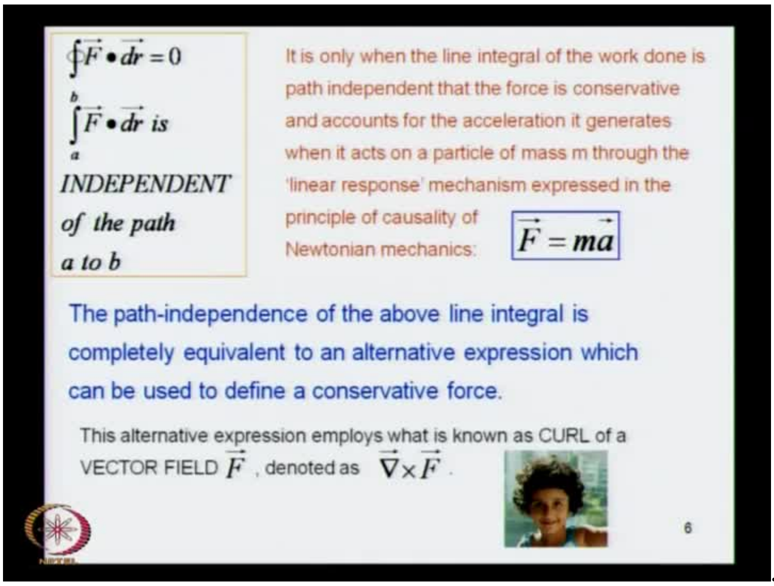

And when this circulation vanishes, then the two relations f equal to minus del psi and f equal to mass times acceleration - these give you essentially the same physical quantities. So, the independence of the path integral is a necessary and sufficient condition for this to happen. The origin of the idea, of course, is quite different. This f equal to ma comes from the principle of causality from the linear response formalism which is contained in the heart of Newton's second law and this idea comes from our consideration of what a conservative field is.

So, the physical ideas which are involved have a completely independent basis, but they generate exactly the same path force in a situation when the force field is conservative.

So, now, what we will introduce is an alternative expression to represent the same idea and this will be in terms of a new quantity which we have not introduced either in this course and this is called as the curl of a vector. And if you see the picture over here, that name curl already suggests that there is some kind of curliness and you already get some idea in your mind. So, you will find that, this is given a very precise meaning in our formulation. So, let us first give the definition of the curl of a vector and I suggest that you follow this definition with me that the curl of a vector is defined for a vector point function. Now, you know what a vector point function is.

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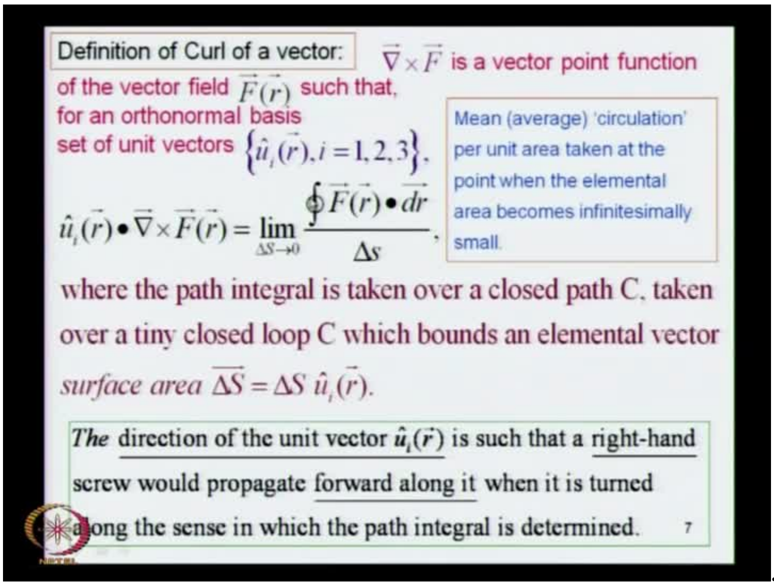

The quantity which is being defined is a vector field, which means that it has got a well-defined value at each point in space and that value is a vector by definition. So, its components transform according to the laws of transformation for a vector, the cosine law. So, it is a tensor of rank one, which has a well-defined law for the transformation of its components when you rotate a coordinate system. So, that is a criteria we have done that in an earlier unit. So, you have got a force field, a vector field f of r, which you can describe in some orthonormal set of base vectors.

So, you have got three vectors which are orthogonal and normalized. That is indicated by this caret on top of these unit vectors. It could be Cartesian unit vectors, but they do not have to be. They could belong to any coordinate frame of reference; it could be a spherical polar system or a cylindrical polar coordinate system for example, or no matter what. And any vector is completely defined in terms of its components; if you give all the three components, then the vectors gets uniquely defined.

So, what we do is, we define this curl of a vector which is written as del cross f. So, it looks like the cross product of 2 vectors, but that it is not. The reason, of course, is the gradient is not a vector; it is a vector operator. So, this is not to be read as a cross product of 2 vectors; you can always read it as del cross f or by its name; it is called as a curl of f.

Now, this will generate a vector field and if you give a definition for its three components which are mutually orthogonal to each other, then you would have defined the complete vector field. So, its component along the i th unit vector is given by the projection of the curl of this vector on this unit vector (Refer Slide Time: 11:27), and this projection gives you that corresponding component, and it is given by the right hand side uniquely.

What you do is construct the circulation of this field. We have already introduced this idea of a circulation. This would be the work done, if f was the force, but f could be any other vector field as well. So, it does not have to be the force. So, whatever is a force field or whatever is the vector field, you always can define the circulation for that vector field. You define it over a closed path, just the way you defined the work done over a closed path, but this will not be the work done, if the vector field you are talking about is not a force; it could be some other vector field. Then you divide it by the area which is contained in the loop because the circulation is a line integral; it is a path integral taken over a loop.

And this loop contains a certain area and I will tell you how this area is to be constructed because it must have a certain relationship to this unit vector u i. The relationship is indicated over here (Refer Slide Time: 13:02) that the path integral which is the circulation it is taken over a closed path C, over a very tiny closed loop C, which bounds an elemental vector surface area delta s which contains the magnitude of the surface element times the direction.

So, it is this direction which is coming in over here. So, obviously, this area will be orthogonal to u i and you must construct this so that the right hand screw convention is followed. So, I will elaborate on this so that there is no ambiguity about it. So, there is a certain relationship between the direction of this unit vector and how this area is chosen. This is a magnitude, the left hand side, of course, is a scalar; this is the component of the curl of the vector; the right hand side is also a scalar in which the numerator obviously is a scalar, and the denominator is the magnitude of the vector area.

The area, a tiny infinitesimal area will be a vectorial and its magnitude will be a scalar. So, that is that scalar quantity which is used over here. So, divide the circulation and then you construct the limit as delta s going to 0. So, you can see that you have divided the circulation by area. So, the quantity you are looking at a circulation per unit area and this must be taken at a point - how do you do that by letting this elemental area tends to 0, so that the area contained by that circulation will become infinitesimally small. And when that happens, you would get a property which belongs to a point because the area will then shrink to a point.

Now, the direction of the unit vector u must be such that the right hand screw would propagate, forward along with when it is turned along the sense in which the path integral is taken. Because this path integral is taken along a path and you can take it along this path. You can see this pointer you know going round in a clock wise direction, but I could also take this path in the anticlockwise direction. If I were to take this in the anticlockwise direction, then the relationship which the right hand screw rule provides between this surface element delta s and this unit vector will be just the opposite. You will end up defining the opposite unit vector.

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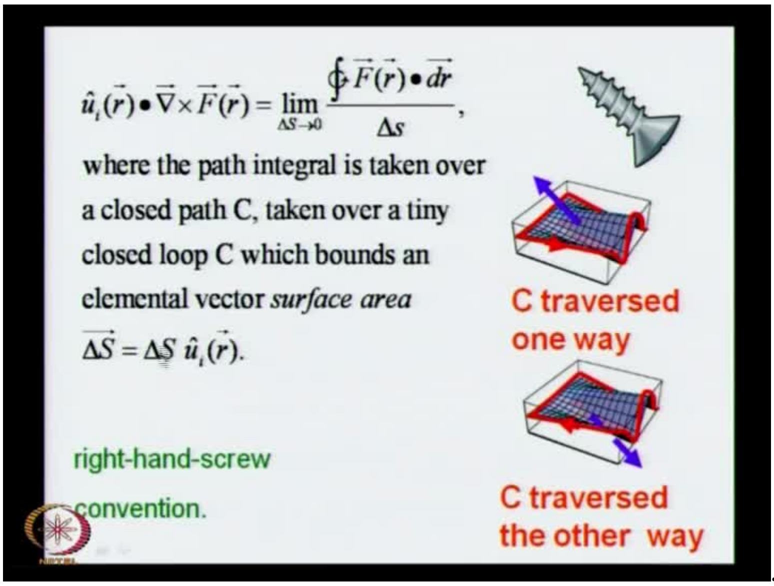

So, let us see that. Let us take a look at some of these pictures so that it will become very clear. So, you have got a certain surface element over here. This does not have to be flat; it may have some curvatures. It could be like the handkerchief that we always talk about; you pinch it somewhere in the middle or at 1 point or at 2 points or 3 or 4 points; let it wiggle across; so, it will have some ups and downs; it does not matter, but then it has got a unique boundary and this boundary can be traversed in one way as shown in this. So, you go along the boundary like this, then go along this edge, come down, and then follow the edge on the first side, come along this edge, and this is the direction in which the path C is traversed (Refer Slide Time: 16:48).

Now, if this is the path, then if you take a right hand screw and advance it along this path, then it will have a forward motion going along this. right. So, this is the relationship. This u i is the direction of this surface element and this direction is intimately connected to the direction in which this edge of this surface element is traversed (Refer Slide Time: 17:20). Because if you traverse it in the opposite way, then the right hand screw would traverse in just the opposite direction. So, if this is your path, then a right hand screw would move forward in this direction. If you take another path, then the right hand screw would propagate in just the opposite direction. So, this is the connection between the direction u and the direction in which this circulation is built because it is constructed as line integrals.

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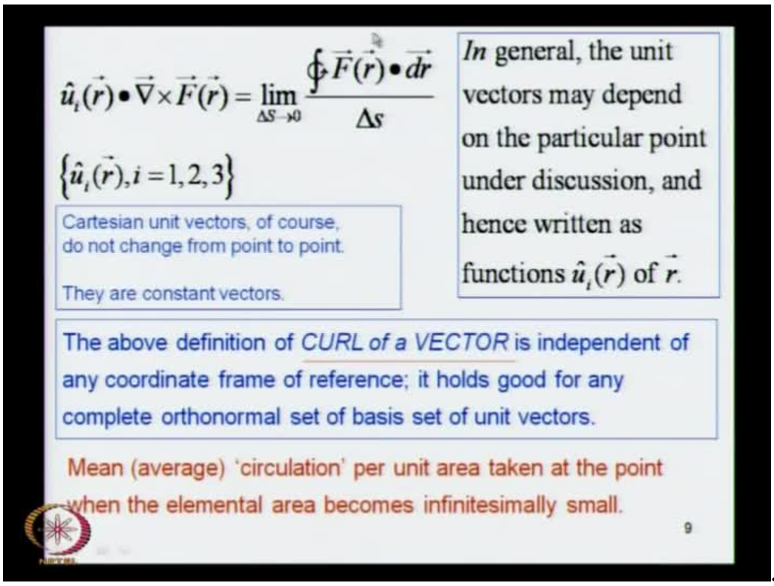

Now, we can take a certain basis set over here: the Cartesian unit vectors are often a favorite choice; they are constant vectors the e x e y e z. You can take any other basis set; it really does not matter. But what is going to happen is that if you choose the polar coordinates? Then the unit vectors change from point to point. They are not constant vectors. So, the unit vectors will be returned with an argument of the point. They are independent of the point only for the Cartesian unit vectors, but in the case of any other coordinate system, they will change from point to point. So, in general, the unit vectors are returned as a function of r - Cartesian coordinate vectors. The Cartesian unit vectors are special case which do not however change from point to point.

Now, this definition of the curl of a vector, this is in terms of an orthonormal set of basis. It will apply to the Cartesian unit vectors, but it will apply to any set of unit vectors and it is therefore, completely independent of a particular coordinate frame of reference. Now, you have to take this average circulation. Why average? As I mentioned earlier, you construct the circulation; divide it by the area; that is what gives you the average per unit area. And then, take this over an infinitesimally small elemental area.

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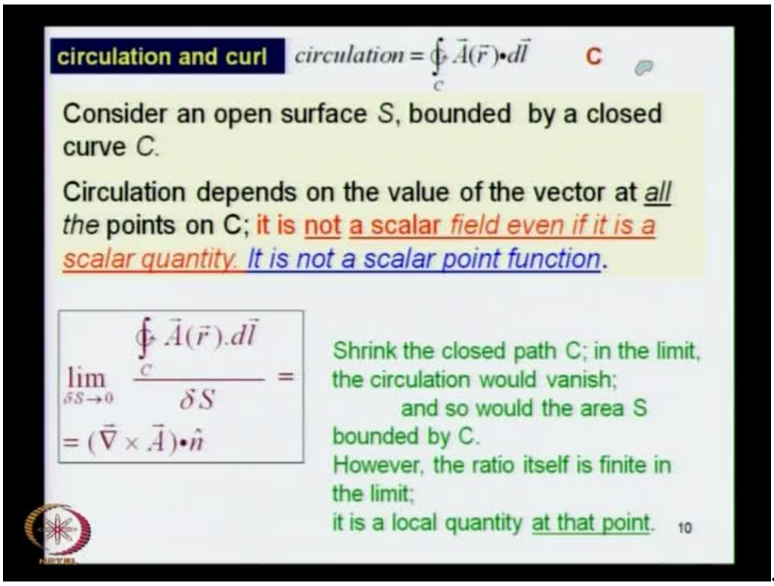

So, let us do that step by step. You have constructed this path integral and let us say that you have got a certain loop which goes round at tiny piece of area. You must note that the circulation requires this A dot d l to be constructed at each tiny piece of elemental you know element of this path and then you have to add up this. So, when you do this addition in the limit, this addition generates the integral. So, this is the property of this entire loop. It is therefore not a point function because that loop is not made up of a single point.

It is not a point function and it does not generate a scalar field. The circulation is a scalar, but it is not a scalar point function; it does not generate a scalar field. So, every scalar does not generate a scalar field. The reason is that, to become a point function, it must be a property of the point and a circulation, of course, cannot be a property of the point because you have to add up A dot d l over the boundaries of any edge; it could be the edge of this table, you go from that end to this end, to the third end, and then to the fourth and then come back. right Then, you get the circulations. So, this is the circulation per unit area.

But then, what happens is that, this area which goes in the denominator, when you let that area also shrink to a point in the limit, that the area reduces to 0; both the numerator and the denominator tend to 0, but the ratio remains nonzero. So, 0 by 0 is indeterminate, but you can take the limiting value, and this ratio turns out to be just the component of the curl of the vector, along the direction which is normal to this area.

So, what you can do is shrink the closed path in the limit. The circulation would vanish; the elemental area would also vanish. The ratio would give you a nonzero finite value and this will be a property of that point. So, what is coming out of this is a point function. The circulation is not a point function; the area is not a point function; both are properties of certain extended regions of space, but their ratio in the limit that the area shrinks to 0 gives you a point function.

So, the curl of a vector is a point function; it is a vector point function; when you define it for all the points in that, in a region, you generate a vector field. So, this is the definition of the curl of a vector and this is how you can see that you can take a certain area, and let it shrink to a small point.

And when that happens, you will get the numerator and the denominator both vanishing, but the ratio giving you a nonzero finite component of the curl of the vector. So, this is more precisely, since we must take the limit as the elemental area shrinks to a point, this is called as the limiting circulation per unit area; not just circulation per unit area, but the limiting circulation per unit area. What is this limit referring to? It is referring to the limit that the area shrinks to a point and the path which goes around it then shrinks as well along with this.

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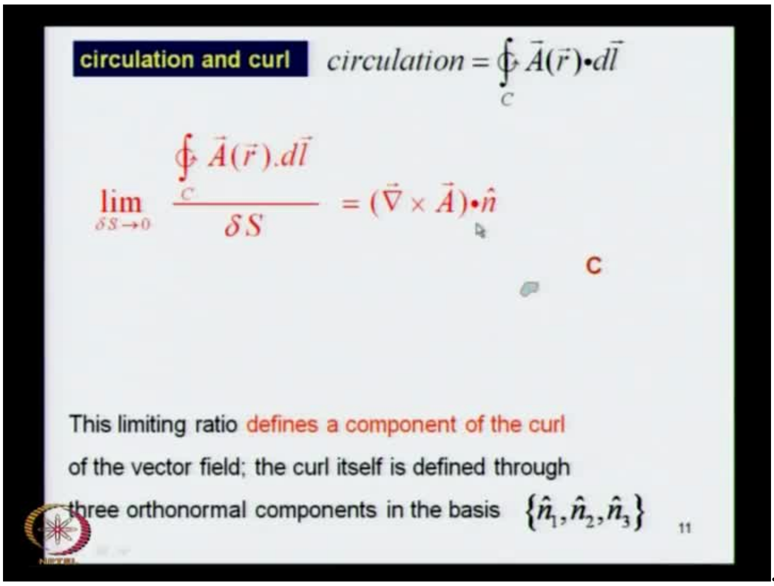

So, this is our this is the connection between circulation and the curl. The curl is precisely defined in terms of the circulation in the limit that the area shrinks to a point and when you this relation defines the component only along one direction which is the directional along n which is normal to that surface element, but you must define this for 3 directions which are independent of each other.

So, what you do is take a basis set of orthonormal unit vectors or 3 linearly independent vectors. You can always orthogonalize them and then also normalize them using something like the Schmidt orthogonalization procedure. So, you have got a basis set of three unit vectors which are orthonormal to each other, and then you define the components of the curl of the vector along these three directions n 1, n 2, and n 3. Then you have the complete definition of the curl of a vector which defines a vector point function. When you do it for the entire region, you get a vector field.

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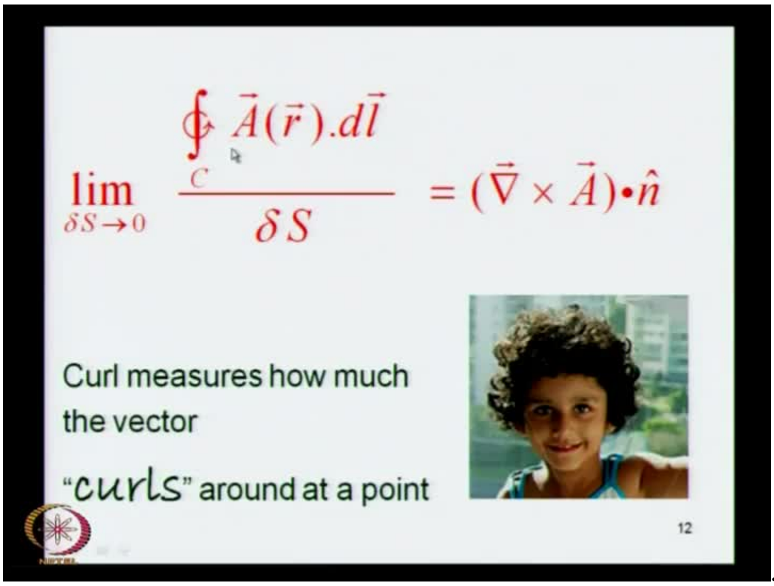

So, what it does is - it gives you because it is coming from this circulation, it is already generating in your mind some idea of a curliness. Because you would need to get the circulation, you need to go round on a closed path. Round does not mean a strict circle; it means a closed path, but you do have to go around it.

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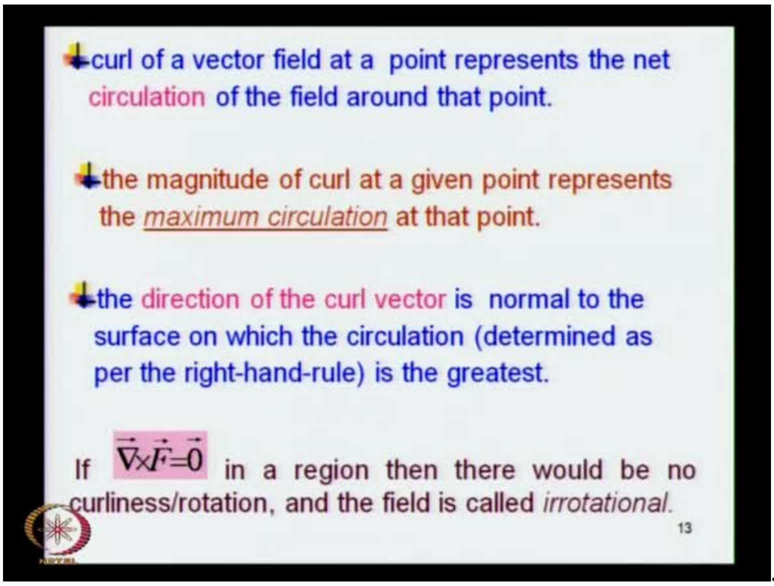

So, these are the key points that the curl of a vector field at a point; it represents the net circulation of the field around that point; the magnitude of the curl at a given point represents the maximum circulation at that point because the magnitude of a vector field along a certain direction will be maximum, if that vector field is in that particular direction; otherwise, you will get reduced components along different directions.

The direction of the curl of the vector is normal to the surface on which the circulation is the greatest and this is determined by the right hand screw convention which I defined earlier. And we introduced a term called irrotational field, that if the curl of the vector vanishes if it generates the null vector, then we say that the vector field is irrotational because the curl of the vector not being a null vector generates the idea of a curliness. It gives a rotational attribute, and if that is missing, then you will call it as irrotational field.

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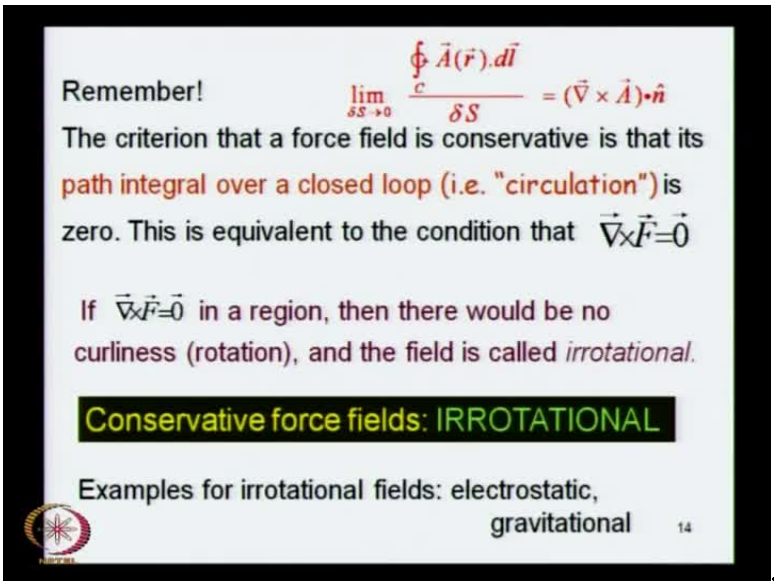

So, this is our basic definition. And now, you immediately see their equivalence. This criterion, if you construct this path integral and if this path integral was for a vector field which is actually a force. Now, this is this our definition holds good for any vector field. What if we consider that vector field to be a force field? Then the numerator gives us the work done over a closed path, and the criterion for a conservative field is that the work done over a closed path is 0.

So, if the numerator vanishes, then the components of this curl in that direction vanishes, and if this happens for three orthogonal mutually independent directions, then of course, the curl of the vector itself vanishes. In other words, the vanishing of the curl of a vector is a necessary and sufficient condition for the corresponding force field to be conservative. So, now we have an alternative idea to express what is meant by a conservative field.

We can say, a conservative field is one for which the line integral is path independent. We can say that, it is one for which the path integral over a closed loop goes to 0, or equivalently, it is a force field whose curl is 0 and it is an irrotational force. So, this is a completely equivalent definition of a conservative force.

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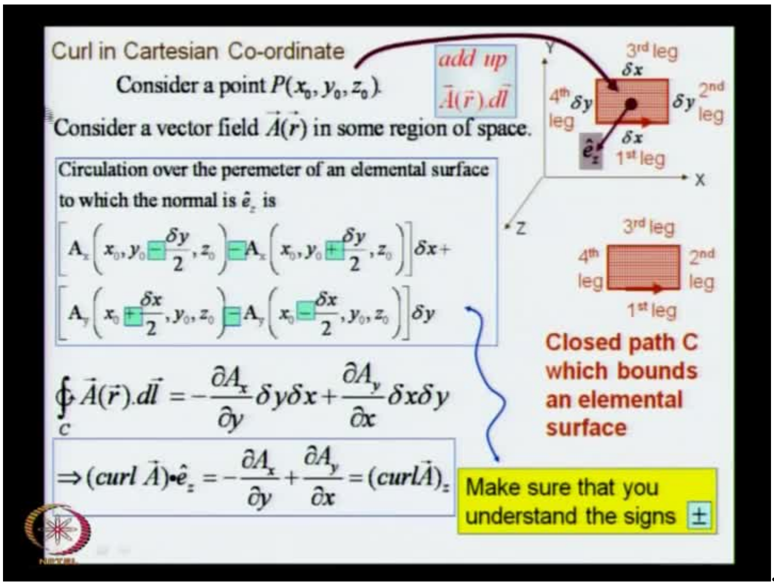

So, this is the reason it is called as an irrotational field. And common examples of this, of course, are the conservative fields that you are already familiar with, like the electro static field, the gravitational field - all of these are examples of irrotational fields and they have a vanishing curl. Now, let us try to get it, get an explicit expression for the curl of a vector in the simplest of coordinate frame of reference namely the Cartesian coordinate frame. So, the first thing we do is to consider a point P whose Cartesian coordinates are x 0, y 0, and z 0, and this is a point P. So, this has got this black dot over here (Refer Slide Time: 30:41); this is the point P. Its Cartesian coordinates are x 0, y 0, and z 0, and let us say that in this region of space, a certain vector field a is defined at every point in this region.

So, this vector field generates this vector A of r which changes from point to point or which may change point to point. There may be some neighboring points at which it does not change, but that is besides the issue. In principle, it could actually change from point to point and this vector field is generated by this A of r. And what we need to do is to construct A dot d l which is the path integral and add it up over a closed path.

So, let us consider a closed path which begins over here at this corner (Refer Slide Time: 31:40). It goes from here to here and then from here to this point; then from this point, it comes to this point, and then back to this. Now, if you follow this pointer, you will have a closed path which is in the x y plane, it is orthogonal to the e z direction.

So, on the first leg if you construct this A dot d l, you must construct A dot d l on the first leg from this point to this point; on the second leg from here to here, and the third leg from here to here, and on the fourth leg, it will come back to the point at which it really began (Refer Slide Time: 32:30). Now, let us determine the circulation. Then, we will take the circulation per unit area in the limit that, that area becomes infinitesimally small.

So, what is this? You have to multiply A dot d l. Now, d l is along e x. right So, the only component that will contribute is the x component. And where would you take this x component? You will take it on this lower edge; this the first leg, and this first leg is situated at a value of y which is y 0 minus delta y by 2. So, if this edge has got a value delta y, then half of this delta y is above y 0 and the other half is below y 0. So, the first leg is at a value of constant y which is y 0 minus delta y by 2. So, you take the value of A x at y 0 minus delta y by 2 and see how it changes along this path from y 0 minus delta y by 2; from here to here (Refer Slide Time: 33:52). So, I have taken from minus delta y by 2 to here. So, actually I have taken this to be first leg; so, does not matter.

I will re label this as the first leg; this as the second; this as the third and this as the fourth (Refer Slide Time: 34:06). So, it really does not matter. So, you construct this quantity which is A x times delta x; delta x is coming from this A x e x; A x dot e x. right the dot product of the this unit vector A along this displacement is delta x; it is along e x. So, you get the component A x and then the size of this delta x; but then, you have to take the difference from y 0 plus delta y by 2 which is on this leg. There are two legs along which x is changing. x goes from this point to this point, and then from this point to this point along these two legs, and along these two legs, it is y which is changing from the lowest corner to the upper most corner. And here it goes from the upper most corner to the lower most corner (Refer Slide Time: 34:56 to 35:08).

So, keep track of the sign. So, A x you have y 0 minus delta y by 2. You have got a minus sign over here and you must watch out for these signs. There is a minus sign here, a minus sign here, a plus sign here. So, why is there a minus sign here? (Refer Slide Time: 35:27) because this is taken along this leg. So, on this leg, the value of y is y 0 minus delta y by 2. Why is this, a plus sign? There is a plus sign over here because this is constructed at the third leg over here. So, on this leg, the value of y is y 0 which is the value at the center plus half of this delta y; so, this sign is plus. Why is there a minus sign over here? Because when you go from left to right, x is increasing whereas, on the top edge, when you go from right to left, x is decreasing. So, you get a minus sign over here.

So, likewise you have a plus sign over here, a minus sign over here, and a minus sign over here (Refer Slide Time: 36:26). So, you have to be very careful about these signs. And now if you construct this over this entire closed path, then, this if you take this difference between these two quantities, this is the difference between the values of A x at the top minus the value of A x at the bottom, with a reverse sign. So, there is this reversal of sign and this difference is nothing but the rate of change of A x with y multiplied by delta y, which will give you the difference in the value of the x component of A. So, is that algebra clear? That is good.

So, you get minus del A x by del y times delta y and this delta x is already sitting over here, and when you take the other two edges which is this edge, when you go from the bottom to the top and over here you go from the top to the bottom, you get similarly delta A y del A y by del x times delta x delta y. But now, you can take delta y delta x as common and take the limiting circulation per unit area. So, you will divide it by the area and that area is nothing but the product of this delta x with this delta y which will cancel this delta y delta x or delta x delta y, and you get del A x by del A y with a minus sign coming from here and del A y by del x with a plus sign coming from here.

So, this gives you the component of the curl along the direction which is orthogonal to this, and if you use the right hand screw rule, then along the path that we have chosen, the right hand screw rule will move forward in the direction from the plain of the figure towards you; so, it will be along the e z direction. Right. So, this is the component of the curl of the vector along the z direction, and similarly, you can define the component at the curl along the y and the x directions.

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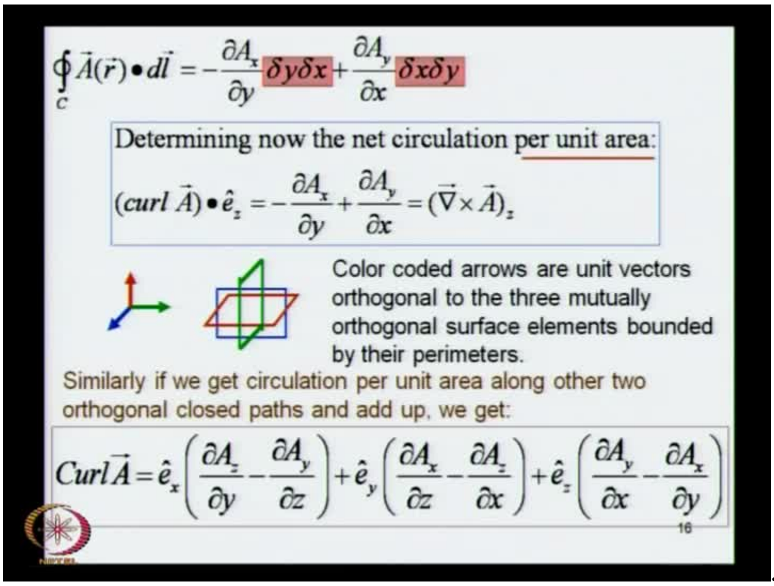

So, as soon as you have done that, you would have defined the curl of the vector completely. So, just be sure that you make absolutely no mistake about the signs. Now, we have the circulation per unit area in the limit that the area shrinks to a point. So, this delta x delta y area would be cancelled and you get the z component of the curl of the vector. This is orthogonal to the x y plain, but then there are three planes which are orthogonal to each other. So, this is one, here is another, and here is another (Refer Slide Time: 39:27). So, there are 3 orthogonal planes for each other and I have color coded them so that each direction… here this is orthogonal to the corresponding area shown in the same color. And when you do this for all the three planes, for all the three directions, you get the complete definition of the curl of the vector in the Cartesian coordinate frame of reference. So, let us do that.

You get The component along e x we have already determined, which is del A z by del y minus del A y by del x which is along e x or this one is along e z; so, that one is coming over here (Refer Slide Time: 40:20) e z times this is del A y by del x with a plus sign and del A x by del y with a minus sign which comes over here. And the two terms you can get by simply making cyclic changes; you take z to y, sorry z to x and x to y, and y to z.

So, if you make cyclic changes, you get the other two terms easily or you can do it term by term for the other two rectangles.

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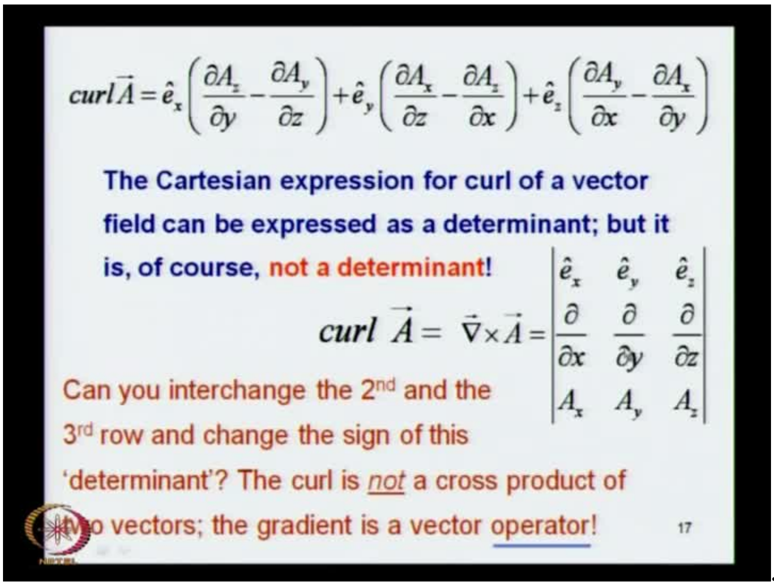

So, we have got now the complete definition of the curl of a vector. And as you can notice, you can write it as a determinant because if you think of this as the determinant, it is being used only as a notation; it is not really a determinant. And it is important to recognize that it is not a determinant, but the notation of a determinant in which you normally expand a determinant. Suppose you expand it along this first row, then you will write this element first and then multiply this and this, and then this and this (Refer Slide Time: 41:30), and subtract it from the previous term. right That is how you expand a determinant.

Now, here, you are really not multiplying these two terms; you cannot. Because this is an operator, this is the component of the vector function. You are actually taking the partial derivative with respect to y of the z component of A. And from this partial derivative, you are subtracting the partial derivative of the y component of A with z, which is what is coming over here (Refer Slide Time: 42:05).

So, the component along e x, this is the component along e x. This is del A z by del y which is coming from here; the partial derivative of A z with respect to y minus the partial derivative of A y with respect to z. So, this is the component over here, and then likewise, you have the other two terms. So, this gives you the curl of a vector. This is really not a determinant, but you can exploit the determinant notation.

The reason, of course, it is not a determinant is because you cannot just go ahead and interchange the 2 rows - the second row and the third row, for example, and you will not get any meaningful quantity with just a negative sign. If it were a real determinant of numbers, you could do that. So, this is not a determinant; you are only exploiting the notation for a certain convenience, and happily you can do it in the Cartesian frame of reference; does not automatically mean that you can do so in other frames of reference; as a matter of fact, you cannot. But in a Cartesian frame of reference, you can exploit this feature. The reason you cannot do it is because this gradient whose components you have written over here, these are not like the components of a vector. The gradient is an operator; it is not a vector; it is a vector operator. When it operates on a scalar function, you get a vector.

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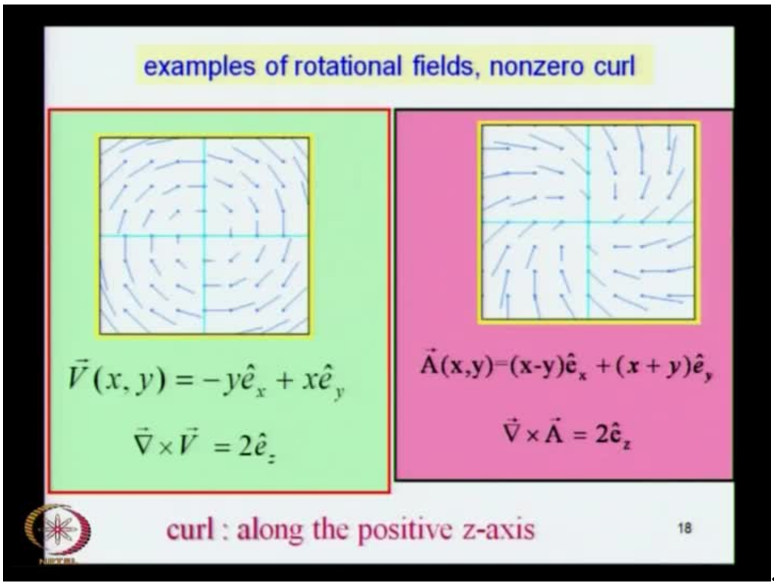

So, here are some examples. If you have got a vector field which looks like what you see in this picture, and this is a vector field whose Cartesian expression is minus y times e x plus x times e y. Then, if you use the previous formula over here, which is this (Refer Slide Time: 44:00), and if you use this relationship, then it is very easy to deduce that the curl of this vector is twice e z. So, this is the matter of simple algebra. You can work it out. Here is another example that, if you have got a vector; this is the picture and this can this is a pictorial expression of a vector field given by this x minus y e x plus x plus y e y and you take the partial derivatives term by term, fit it to the Cartesian formula, and find out what this curl of the vector is, and you find that this curl is also twice e z. So, there are these different functions; the fields are different but they give you a similar curl.

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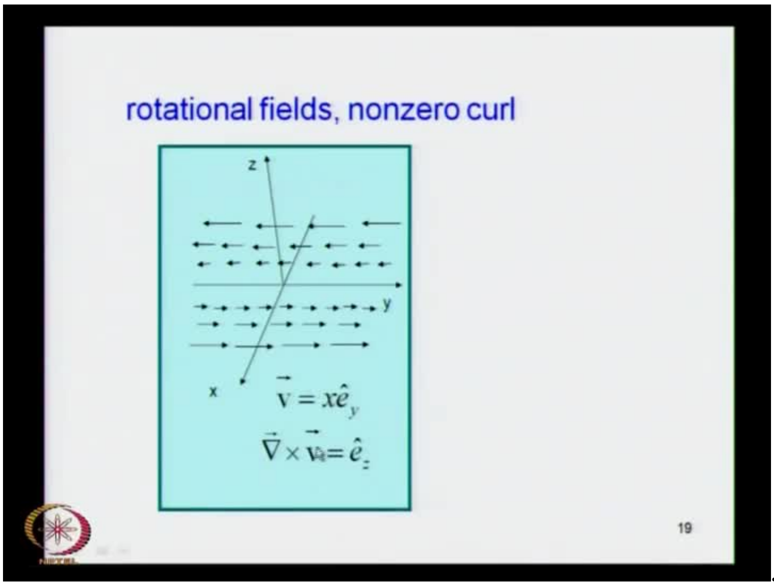

So, here is another example of a rotational field. A field will be rotational if its curl is not 0. So, suppose you have got a velocity field and there is a certain line beyond which the velocity field reverses, and over here, it goes in what this direction. So, if you construct the line integral over a closed path, you can already see that it will not vanish, as that elemental area shrinks to 0, and that is explicitly determined by taking the formula for the curl of this vector which turns out to be e z; it is not 0.

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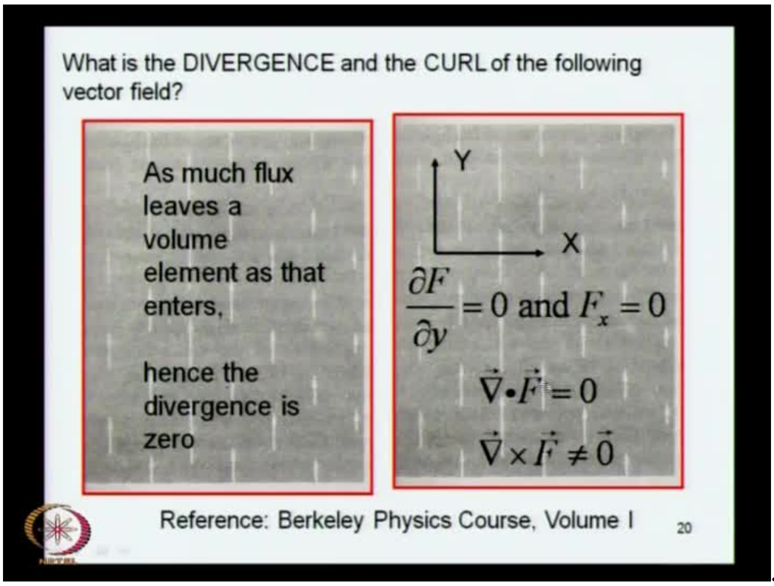

Let us do some simple exercises. If you look at this, this figure over here, and we have these figures from Berkeley physics course, volume one - these are very nice examples which I enjoy from the Berkeley physics course. And if you look at this, you already see that whatever comes in goes out. So, the net divergence of this field will be 0. What about the curl? Can you see that it is not 0 because del F by del y is 0 and F x is 0. So, the divergence of the force we have seen to be 0, but if you take this path integral over closed paths, then you see that the length of these arrows is changing as you go across from right to left. So, as a result of that, this circulation will not be 0, and therefore, the curl of the vector will not be 0. What about this field? Now, this looks like a radial field and you obviously see that the divergence is not 0.

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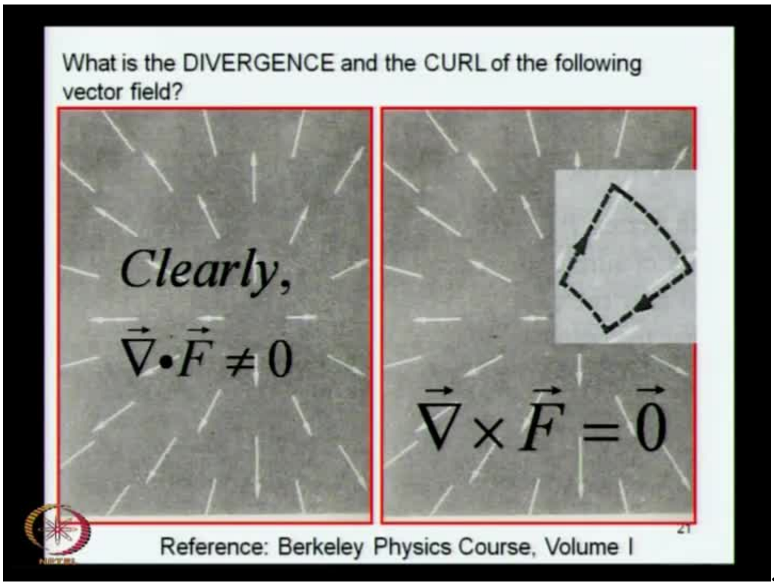

What about the curl? It should be very obvious, that in this case, the curl will be 0 because this has got a rotational symmetry. So, if you construct the path integrals over closed paths and then let this area element shrink to 0, then the integrals, the line integrals over this path and over this path would vanish when the elemental area shrinks to a point because they will be line integrals over just the same path in opposite directions, in the limit that the area shrinks to a point. So, the circulation vanishes.

This is the typical property of a rotational field, wherever whenever you have spherical symmetry. You will find, if a force field has got a spherical symmetry, a center of symmetry, then it will be an irrotational field and it will be conservative.

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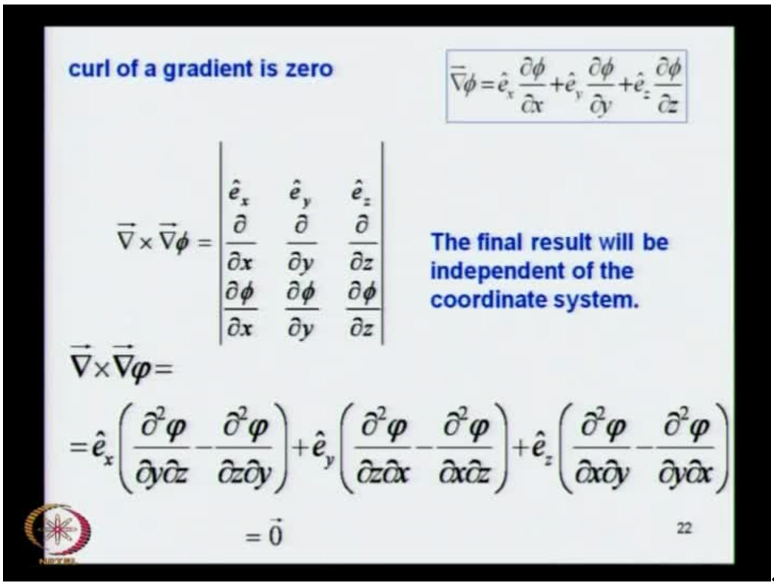

Now, likewise, if you have a vector which is expressed as a gradient of a function, then it will always be irrotational. The curl of that gradient will always be 0. So, these are some general rules which are nice to keep at the back of your mind that the curl of a gradient vanishes, and you can do it very simply in the Cartesian geometry because here, we know that we can use the determinantal expression; not a determinant, but the determinantal notation. And if you find that what you are doing is - you are taking the partial derivative, the second order partial derivative of phi with respect to y and z; first with respect to z and then with respect to y over here; first with respect to y and then with respect to z.

But these two are independent degrees of freedom, and therefore, these partial derivatives can be obtained in any order, and they will give you essentially the same. So, when you subtract one from the other, the second order partial derivative vanishes, and that means that the curl of a gradient always vanishes. So this is del 2 phi by del y del z minus del 2 phi by del z del y, but the order in which you take these derivatives really does not matter.

So, this will give you a null vector and the corresponding vector field is irrotational. The final result, of course, cannot depend on a coordinate system. So, this result although we have obtained using the Cartesian expression for the curl of a vector, the end result will be independent that the curl of a gradient vanishes. So, you just forget about this intermediate step, and no matter how you express the curl in and any other coordinate system which we are about to learn, you will find that it must vanish because any end result cannot depend on the choice of a coordinate system.

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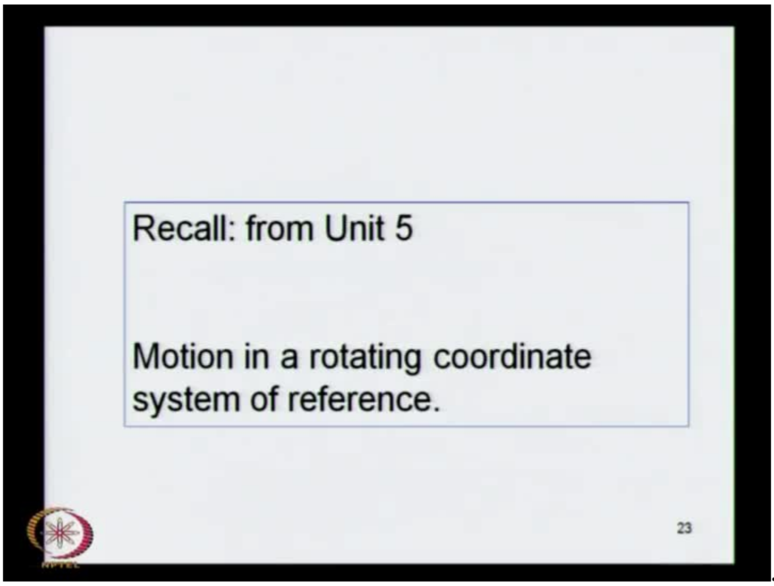

Let me also remind you from unit 5 which we did some time back, in which we studied motion in a rotating coordinate system of reference and I want to recapitulate one or two ideas from this so that you understand the term curl very clearly. What you will see is that from a linear quantity you get a rotational quantity when you apply the operator curl, which is why it is called as a rotational operator.

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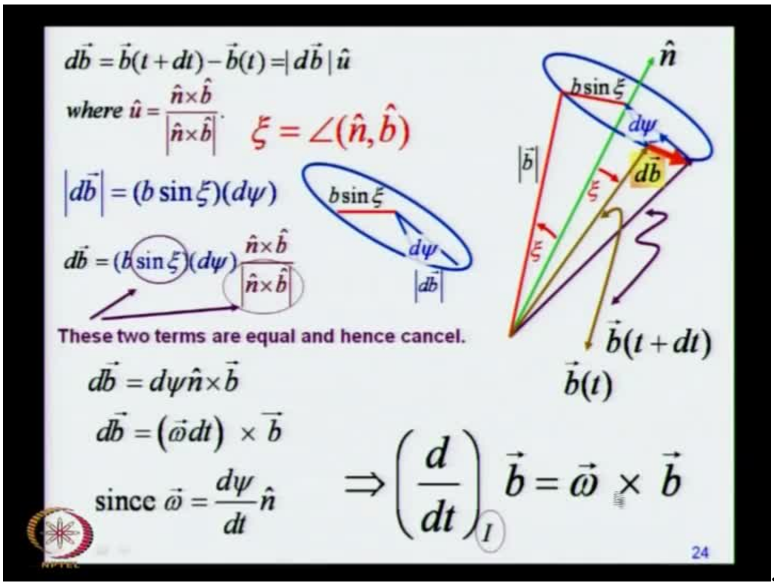

So, when we studied dynamics in a rotating frame of reference and we looked at a certain point which remained invariant in the rotating frame of reference, and we asked ourselves - how it would look from an inertial frame of reference? So, if it is fixed in the rotating frame of reference, then in the inertial frame of reference it will change because the rotating frame of reference is rotating with respect to the inertial frame of reference. So, these are this is the diagram that we had constructed that if you have a frame of reference which is rotating about an axis n along this direction of the red arrow, then as if a point remains fixed along with this rotating frame and it goes from here to here (Refer Slide Time: 51:33) and its new position in the inertial frame is given by b at a later time t plus d t, and the displacement is d b because this is obtained by subtracting from b t plus d t - the position of the vector b at the previous time t.

And what we did by was, by doing some simple algebra, we showed that this magnitude of d b, this one - the magnitude of the red arrow, which is nothing but this radius times this arc length and the radius is b sine psi, where psi is this angle which this rim of the cone subtends at the vertex. So, b sin psi is this radius. And this distance, this radius times d psi gives you the b sin psi which is the arc length over here.

We also saw that the sine psi is the same as n cross b. This is the unit vector n, this is the unit vector b, and the angle between them will give you, the n cross b will give you the sin of psi. So, we have done this in details when we did the unit 5 and this is just to remind you of some of the basic definitions over there so that you understand the term curl of a vector. And what we found is that, this d psi can be written in terms of the angular velocity because angular velocity is nothing but d psi by dt along the direction n.

So, this gave us n operator equivalence that when you operate on an operand b by the time derivative operator, then it is completely equivalent to taking the cross product of that vector with omega. So, the process of taking this cross product is completely equivalent to the process of taking the time derivative in the inertial frame of reference.

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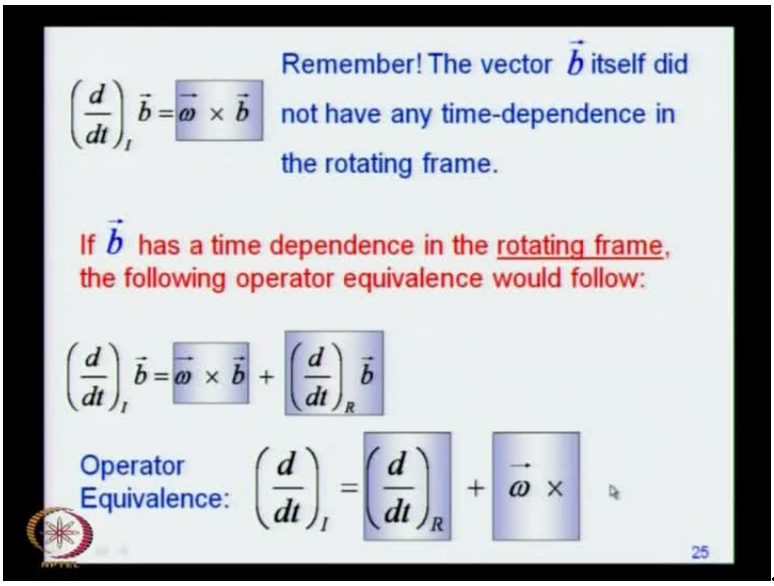

Now, we did this for the vector b which was not itself time dependent in the rotating frame of reference. If it was, then you will need to add the time derivative in the rotating frame of reference of that vector b, and then the operator equivalence for the time derivative operator is that the time derivative operator in the inertial frame of reference is equal to the time derivative operator in the rotating frame of reference plus this cross product. So, these relations we had obtained in some detail when we did unit 5.

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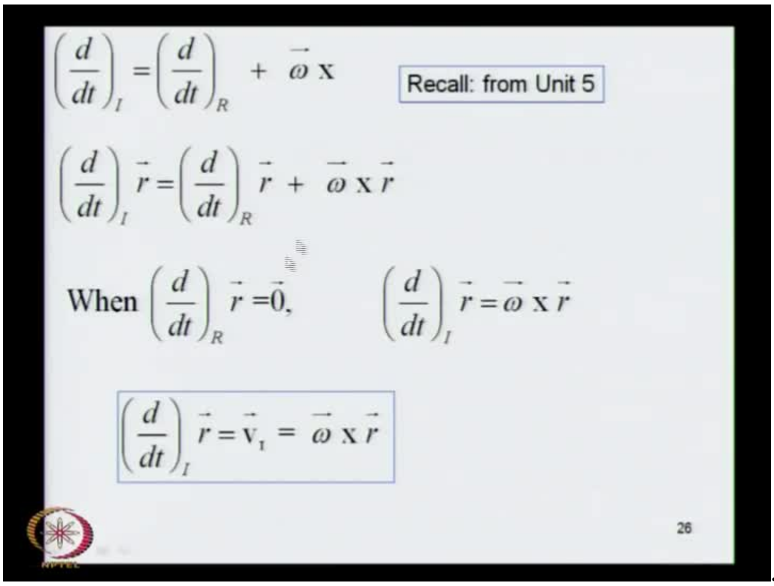

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Now, let us use this operator equivalence on the position vector for the special case that the position vector is a constant in the rotating frame. So, its time derivative in the rotating frame vanishes. So, this term would vanish, and you have d by d t in the inertial frame of a position vector which is given by the left hand side; here, will be given by the right hand side which is omega cross r. So, on the left hand side, you have got the time derivative; on the right hand side, you have got the cross product; so far, so good.

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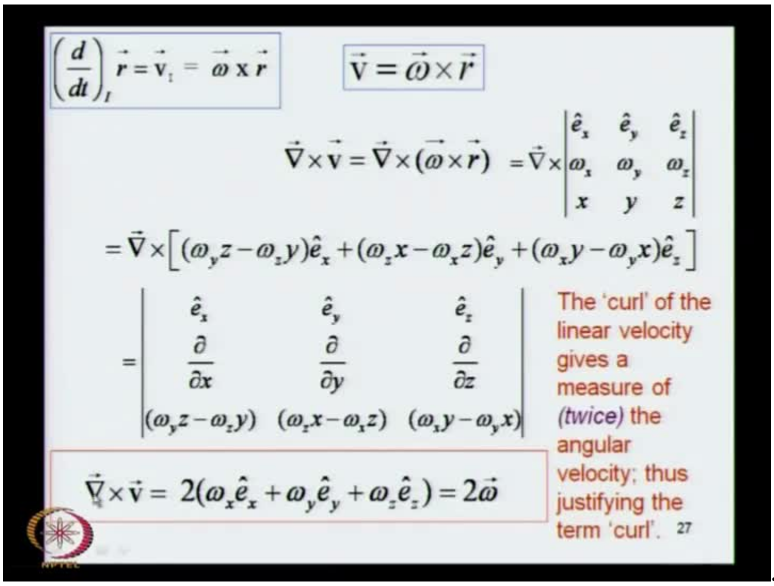

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Now, let us take the curl of this. So, we take the curl of this cross product. Now, construct this cross product. Now, this is the cross product of 2 vectors. There is no difficulty using a determinantal expression for this. There is no approximation or any play which is going on over here. And you have must take the cross product of this determinant. So, let us do that. You are taking the curl and you expand this determinant. So, you get the component along e x is omega y z which is coming from here, minus omega z y which is coming from here, and likewise you have got the other 2 terms - one along e y and the other along e z. So, this is now the expression for the curl of the vector.

Now, let us use the determinantal notation. This is not a determinant, but this is a determinantal notation, and now, let us get this e x e y e z - the determinantal notation. For the gradient operator, we write the del by del x, del by del y, del by del z, and the components along e x is omega y z minus omega z y, which is here - omega y z minus omega z y. Then the component along e y is here - omega z x minus omega x z, and the component along e z is here omega which is omega x y minus omega y x. And now, all you have to do is to take the partial derivative of this with respect to y and partial derivative of this with respect to z. So, that should be very easy, and what it gives is twice omega; that should be obvious.

What we have got is, by taking the curl of the linear velocity, we have got the angular velocity which is the rotational velocity. So, from a linear property over here, this velocity is a linear property, we get a rotational property which is the angular velocity; yes, you do get twice that; so, it is not an equality; nevertheless, does not change the fact that from a linear property, you get a rotational property. And that is a reason this quantity which is defined here as del cross v is called as the curl of a vector because from a linear quantity, you get a rotational property- an angular property.

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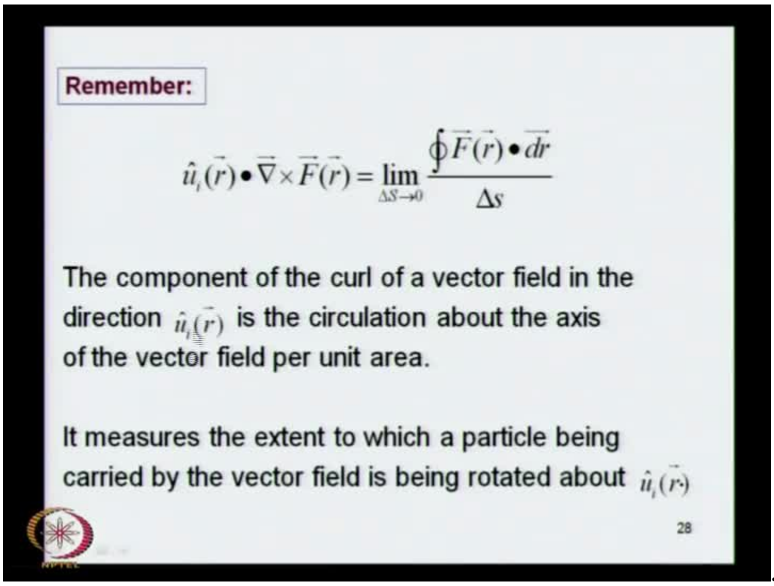

So, what it does is - it actually measures the extent to which a particle being carried by a vector field is being rotated because that sense of rotation, that angular you know piece of connotation which is coming in becomes implied when you take the circulation per unit area in the limiting value that this curl of a vector, when you find its component, it gives you a measure of the extent to which a particle being carried by the vector field is being rotated about that particular axis. In this case, the axis is u i about which this rotation is taken.

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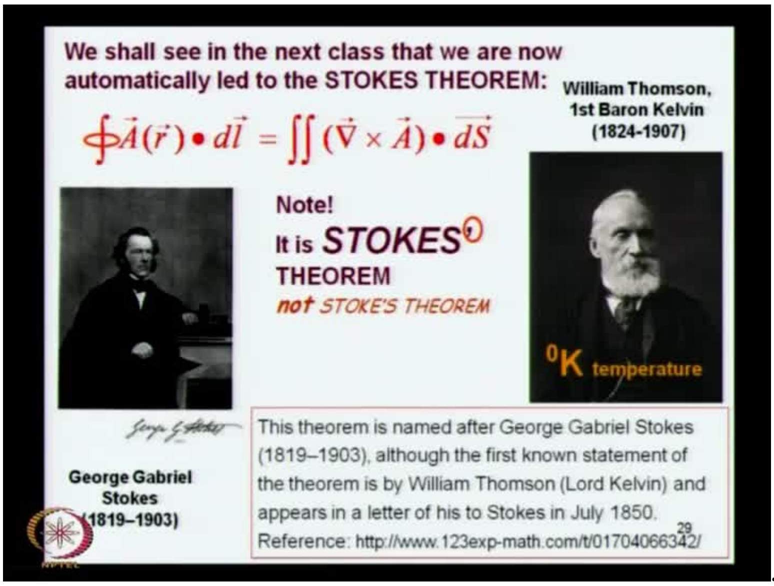

So, this is called as the Stokes’ theorem and this is something which I will discuss in the next class. We have defined the curl of a vector and we will use the definition of the curl of the vector to define the Stokes’ theorem. Notice that Stokes’ is written with an apostrophe not between e and s, but after s. So, s is the part of his name. So, the apostrophe is not here. The statement of the Stokes’ theorem is given here; we are yet to prove it; we will prove it very easily because we have defined the curl of a vector.

So, we know the definition of this quantity. Then all we have to do is to dot it out with the elemental surface area and construct this surface integral. And then, what will come out of it is the fact that this circulation over a closed path is equal to the surface area of the curl of the vector which is taken according to that right hand screw convention, and so on. So, I will prove this in the next class. So, this called as the Stokes’ theorem.

Here is a picture of George Gabriel Stokes’; nevertheless, I should tell you that the Stokes’ theorem was in fact first formulated by William Thomson or better known as Lord Kelvin, after whom that the absolute temperature scale - the Kelvin scale of temperature, is known. And Kelvin, in fact, established what we call as the Stokes’ theorem and Kelvin wrote about it in a letter to Stokes’ who probably popularized it and now, everybody calls it as Stokes’ theorem, but originally it was formulated by William Thomson, better known as Lord Kelvin.

So, this was in July 1850 and we will take a break here, and when we come back, we will do the proof for the Stokes’ here.

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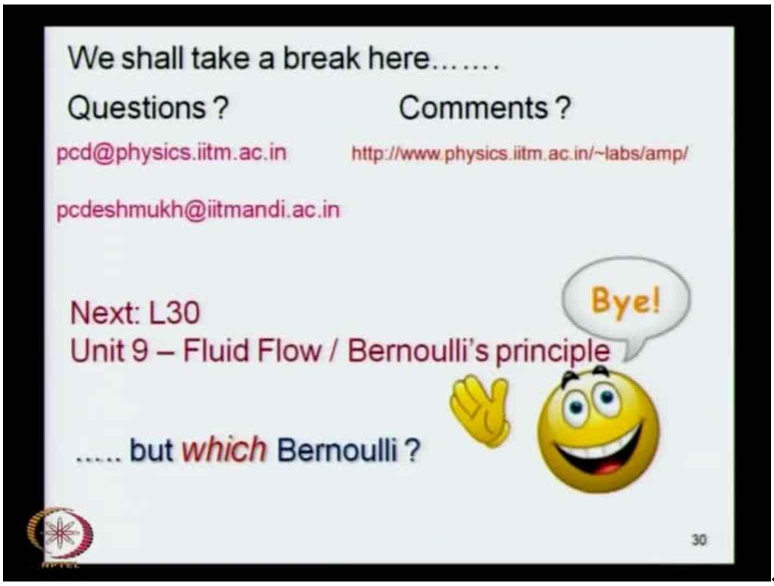

So, if there are any questions, I will be happy to take any questions or comments. And then, we will of course, go on to prove the Stokes' theorem, and then we will discuss the Bernoulli's principle, but of course, there were several Bernoulli's and they were all brilliant, but that is for the next class. So, if there is any question or comment, I will be happy to take; if not, we take a break; you can always send your question by E-mail. So, ready for the break.