Select/Special Topics in Classical Mechanics

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Module No. # 01

Lecture No. # 03

Equations of Motion (ii)

Greetings, once again welcome to the third lecture on Special or Select topics in Classical Mechanics. This is the second lecture on the equations of motion, which is our first unit.

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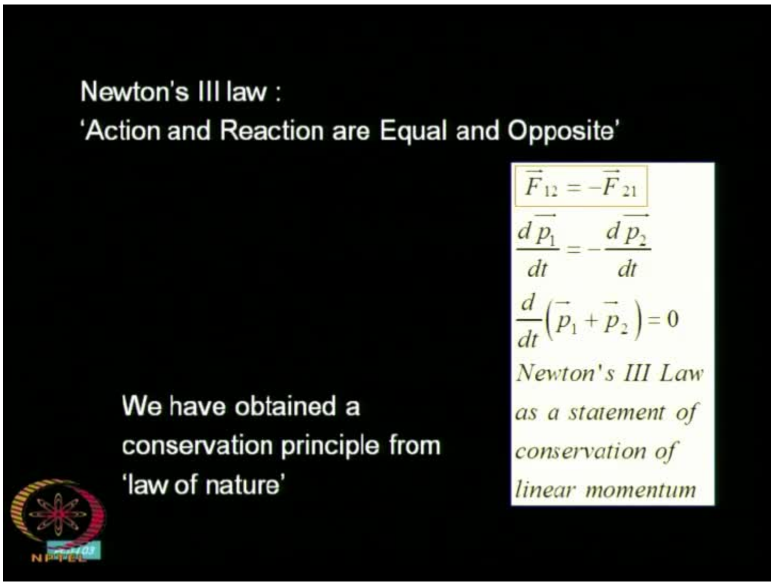

We will begin with where we left last time. It is the consideration of Newton's third law and this is stated as - action and reaction are equal and opposite and this can be stated exactly on the same footing as Newton's first law and the second law. Now, there a few important points which I will recapitulate quickly.

First - the first law of Newton, which is the law of inertia or Galileo's law of inertia. It is a separate law by itself. It cannot be considered as a special case of the second law by putting f equal to 0 in f equal to ma. So, whenever you make use of the second law, the first law is already implicit in it. In the sense, implicitly is referenced in it because, unless you identify an inertial frame of reference, you do not touch the second law. So, you first identify a frame of reference in which, motion is self-sustaining. This is what defines equilibrium. Once you recognize this, you can ask what is responsible to change the equilibrium. Second law answers this and there is some physical interaction, which is called as Newtonian force. So, this comes into consideration only after recognition of the first law of inertia. This is sometimes stated by saying that when you want to identify whether a frame of reference is inertial, you say that an inertial frame of reference is one in which Newton's laws hold.

That is like a cyclic statement. It is formally correct, but it really does not take you anywhere because you define Newton's laws with reference to inertial frames. If you define inertial frames with reference to Newton's laws, then you are really not out of this cyclic argument. So, you need some kind of mechanism to come out of this cyclic argument and this is what develops a notion of physical interaction in your mind. In the absence of any physical interaction, if you decide that motion is self-sustaining in a certain frame of reference, then it is this frame of reference that you would regard as an inertial frame of reference. In this frame, if you now perceive any change in equilibrium, which manifests as acceleration, then you call the agency, which cause this change as the Newtonian force.

So, f equal to ma comes into picture, only after you have identified an inertial frame of reference invoking the first law. These two laws: the first law, which we got from Galileo's experiments and from his observations. The second law that comes from Newton's reasoning for the explanation with regard to the departure from equilibrium. These two laws were not derived from anything. These are fundamental principles on which the rest of mechanics stands. So, this provides you with the platform and everything else in mechanics can in fact be derived out of this. On the same footing, one could state the third law, which is there on the screen. In front of you that action and reaction are equal and opposite.

This can also to be stated exactly on the same footing as a fundamental principle. It is not derived from anything else, but which an equal partner along with the first and the second law in providing you with a foundation on which the rest of mechanics stands. So, one can certainly learn the third law exactly on the same footing.

An immediate consequence of the third law is what you see on the screen. We discussed this briefly last time. If you write action and reaction as equal and opposite in this expression over here, which is stated mathematically as F 12 equal to minus F 21. It immediately follows that the total momentum is conserved; its time derivative vanishes and therefore, it is a constant of motion. In other words, we get a conservation principle from a law of nature. It is the third law that I am referring over here. The third law takes the place of a law of nature. This gives us a basis to determine a conservation principles; namely the conservation of linear momentum.

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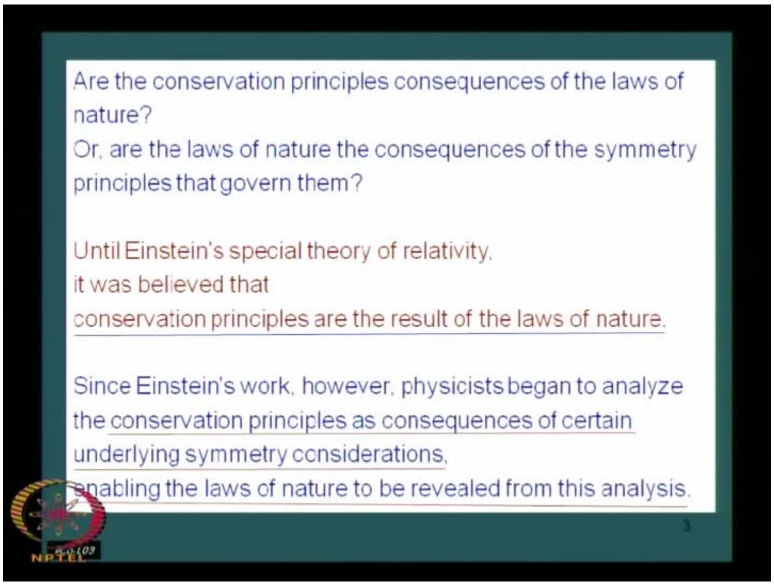

Now, let us ask this question. Are the conservation principles consequences of the laws of nature? The answer would be yes, if we accept the logic. What we just discussed is exactly what we did. We began with the law of nature; namely the third law of mechanics. From that we deduced the conservation of linear momentum. We can also ask - are the laws of nature the consequences of symmetry principles that govern them? Now, this is a new question, which some of you may not have considered earlier are the laws of nature consequences of the symmetry principles that govern them and this is a very fascinating question that can in fact be asked. The mindset of physicists really underwent a sea change, since Einstein's special theory of relativity. This was in 1905 and this came out of his discussion in understanding of the electromagnetic phenomena and electrodynamics and so on.

I will be discussing some of this in later unit, when we come to special theory of relativity and towards the end, when we also introduce ourselves to electrodynamics. Until Einstein's special theory of relativity, the conservation principles were considered to be results of the laws of nature, as we just saw. Since Einstein's work, physicists began to analyze conservation principles as consequences of certain underlying symmetry. Now, this is a completely new approach, but not anymore, it has been with us since, Einstein's thought process. So, not new in that sense of the term, but new in terms of the context in which Newtonian mechanics is normally discussed.

What this consideration allows you to do is you are able to actually deduce laws of nature by examining these connections between symmetry and conservation laws. These connections reveal you the laws of nature in which a physicist is interested in. Therefore, it gives you a mechanism, a tool, a pathway or some sort of a road map to discover new laws of physics. That is always a fascinating issue because we do not even know all the laws of physics. We do know a lot, but whether we know everything are there; some other laws, which we are yet to be discovered. The physicist's quest will be directed towards discovering these additional laws, if there are any. We do not know whether there are or there are not.

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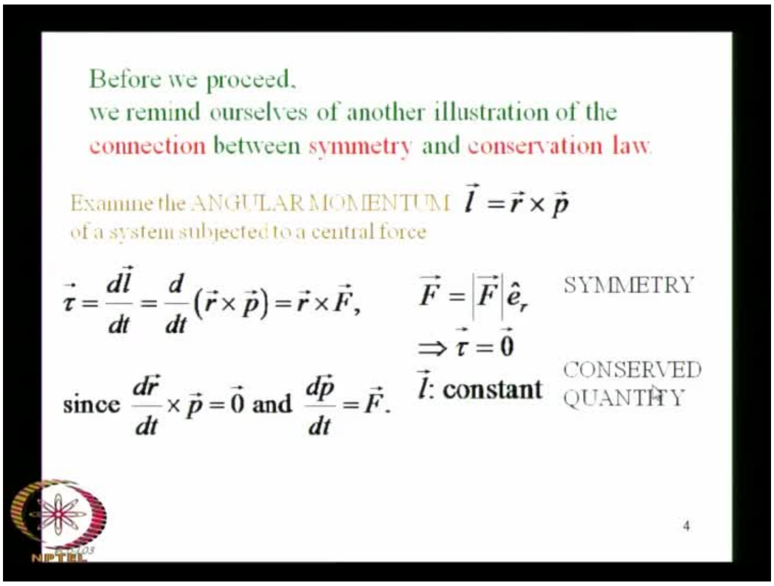

Before we proceed, I will like to show you another connection between symmetry and conservation law. So that you know that this starts becoming a part of your hour thought process. Consider the angular momentum and you know it is defined as r cross p in classical mechanics. Obviously, there is a certain point of reference with respect to which you measure the angular momentum, either a point or an axis or whatever. We are going to consider angular momentum of a system, which is subjected to a central force. Now, what is a central force? A central force is one, which is directed along the radial line and that is called a central force because it has that central field symmetry.

Now, if you look at the torque, which is a rate of change of angular momentum. I write d by dt - the time derivative of the angular momentum, which is r cross p and this will be dr by dt cross p plus r cross dp by dt. There will be two terms in this and out of the two terms, dr by dt cross p will vanish because dr by dt is the velocity, which is collinear with the momentum. So, the cross product of these collinear vectors vanishes and the second term, which gives you r cross dp by dt gives you r cross F, which is the torque.

Now, our interest is in a particular force, which has a special symmetry. This is the radial force, it is directed along the radial line. So, this force has a certain magnitude and a direction, which is along the position vector r. So, this force becomes collinear with the position vector r and again, the cross product of these two vectors vanishes. So, you immediately see that the torque would vanish for the central force. If the torque vanishes, then dl by dt becomes 0 and l becomes constant. So, the angular momentum is a constant of motion. For this particular case, what have we seen is that if there is a radial symmetry or if the force is a central field force, then the angular momentum is conserved. So, we are meeting once again for a connection between the symmetry and conservation law. So, the fact that F is along e r is a statement of symmetry, l is constant, which is a conservation principle.

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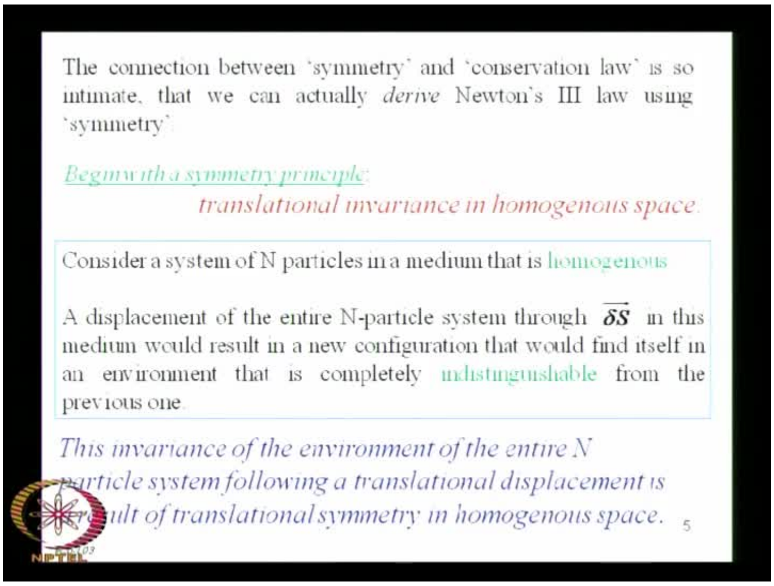

Now, we are going to discover that relationship between symmetry and conservation law is in fact, so intimate that it allows us to actually derive Newton's third law using symmetry argument. Now, this is a very novel approach from the point of view of traditional orthodox Newtonian mechanics. It is not new in the historical sense. People have known this for more than 100 years. So, it is not new in that sense, but new from the point of view of traditional orthodox Newtonian mechanics.

Newton's third law was always regarded as a fundamental law of nature on the same footing as the first and the second law. It is not derivable from anything else, it has its own emphatic existence, which can be verified, but not derived. It cannot only be verified; actually you find no exception to it at least in the domain of mechanics. Now, the mechanics that Newton addressed was obviously, mechanics, which addressed interactions between two particles, but the electrodynamic interaction was not considered at Newton's time. So, once you start including electrodynamic interactions, then some of these ideas have to be reinterpreted. I will get to that as the course progresses, at a later time.

We are going to leave out the electrodynamic interaction for the time being. In the context of the kind of mechanics that Newton dealt was everything outside electrodynamics and other interactions, including the nuclear interactions. Newton's third law was always considered to be on the same footing in the traditional orthodox mechanics. Now, we will actually derive it using a symmetry argument. Let us see how this discussion will go. What we do is to begin with a symmetry principle. The symmetry we invoke is translational invariance in homogenous space. So, let me comment on each of the terms that we are referring. Here, translational invariance to translate is to have a lateral displacement. You have a certain object and you move it sideways or in this direction or in this direction.

These are translational displacements. As opposed to this, this is rotational. So, this is translation, this is rotation. So, these two are different operations and you can change the configuration either by translation or by rotation or by a combination of both. While executing the translation from this point to this point, you could also turn it. There could be a combination of both. What we are going to do? To consider translational displacement, we need the space to be homogenous and the meaning of the term invariance will automatically come, when I comment on the homogeneity of space.

I am going to consider translational invariance, in such a space, whose properties are such that they do not change, whether you consider the properties of that space in this part or in this part or in this part. I could consider a translational displacement from here to here to here. The properties of the medium, the properties of space will not actually change. Consider the displacement of this water bottle and move it from here to here. I moved it in space and the medium over here is the mass of air that is around. It is also the same. It is not carbon dioxide and oxygen; it is the same here. So, I could say that the properties of space have not changed. If you look at very microscopic level, they may have changed because who knows, what exactly the constitution of a mass of air over here, compared to what it is over here. There could be some minute differences.

The differences may be in the relative proportions of oxygen, nitrogen. There may be a minuscule difference. If not the kinetic energy of all the molecules, the individual kinetic energy of all the particles could actually be different. When we talk about this invariance, we are referring to such a system, which has identical properties. Whether you consider this part or this part or this part, these properties must be entirely homogenous. There must be absolutely no change of any kind, even at the microscopic level. Is it clear? So that is the kind of invariance that we are talking about. This invariance is referring to the fact that the properties would not change, when you go from here to here to here, under any translational displacement. So, this is the kind of medium that we are talking about. There is a symmetry, which is involved. If you keep an object over here, it will find itself in an environment. If you displace this through a translational vector, it would find itself in a new environment, which will be identical to the previous environment. The identity of the environment after the displacement is the invariance.

We are going to consider a symmetry principle, which refers to a translational invariance in homogeneous space. Now, we understand what a homogeneous space means. In such a space, we consider an N particle system; a system of N number of particles. We are going to consider a displacement of the entire N particle system through a small vectorial displacement, which is delta S. So, there is a vector arrow on top of this displacement vector.

The entire N particle system is moved that is N block from one part to another. The difference vector being delta S, essentially means that each of the N particles exactly undergoes the same displacement. The end particle system is now going to find itself in a new environment. Obviously, it has moved, but the new environment is identical to the previous one. We are referring to homogeneous space, in which translational invariance is preserved. So, this is the kind of displacement, we are talking about.

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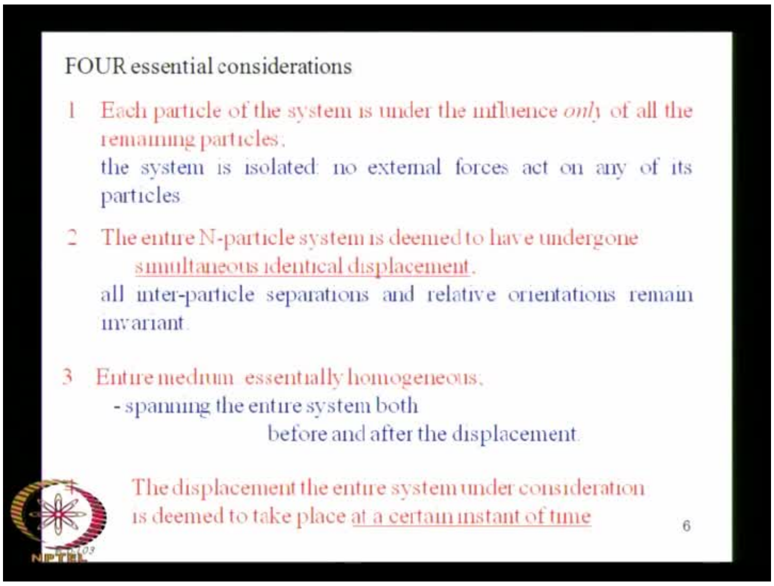

Actually, there are four essential considerations in this kind of a displacement. First of all, each particle of the system is considered to be under the influence of the remaining N minus 1 particles. In other words, we are talking about an isolated system. There are no external forces. It is not that there is no force on any particle, there is a force on each particle, but whatever force is there on each particle, it is strictly generated by the remaining N minus 1 particles. Other than this, there is no other force. So, there is no other external force.

Second, the entire N particle system is deemed to have undergone simultaneous and identical displacement. I want to rub this point because it is an important consideration. It means that all the inter particle separations will be preserved, all the inter particle orientations will also be preserved. Whatever the orientation of the third particle, it has a reference to some other pair of particles. Relative orientation will not change, when you consider this displacement.

The medium is homogenous and it spans the entire system, both before and after the displacement. It is not that some part of the system remains in the previous medium and then it goes to some other part, where the properties have changed. So, the entire medium is considered to be homogeneous. Finally, the displacement of the entire system is considered to have taken place at a certain instant of time. Now, where does time come into the picture and why does it come into the picture? It is an important consideration because it will alert you to the fact that if you were to think of such a displacement. Suppose, I ask you to take a set of N particles and I give you a homogenous space. I ask you to displace the entire N particle system to a new configuration, which is displaced from the previous configuration. How would you do it?

Suppose, you have a few objects laid out on this table. I can move this laptop from here and keep it over here. I can move this wristwatch from here and keep it at the same relative distance from here. So that you know, it will look identical at the end. If I were to do, it would take me sometime.

You said that there is no time limit, t in the displacement.

It is done at a certain instant of time.

Isn’t just a restatement of the second condition?

It is a simultaneous identical displacement. You are quite right. It states it differently; it underscores the fact that at a specific instant of time, which really means that if you were asked to displace a N particle system in this kind of manner. You would really not be able to do it. You would physically not be able to do it. This can only be a mental thought process.

Sir is it an ideal case or is it a practical case?

This is not practical and that is exactly what I am trying to emphasis. Here, you cannot do it physically. If you were asked to do it, you would not be able to do it. You can only think of this as a mental thought process.

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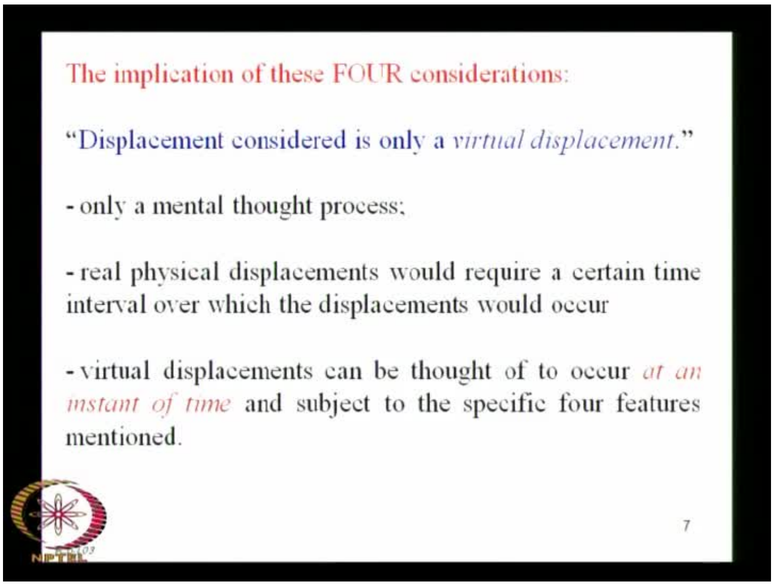

The implications of this are the following: the displacement considered is only a virtual displacement. It is only there as a phenomenon in your mind. It is a mental thought process and such displacements are called as virtual displacements. This is the meaning of a virtual displacement. It is not a physical displacement. It cannot be really executed and it is not practical as you point out.

I do not like to use the word ideal or non-ideal. I think referring to it as not practical is a practical way of saying it. This process is not practical; you cannot actually do it. So, it is only a mental thought process. Since real physical displacements would require a certain amount of time, over which, these displacements can be executed. So, it is not a real physical displacement. It is called as a virtual displacement and in our mind; nothing stops us from conceiving such a situation. We are interested in the properties of the space. What is it that this leads us to? It is the intellectual thought process, which we have engaged ourselves with and we will see that it has amazing consequences.

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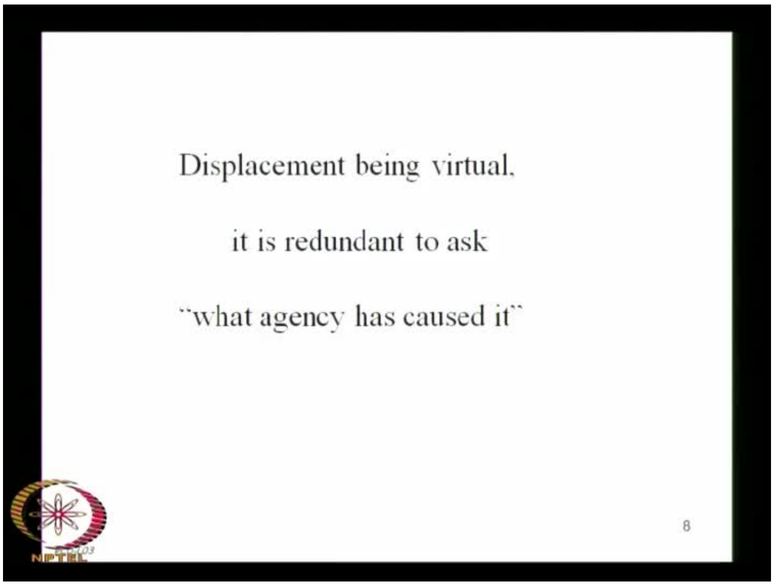

It is obviously redundant to ask, what agency has caused it. You do not need a physical reason for it. It is not even a physical displacement. If I were to move this bottle here, I need to answer how the bottle got here. I need to explain that I pushed it, but since this is a virtual displacement, the question of what agency caused this displacement is irrelevant. So, we do not get into that argument at all.

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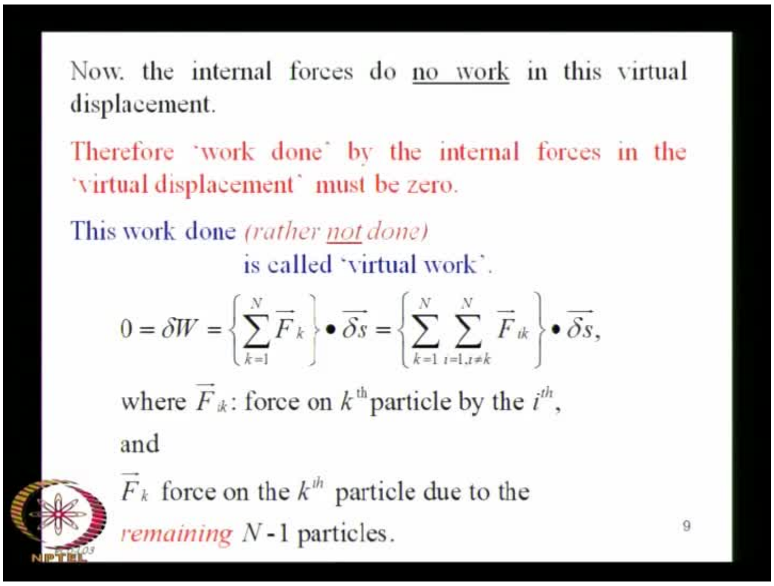

Now, internal forces cannot do any work in the virtual displacement. It should be self-evident. Internal forces would do no work in this displacement and the work done by the internal forces would therefore be 0. As I mentioned in the parenthesis over here, no work is really done. If you were to determine this work done, it would turn out to be 0 and this work is what is called as the virtual work.

Now, the terminology should be self-explanatory because it is the work done or rather not done by an agency, which is not relevant to the process of the virtual displacement at all. So, we write the virtual work done to be 0, which is delta W. It is the component of the net force along the displacement. If F k is a force on the kth particle, you sum over all the particles, k going from 1 to N. When you take the scalar product of this with the displacement vector delta S; this scalar product must vanish. You can write the force on the kth particle by the ith particle i summed over all the particles from 1 to N except i equal to k.

So, this is the force due to all the remaining N minus 1 particles. I emphasis this point; that these particles are interacting with each other, but there is no external force acting on it. It is not that there is no force acting on each particle, but whatever force is there, it is imparted only by the remaining N minus 1 particles and no other forces are involved. So, 0 is equal to delta W and this relation is now quiet clear. This is an important result.

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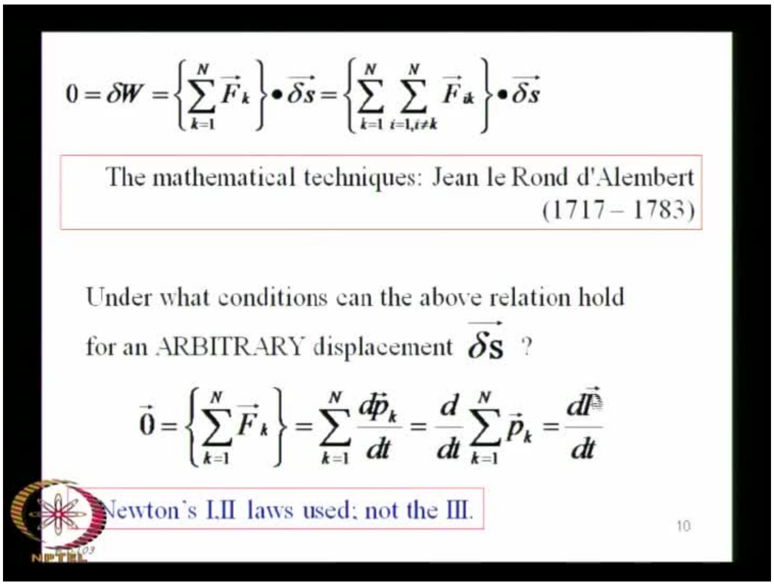

Now, let us see the consequences. The particular technique that we are using is the idea of virtual displacement and virtual work. This was introduced by d'Alembert. I do not have any French student in the class, so I can getaway pronouncing Alembert's name any way I like. I am sure that either t is silent or b is silent or m is silent or I do not know, so pardon me, if the pronunciation is not correct. I can assure that it is not correct and I have no idea of what the correct pronunciation would be like. If it happens to be correct, it can be the biggest accidents I have ever had.

These techniques were introduced by d'Alembert and this is sometimes also referred to as d'Alembertian's principle etc. There are various names or the principles of virtual displacement or the principle of virtual work. All these things go together and d'Alembert is the person, who proposed this mathematical technique. Now, let us ask under what condition can this relation hold, regardless of the displacement vector delta S? We never specified what delta S is. We simply said that as a tiny virtual displacement. Did we say that this displacement should be toward east or toward west or toward north or south any direction? It is completely arbitrary and no matter what this delta S; displacement is in whatever direction. It is this relation that must hold. It can hold, if delta S and this summation of all these forces is in this beautiful bracket in this first term.

If this delta S and this net force were orthogonal to each other, the scalar product would vanish. It would happen only for a particular direction delta S, which is orthogonal to this net sum of the forces. If you consider some other direction delta S, then you cannot invoke the 0 value of cosine theta to explain that expression. If this relation has to hold no matter what the direction of delta S is or no matter what its magnitude is and no matter what its direction is. Regardless of the magnitude and the direction of delta S, if this relation is to hold, then there is only one possibility that the sum of the forces must go to 0.

Now that conclusion cannot escape us. So, the sum of these forces in this beautiful bracket or what is expanded in this larger beautiful bracket, in which the force on each of the kth particle is written as a sum of the forces on the kth particle due to all the other particles i going from 1 through N into i not equal to k. So, this overall force must vanish. It guarantees that the relation will hold. Now, we have a very fascinating result. Look at this; you set this net force equal to 0. This conclusion cannot escape us and we will accept Newton's second law. We know that having accepted Newton's second law, we would have implicitly accepted Newton's first law as well. Newton's first law is in fact, the first law and the second law is the second. There is a certain sequence in the thought process. You cannot get to the second law without identifying an inertial frame of reference, which is the first law, which is the statement of equilibrium. So, we would have used Newton's first law and second law.

Using the first and the second law, we can write for each force on the kth particle as a rate of change of momentum of that kth particle, which is the statement of the second law. I am sure enough that you are using it and if you simplify this relation, you can take the derivative operator outside operating on this total sum, which is the total momentum of the system. So, the total momentum must be constant, since its time derivative vanishes. It is a result, which has now come from the homogeneity of space.

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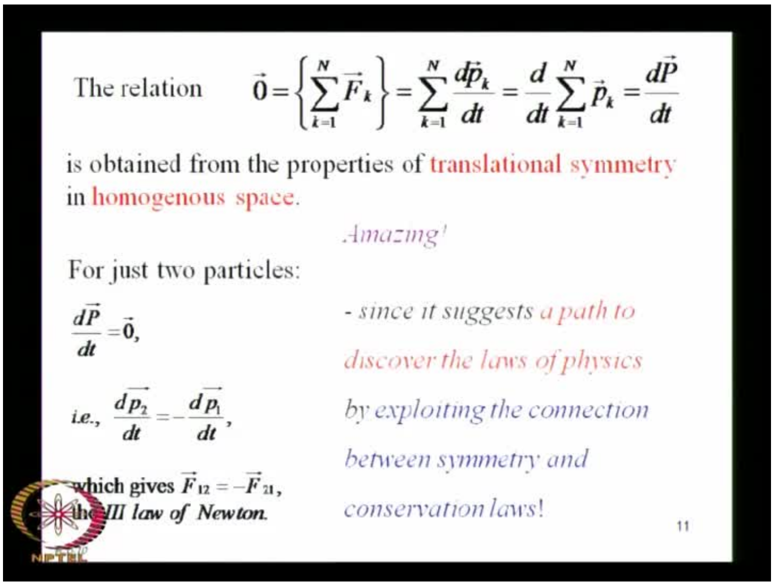

From the consideration of translation invariance in homogeneity of space, what does dp by dt equal to 0 boil down to? If you take a two particle system, dp by dt equal to 0. It means that the rate of change of momentum of particle two must be exactly equal and opposite to the rate of change of momentum of particle one and that is the same statement as Newton's third law. So, this is really amazing because we have actually deduced the third law. We did not begin by a statement that this is a fundamental principle. We accept Newton's first laws of fundamental principle and then we accept Newton's second laws of fundamental principle. Then we go to symmetry and we consider virtual displacements. We consider the virtual work as done and from the properties of invariance, we actually deduce the third law. What we have discovered ourselves is a path; given symmetry consideration, you can actually discover law of nature and you can take it far enough.

Not just Newton's laws, there are other laws of physics. Newton's law does not provide a complete set of laws in physics. They provide in the domain of classical mechanics, but even in classical mechanics, it has certain restrictions. If you get into electrodynamics, then Newton's law cannot be used in the way that we normally refer to them. When you get two electromagnetic fields, you have to take the momentum into account, which is carried by an electromagnetic wave. That is something, which Newton never thought about and there was no need to do that.

At Newton's time, the laws of electrodynamics were not known. So, this came much later and therefore, the Newtonian laws actually have to be reinterpreted to accommodate the electrodynamic interaction. We do have a technique, by exploiting the connection between the symmetry and conservation laws. We can actually discover new laws.

Can we deduce the first and second law with this?

No, we cannot. There is nothing that I know, from which we can deduce the first law or the second law. First law is a fundamental law. I do not think that it can be deduced from anything. The second law is a statement of cause and effect relationship. It is a linear relationship between cause and effect, but one can do away with this relationship. So, one can do away with the second law in an alternate formulation of mechanics, which I am about to introduce. So, you can do away with it, using the principle of variation. I will be introducing shortly and then, you will not really need f equal to ma to solve problems in mechanics. You will have an alternate way of solving problems in mechanics, but I will get to that.

If one can do away with it, it has to be accepted as a fundamental self-evident principle, which you are free to challenge and conduct experiments to verify it. The fact that you find no exception to it elevates that to the status of law of nature. So that is what makes it a law of nature. At this point, I would like to comment once again on the question, which Sidharth had asked last time as to why is it means what is the difference what is it that you call as the law of nature? Do you call symmetry as a law of nature? As I pointed out in the brief answer that I gave last time. The expression for law of nature or law of physics is used, when you talk with description of a process, which explains the evolution of the mechanical state of a system. Given a state of a system in classical mechanics, this system is described by position and velocity.

What is the law, which will give you an explanation for how the state of the system changes with time? When you do it, you are talking about what you call as a law of physics or a law of nature. Symmetry, typically refers to changes that you can bring through certain geometrical transformations like rotation about an axis or displacement. These are symmetry considerations, but these are geometrical symmetry considerations, which is the context, in which one normally talks about symmetry. However, there are other symmetries, which are known as dynamical symmetries. I will come to them a little later, but for the time being, we will restrict ourselves to geometrical symmetries.

Symmetry is the property, which leaves a system under certain invariance, after you carry out some geometrical transformations. So that is where you use the term symmetry. You do not refer to it as a law of physics or law of nature, but you call it as a symmetry principle.

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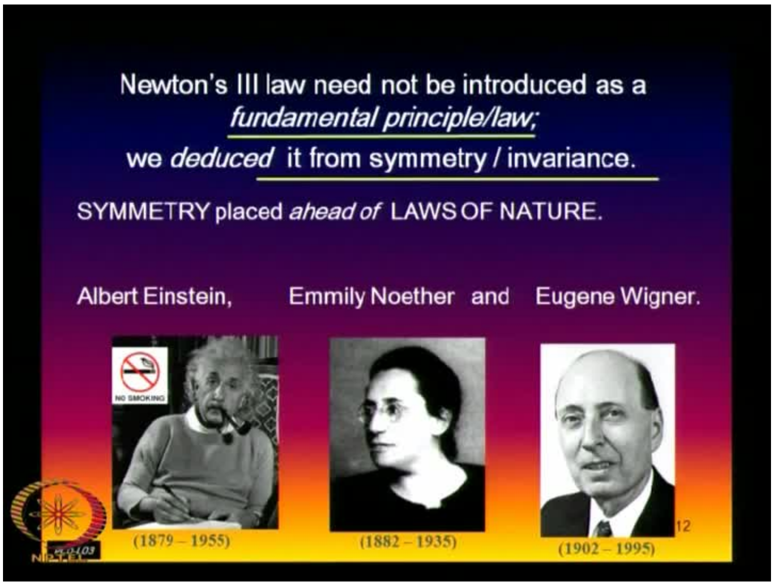

What we figured out is that it is not really necessary to introduce the third law as a fundamental principle because we have actually deduced it from a symmetry principle. It is derivable from something else. What we have done is to place symmetry ahead of the laws of nature. This is an approach, which I mentioned.

Earlier, it began with Einstein and it is contained in a very deep theorem known after Emmily Noether. This is known as the Noether's theorem. It states that for every symmetry principle, there is a conservation law. Associated with every conservation law, there is symmetry; there is a one to one relationship between symmetry and conservation principle. This is a popular statement of the Noether's theorem. In its rigorous form, one can state it using field theory, which I will not even attempt as part of this course. In nutshell, this is what it amounts to. There is a close connection between symmetry and conservation principles.

What Wigner did was to elucidate these connections in a very beautiful, very lucid manner and very rigorous manner because he considered all the symmetry operations. When you consider all the symmetry operations and put them in a set, you get a mathematical group. Using group theoretical arguments, one can bring out the relationship between symmetry and conservation laws in a very nice manner. So, Wigner's name should also be mentioned here. He elucidated the connections between symmetry and conservation principles using group theoretical methods in a very lucid manner.

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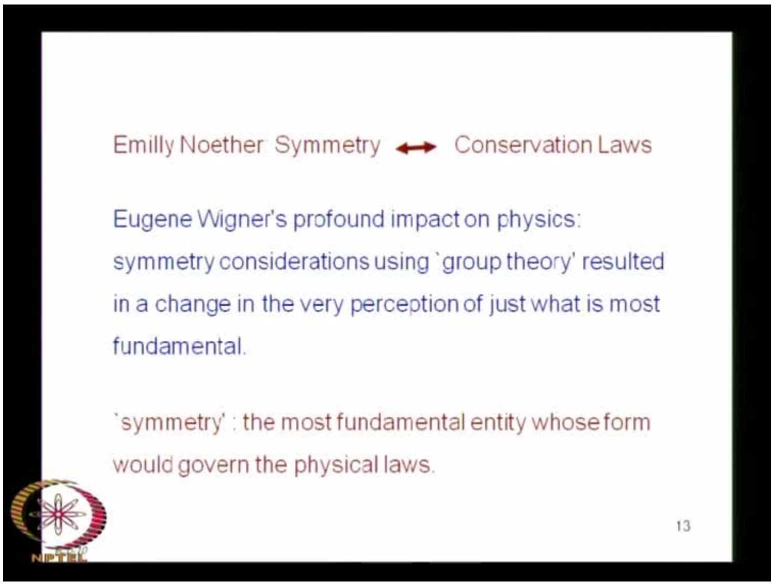

This is the statement or this is relationship between symmetry and conservation laws. This is what is at the heart of the Noether's theorem. Wigner explained this and elucidated this very nicely using group theoretical argument. Symmetry is now regarded as the most fundamental entity, whose form would actually govern physical laws. Therefore, you can hope to deduce physical laws by examining these relationships.

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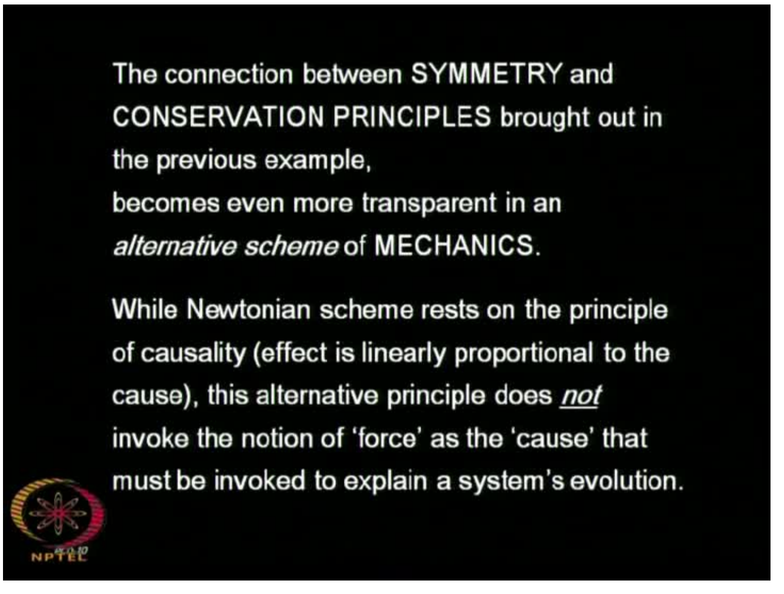

We have seen this connection, but this connection becomes even more transparent, if you formulate the laws of mechanics using a different approach. The difference is with reference to the Newtonian scheme. Now, what is at the heart of Newtonian scheme? It is the recognition of equilibrium and its connection to an inertial frame of reference. The intimate relationship between equilibrium, as a property of an object, whose motion is self-sustaining in an inertial frame of reference is the first law. The cause effect relationship occurs, when there is an interaction to which, this subject is subjected.

If this subject is imparted some kind of force or if it is involved in interaction with any external agency, then its equilibrium will be disturbed. The manifestation of this disturbance is acceleration. The velocity would change and if it changes, it would change at a certain rate, which is dv by dt. This is what you call it as acceleration. This is the effect of the force, which acted upon the object. So, acceleration is directly proportional to the force and the proportionality is the inertia. So, f equal to ma comes out of that

Force is interpreted as a cause of acceleration. This is a linear response theory and as you can see, acceleration is the response to the cause, which is the force. This linear response theory invokes the idea of force, but there is an alternative scheme of mechanics, which will make no use of force. I just mentioned that you can develop a scheme of mechanics, in which, you really do not invoke f equal to ma. Leave f equal to ma, you do not even invoke the concept of force at all. This alternate scheme of mechanics makes no reference to force. It will not use Newton's second law and it does need the first law.

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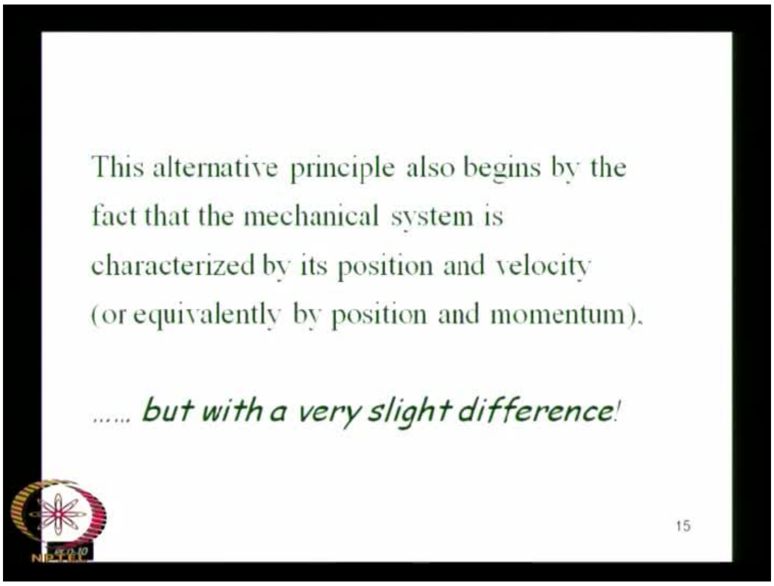

This mechanics has this starting point, which is similar to the starting point in Newtonian mechanics. In Newtonian mechanics, a system is described by position and velocity. Thus, we agreed that we must first identify the system. A mechanical system is characterized by its position and velocity. The question in mechanics is posed by asking, how does this evolve with time? How does the position change with time? How does the velocity change with time? These are the questions in mechanics. What answers these questions is the law of mechanics or it is the law of nature that is what you are trying to explore.

The alternative scheme that we are allowed to introduce also begins with the same fundamental recognition. It is of the fact that a mechanical system is described by position and velocity or equivalently by position and momentum. So that alternative within this scheme exist. If you describe it, in terms of position and momentum, you will then ask how the position changes with time. How does the momentum change with time? So, you will look for equations for dq by dt. If q is the position and for dp by dt, p is the momentum. If you describe it, in terms of position and velocity, you will look for dr by dt and dv by dt, which is the acceleration.

The equation of motion gives you rigorous mathematical relationships between position velocity and accelerations. So, we are going to have the same kind of beginning, which is to describe the mechanical system by position and velocity with a small difference. If you have a well-defined function of position and velocity, you have a function, which is a dependent variable of an independent variable. So, given x, you know f of x. That is a function; this is a relationship that comes to our mind, when we talk about a function. So, we will make use of a function of position and velocity.

If this function is a well-defined function of position and velocity, then given the position and velocity, this function can also be known easily. This statement of this alternative scheme is a mechanical system. It is described by a function of position and velocity, rather than, by position and velocity itself. In a certain sense, it is obviously determined by position and velocity. So, the starting point conceptually is the same, but you write it in terms of a function of position and velocity, rather than q and q dot itself. Here, q is the symbol for position, q dot is the symbol for velocity, q dot is just a dot placed on q or it stands for dq by dt, which is the time derivative of q.

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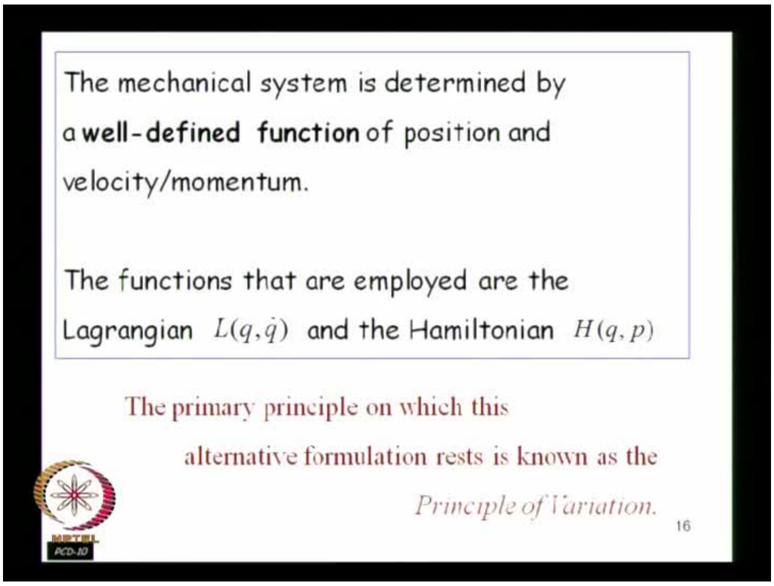

You can also express the mechanical system by a function of position and momentum. It can either be a function of position and velocity or a function of position and momentum. So, the functions that we have in mind are the following: q is the position, q dot is the velocity and a function of position and velocity that we have in mind is what is called as a Lagrangian. This Lagrangian can obviously describe the mechanical state of a system. It is equivalent in saying that position and velocity characterize the system. Instead of doing it directly, it is characterizing the system in terms of the Lagrangian. We will have to find out what kind of a function is.

We will discuss that now. If you were to describe the system by q and p by position and momentum, you can equivalently describe it in terms of a function of position and momentum, rather than position and momentum itself. This function of position and momentum is called as the Hamiltonian. There are some additional writers, which I will not get into; at this point. This is the basic description of a Lagrangian and Hamiltonian.

Now, the principle that we are going to invoke is called as the principle of variation. What was the principle, which was at the foundation Newtonian mechanics? It was the principle of causality and determinism that the force is the cause of the effect, which is the acceleration. The cause determines the effect and this is the principle, we were invoking. So, the principle of causality or the principle of determinism or the linear response of a system to an agency are the principles, which go into the Newtonian formulation of mechanics.

In all of this, force has to play a role in Newtonian mechanics. This is the very cause, which is invoked in the principle that is used in Newtonian mechanics, which is the causality principle. The principle of causality refers to the force. So, the idea of force is intrinsic to Newtonian mechanics. This alternative formulation of mechanics does not require. It will invoke a different principle; not the principle of causality, not the principle of determinism. It will not invoke force and what it invokes is called as the principle of variation.

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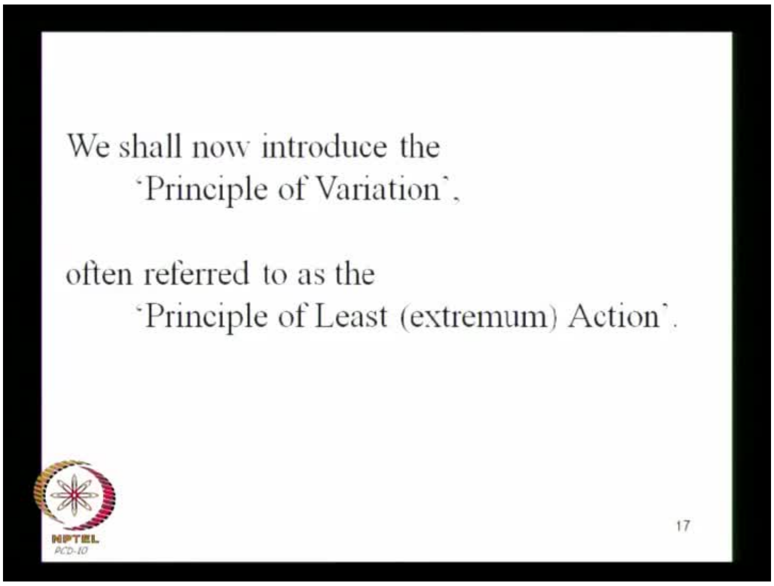

This is also referred to as the principle of least action. More correctly, this is called as the principle of extremum action. It can be stated even more rigorously, by referring not just to an extremum, but to what is called as the stationary point. Some of these things will become a little clear, but let us not go ahead. This alternative formulation is based on a principle, which is called as the principle of variation. It is popularly known as a principle of least action, although it needs some sharpening.

When you go to more subtle applications? So, some sharpening will have to be done. It is popularly referred to as the principle of least action. What might come to your mind, when you talk about principle of least action is to become lazy; go to sleep. This is not about that and it is probably a larger and a stronger law of physics or a stronger law of nature, if not a law of physics. Anyhow, this is the principle of least action.

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What will principle of least action or extremum action do in its various incarnations? It will be applied to everything that is happening around it. Using that you will be able to answer the same question. You did that when you used Newtonian mechanics. We agreed that using Newton’s laws. You do not have to learn anything else. If you want to explain the trajectory of a ball, which is hit in cricket or you want to explain how rockets raise against gravity or you want to explain why water flows in a particular way or any mechanical phenomena. No matter, whatever you want to explain, you can explain it in terms of Newton’s laws.

Likewise, you can explain all of this using the principle of least action. You will not require f equal to ma. This principle of least action will help you to answer all of these questions. Every mechanical phenomenon that takes place in nature is explainable. You can explain it using the principle of least action.

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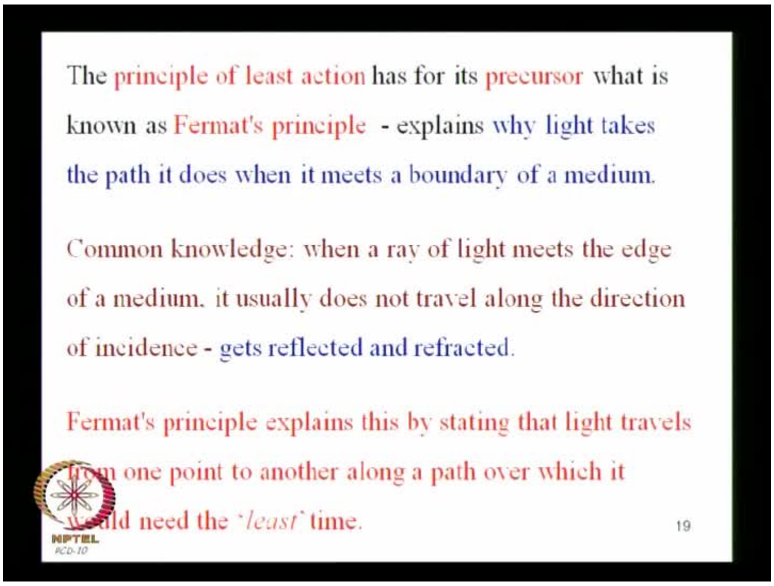

This has a very interesting history. Its precursor is the Fermat's principle and what the Fermat's principle does is- explain, why light takes the path. It does, when it meets the boundary of a medium. We do know that whenever a light ray hits the boundary of a medium, it is reflected or refracted. It does not happen, until it meets the boundary. We can talk about what we often referred to as the rectilinear propagation of light. Light rays travel in straight lines, but when it meets a boundary of a medium, then some part of it is reflected; some part of it is transmitted. Why does it happen? It is explained by Fermat's principle and what this principle states? The light travels from one point to another, along a path to take the least time.

Now, this is again a popular expression of the Fermat's principle because this least time is a more common experience. It can also be the most time. In some case, it can be an extremum, but usually it refers to the minimum time, which is why it is called as the principle of least time. So, this is the Fermat's principle.

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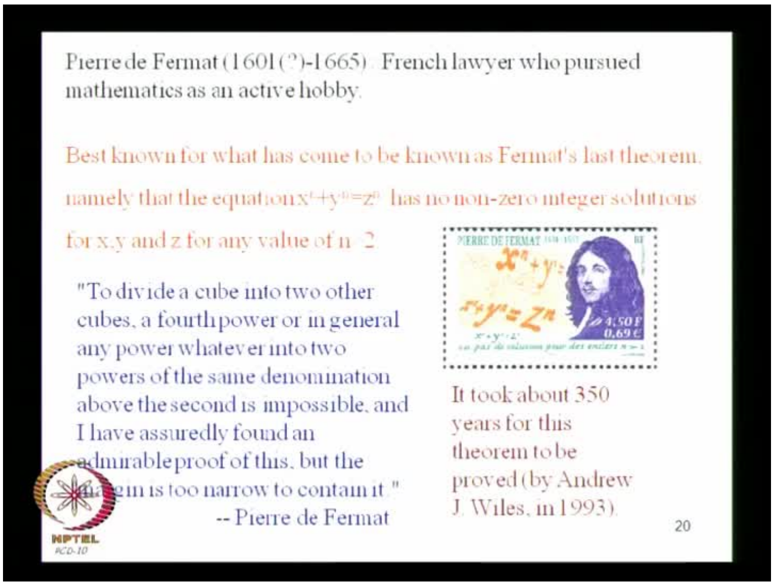

This Fermat was a very interesting person. He was actually a lawyer and mathematics was his hobby. Now, how interesting? When we think of hobbies, we think of either reading or listening to songs, movies, watching games. Fermat was a lawyer, whose hobby was to study mathematics and physics as well. Fermat is best known for what is called as Fermat's last theorem and this theorem is rather well known. What Fermat wrote is - to divide a cube into two other cubes, a fourth power or in general any power; whatever into two powers of the same denomination above the second is impossible.

So, here you have an example. You know that x square plus y square is equal to z square, which is the Pythagoras theorem. So, if you look at the right hand side of this, it is z square in Pythagoras theorem. What you have done is to break this into two similar pieces, x to the power 2 and y to the power 2. Now, can you write z to the 3 as the sum of x to the 3 plus y to the 3? The answer is no. Can you do it for x to the 4 plus y to the 4 equal to z to the 4? The answer is no. You cannot do it for any power higher than 2 and that is Fermat's theorem.

He did not write the proof for it. He said that the he did not have enough space in the margin to write about the proof, but he knew what the proof is and this was in the 17th century. It took 350 years to prove this theorem. This was not proved long ago, less than 2 decades ago in 1993 by Andrew wiles. It took 350 years to prove this theorem and now, the proof is known. This is Fermat's last theorem and this is what, Fermat is best known for.

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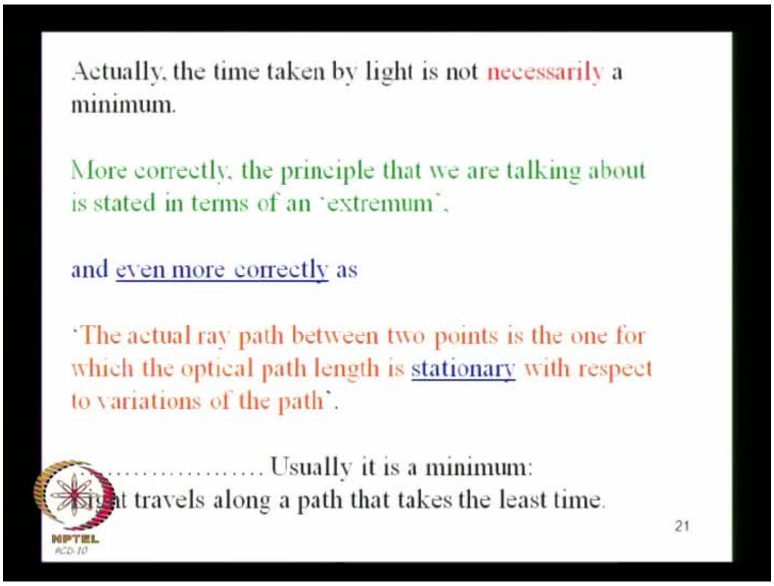

Here, I alert you the fact that the time taken by light is not necessarily a minimum. It can also be a maximum. So, it should more correctly be stated as a principle of extremum rather than least time. Even more correctly, the actual ray path between two points is the one for which, the optical path length is stationary with respect to variations of the path. So, this is a more accurate statement of the Fermat's principle.

If you consider any path and you consider any small departures from that path, then the actual ray is the one or the actual path would be the one for which, these variations would vanish. That is how it can be more correctly stated, but usually it is referred to in the context of the minimum and light travels along a path that takes the least time. So, I guess we want to take a 5 minutes break here, but I will be happy to take some questions.

Sir, can you elaborate on this third statement; the actual ray path?

Yeah, I could, but this is a fairly involved issue when you talk about an extremum or minimum or maximum.

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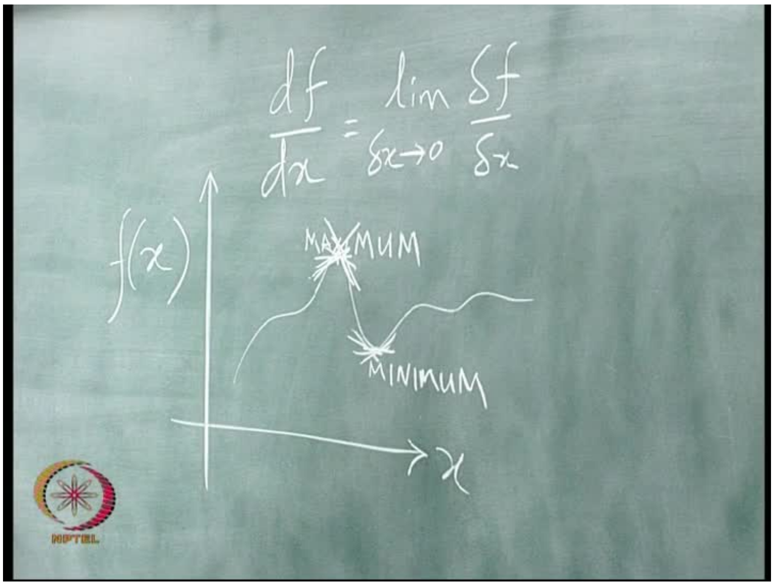

If you plot a function and this is your independent variable, this is your dependent variable and this change. Let us say, it goes as it has got some functional dependence. This function does not change its derivative, which is delta f by delta x in the limit delta x going to 0. If you look at this point, this derivative vanishes and this is the maximum. Same thing happens over here, which is a minimum. So, here you have a minimum and here you have a maximum and this condition that the function does not change.

If you consider infinite displacements about that point and that is what makes that point an equilibrium point, it is a stationary point. If you keep a tiny marble, which is like a point marble over here or over here, it could stay there forever. It describes equilibrium, there is no gradient and there is no change in the force, if this was a potential function. It could be either a maximum or a minimum and in both cases, it is generally regarded as an extremum. Whether it is a maximum or a minimum, it can be determined only by looking at the second derivative. If you look at the second derivatives, you will have slopes, which go like this. Here, they we will go like this. So, they will change in a different way. Unless you look at the second derivatives, you cannot really classify whether it is a maximum or a minimum. As a matter of fact, there are more complex stationary points, saddle points. For example, I will describe them later on. So, these are matters of details.

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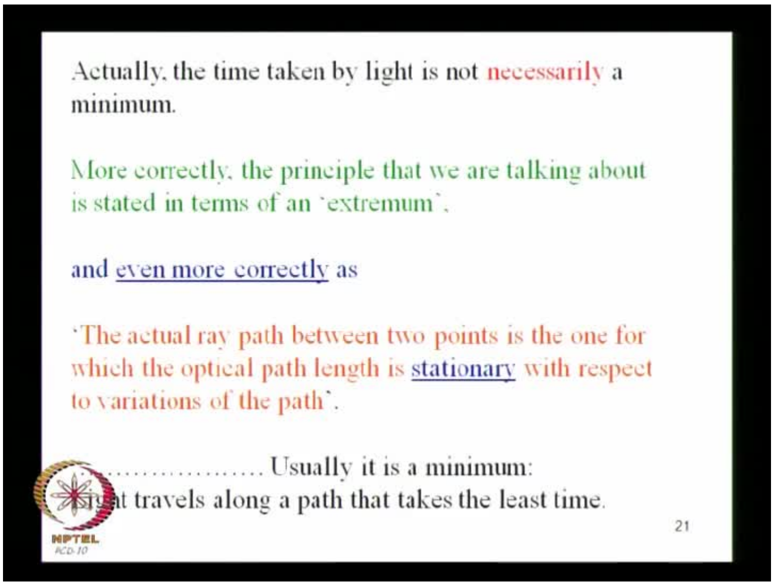

Typically in the context of our discussion, we will be referring to this as an extremum principle. It is commonly known as the principle of least action and I will introduce that in the next class.