Select/Special Topics in Classical Mechanics

Prof. P. C. Deshmukh

Department of Physics

Indian Institute of Technology, Madras

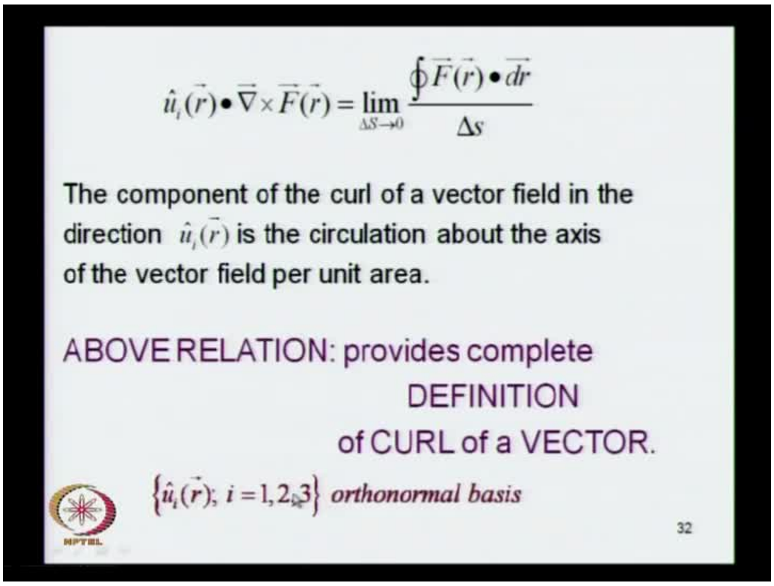
Module No. # 09

Lecture No. # 30

Fluid Flow Bernoulli Principle (ii)

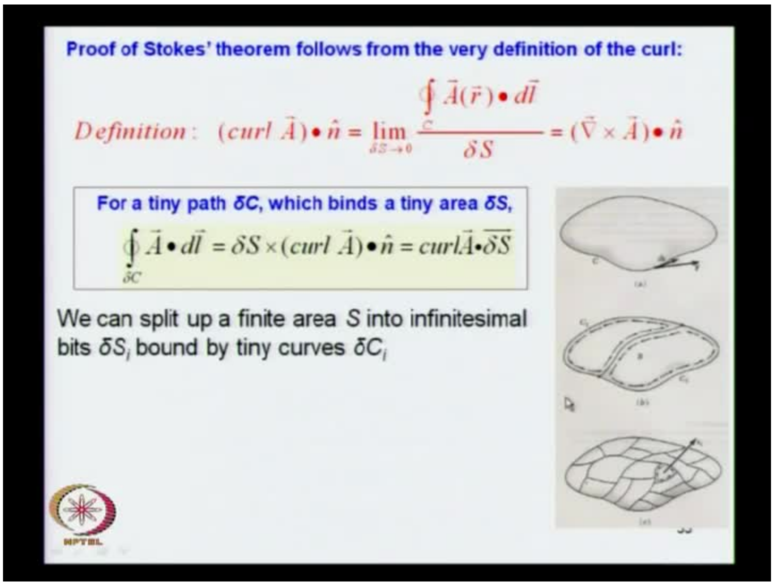
Greetings, so, today we will discuss the proof for the Stokes theorem, we have already defined the curl of a vector and we will find that it has got fascinating applications in fluid mechanics, electrodynamics and so on.

(Refer Slide Time: 00:34)



So, this is how we defined the curl of a vector; we defined it in terms of its components along a unit vector and when you consider three such unit vectors, which form an orthonormal bases, then this gives you a complete definition for the curl of a vector. And it is defined as the limiting value of this circulation per unit area and we have already discussed, how the direction of the unit vector is related to the area that we are talking about, because an infinitesimal area is treated as a vector, having a given direction. And the direction is connected to this vector of the unit vector through the right hand screw rules. So which we discussed in our previous class and with this giving us a complete set of orthonormal base vectors, these components give us the complete definition of the curl of a vector.

(Refer Slide Time: 01:45)

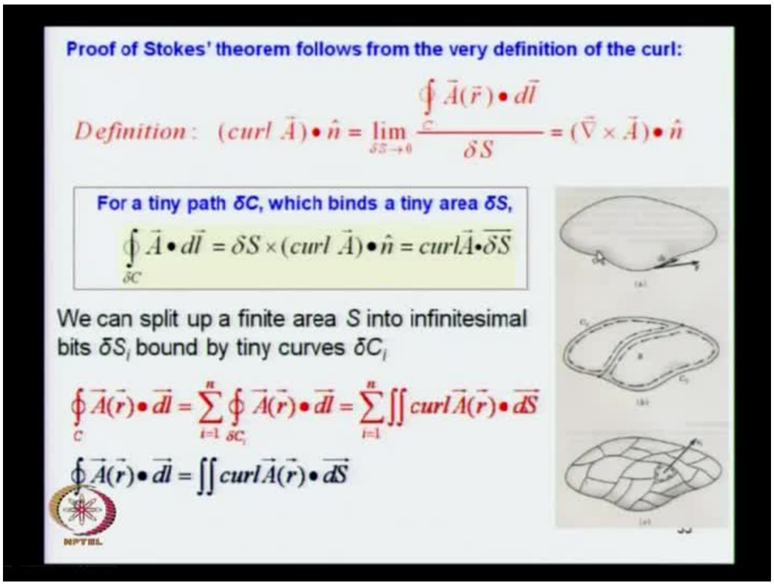


Now, what we do here is, we multiply both sides by the magnitude of this area delta S. So, on one side you get this circulation which is here, and on the other side this curl A dot n is multiplied by the magnitude of the surface element and this magnitude of the surface element together with the direction surface of that surface element, give you the vectorial surface element delta S. So, you get the dot product of the curl of the vector with the vectorial elemental area.

So, this is a very straight forward relationship and what we can do is apply this definition to small segments as well as large segments and this is extremely fascinating application. So, very simple one, but what it does, is it allows us to prove the Stokes theorem in an extremely simple manner. We do not have to do any complicated analysis, it comes straight out of the definition of the curl of the vector which is in this relationship at the top.

So, this is defined in terms of the components of the curl of the vector are defined as the circulation per unit area in the limit the magnitude of this area tends to 0.

(Refer Slide Time: 04:03)



And if you have got a piece of area like this, you can just draw a segment in between and then divide this into two pieces, moreover you can divide each of these two pieces into a number of tiny tits and bits of elemental areas, they do not have to be of any regular shape, they can be quite crooked which is convenient for an artist like me. Because you do not have to worry about the geometry and the shape of the elemental areas, etcetera. So, one could have four sides, another piece could have five sides or it does not matter.

So, as long as you make tiny tit bits of elemental areas and then what you will have is, if you take any of the small pieces of areas. So, you if have a piece of cloth or any hand kerchief or anything that you stretch it out and then you divide into the small segments, you take a tiny piece of that and to this tiny piece, and in this picture this tiny piece is shown over here which as you can see does not have any regular shape or anything of this kind, it is not even flat because the surface could have been pinched and it has some wiggles ups and downs it does not matter.

And this elemental area will have a certain direction corresponding to the right hand screw rule. So, if you traverse this, the boundary of this area along this a right hand screw would propagate upward in this figure which will be the direction of that elemental surface area according to the right hand screw rule.

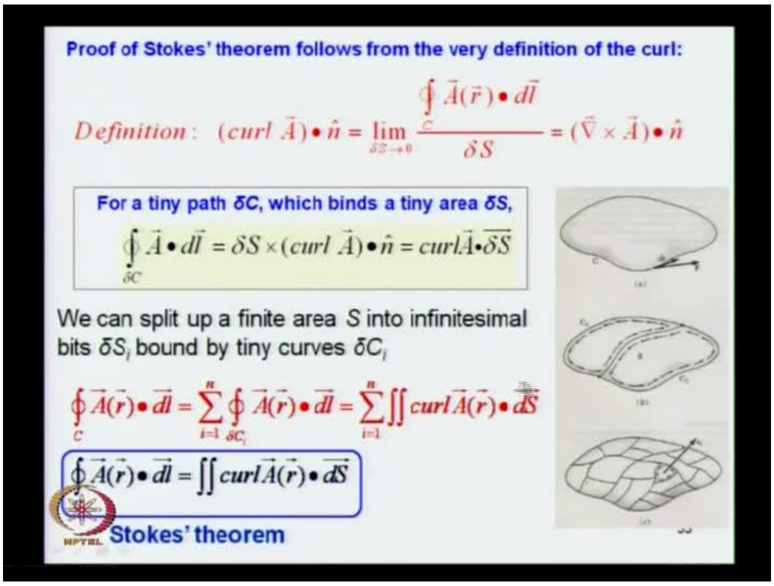
And you can apply this definition of the curl of the vector to this tiny piece of area, but then you could also apply it to any neighboring element like this one or the one on this side or the one on the other side or the one on this side. And you will find that each neighboring pieces will have this path integral, the circulation traversed in opposite directions. Like you see in this figure over here, the piece on the right side is traversed from top to the bottom as you see in this figure and the piece on the left side is traversed from bottom to the top in this figure.

So, the contribution to the circulation by these two pieces will cancel, because dl is a vector elemental displacement along that path and this obviously has opposite directions when you traverse it on the left side or on the right side. So, all of these inner paths will contribute to the circulation, these inner paths will cancel and you will be left only with the net circulation determined along the edge along the outer perimeter of the surface element.

So, what we have done is broken this pieces into these tiny tit bits and given the fact that all of these inner contributions cancel each other, then the net circulation A dot dl over the closed path c, which is this outer path, this is the path c. Notice that this letter c is written over here, this is the outer path which goes along the perimeter of the net area along the outer boundary. And this is written as the sum of these circulations, over all of these inner boundaries of these small tit bits of elemental areas.

So, you sum over i going from 1 through n, n can be as large as you want and all of these inner circulations are over these inner boundaries with the difference that when you carry out this sum from i going from 1 through n, all the contributions of the inner boundaries will cancel each other and then they will add up to a net result, which will be given by this circulation of this vector over the outer boundary.

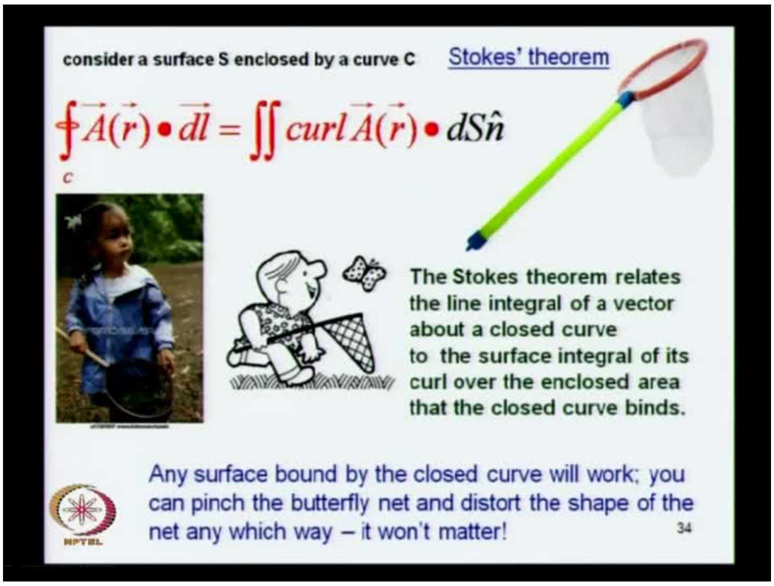
(Refer Slide Time: 08:40)



And by this definition, this will give you the component of the surface integral of the curl of the vector along ds. So, it is a very simple proof and essentially you have this net result. Now, that if you take the dot product of curl of a vector along this surface element and integrate, because when you carry out these summations from i going from 1 through n of all the small elements and let n tend to infinity, then in the limit this sum will generate the corresponding integral; it will generate the corresponding surface integral. So, you have the net result that the circulation over this closed path which is the outer edge is given by the surface integral of curl of A dotted with the elemental surface area over these tiny tit bits, but then you integrate over the entire surface. This integration of the entire surface is what is implied in this summation i going from 1 through n. So, this result is known as the Stokes theorem.

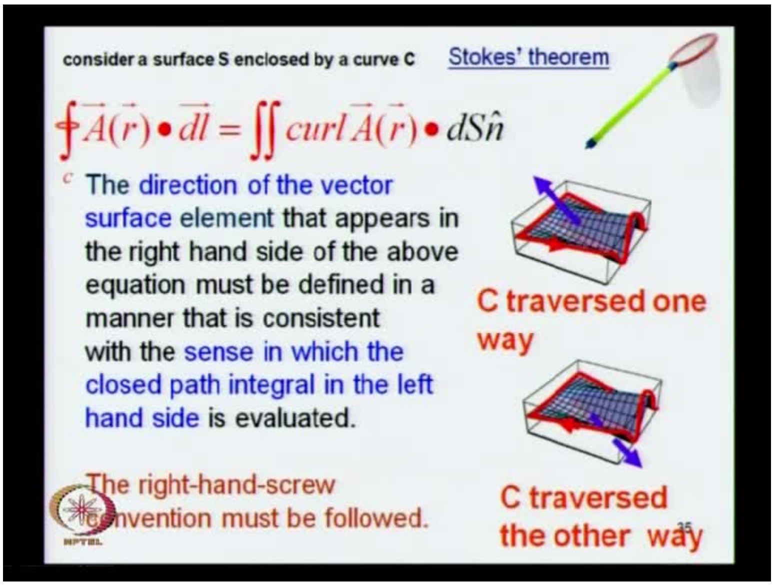
So, we have a complete proof for this and this proof has popped straight out of the very definition of the curl of the vector; we did not have to add anything further to it. And this is an extremely important theorem in vector calculus, which connects the surface integral to a circulation over a closed path, it will remind you for those of you who have studied complex analysis, it will remind you of the Cauchy's theorem and what it does, is it connects the properties along the boundary of a region with what is inside it. So that is a connection and it is very rigorously quantitatively established in terms of the Stokes theorem.

(Refer Slide Time: 10:02)



So, this is our net result. What is amazing is that it can be applied to any shape of the surface element. So, this surface is a butterfly net, it is obviously not a flat surface, but it is the surface nevertheless it could be flat, you could pinch it, pull it up or down or you could pinch a part of it and pull it down; pinch the other part and pull it up, it does not matter what happens to the surface element. Because the rim of this, the perimeter, the outer edge is the same and no matter what shape the surface elements take these individual tiny surface elements on the net take the outer perimeter is the same. So, no matter how the shape is distorted, the left hand side is equal to the same. So, each right hand side will also be the same, no matter how you flex that particular surface.

(Refer Slide Time: 11:22)

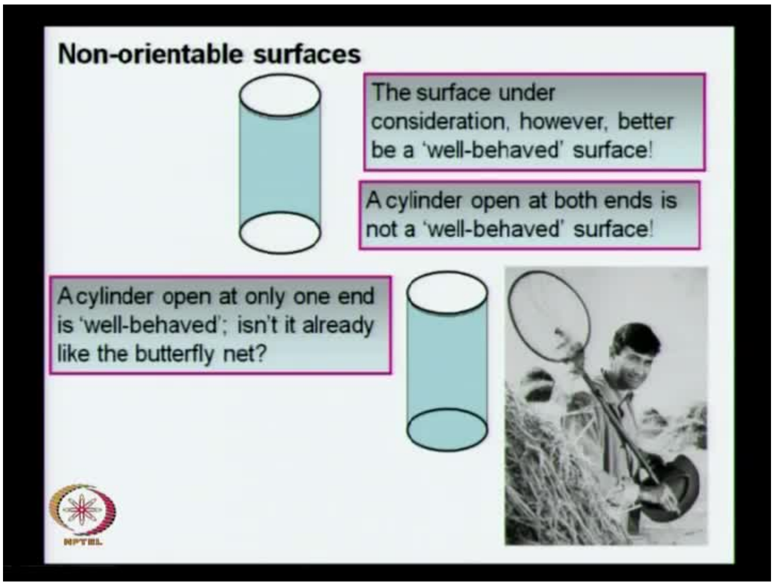


So, this will remind you of this butterfly net, which you can distorted into various shapes. The only thing to remember is the right hand screw convention, because so for is the circulation is concerned, it is along a path which can go either one way as shown in this or in this other figure it is shown in the opposite direction.

And depending on whether you take the upper path or the lower path no matter, how the shape is distorted in between, it may have some wiggles, but the edge of the surface is essentially the same and you must follow the right hand screw convention to link this direction of the surface element and that is given by this arrow. Because a right hand screw would move forward in this direction by this blue arrow. If C, if this path is traversed along this and if this path is traversed in the opposite direction, then the right hand screw would come in the opposite direction. So, these are the only two things that you have to remember.

So, the direction of the path integral is connected to the direction in which this surface is associated, because it is a vector surface element and it has got a direction and it could be either upward or downward, it is normal to the surface that so far so good, but there are two normal at any point to a surface, it could be pointing one way or the other way and there is no ambiguity about it, because that is spin down by the direction in which this circulation path is chosen and then fallowing the right hand screw convention as we have defined.

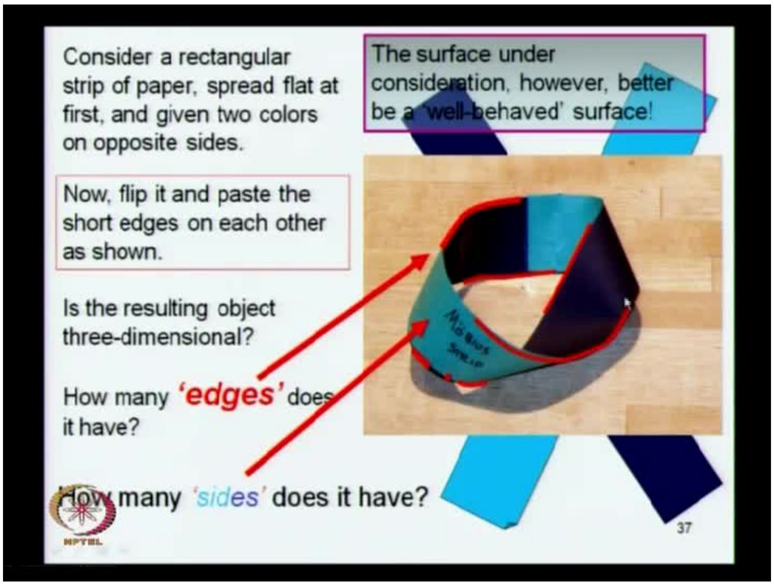
(Refer Slide Time: 13:02)



The only thing is that the surface must be an orientable surface. So, there are non-orientable surfaces we discuss some of this when we did unit 8, but I will like to remind you that surfaces which are orientable can be considered, there are certain non orientable surfaces which we say are not well behaved and though surfaces have to be eliminated from this consideration. So, the examples for this is the cylinder which is open at both ends, this is not a well behaved surface because you cannot pin down a unit direction to an elemental surface on the wall of the cylinder if it is bounded by two edges. Because you can take the paths along these two edges, so that they will generate a conflict so for as the direction of the surface element is concerned; so this is not a well behaved surface.

But if it is a surface which is open only at one end. So, the bottom edge is now closed and you have a top edge and there is no ambiguity with this. So, this is a cylinder which is open only at one end and no big deal about it, because this is anyway just like the butterfly net already, so it is the same example that we consider.

(Refer Slide Time: 14:35)

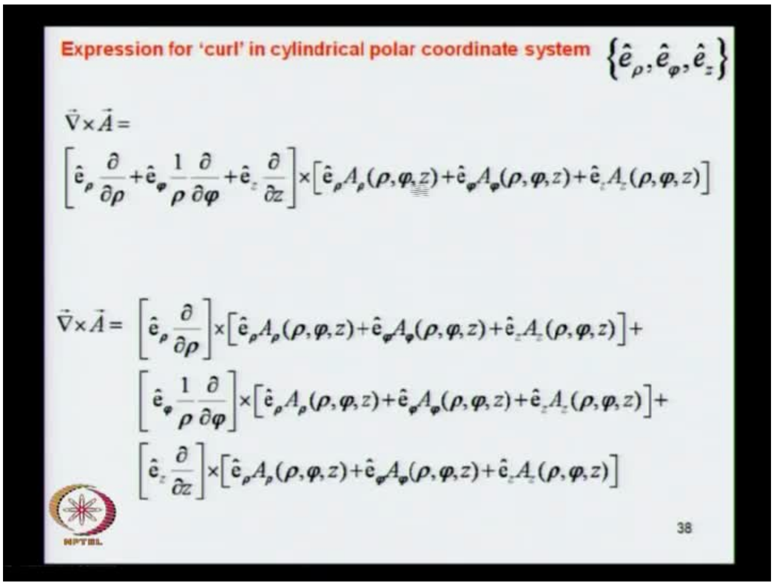


So, there is no difficulty with the cylinder which is open at one end, but for a cylinder open at the top face and the bottom face, then that surface has to be eliminated. Also may be a strip which I showed when we discussed unit 8. There may be a strip we discuss the construction of this may be a strip, all you do is to take a strip of paper stretch it out, twist it and then join the two edges and what happens is that, you have this top edge which if you run your finger at top edge beginning over here, then follow the path and then without lifting your finger you find that you end up arriving at the bottom edge and then you cross it over here and again without lifting your finger you find that you go over to the top edge. So, there are really are not two edges there is a single edge.

Likewise if you put your finger on what looks like an inner surface over here, just under need this arrow and run this finger as I am showing by this pointer inside there may be a strip, but then over here you keep going on that and you find by the time you come over here you find yourself going to what you would otherwise call as the other side.

But since you have not lifted your finger, you are going to have to admit that it is really the same side. So, this does not have really two sides; so this is not a well behaved surface and you cannot apply the Stokes theorem to such geometries.

(Refer Slide Time: 16:10)

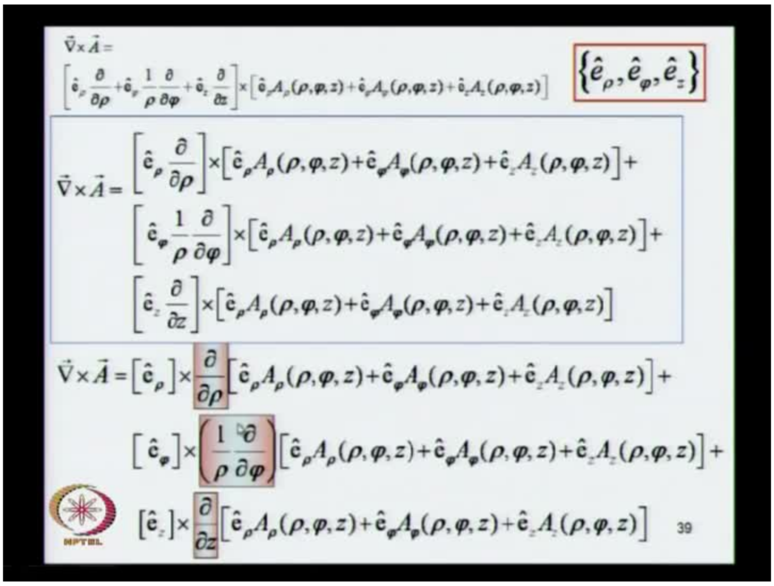


Now, we already got the expression for the curl in the cylindrical polar coordinate system, we got it for the Cartesian coordinate system. Let us get it in some of the other coordinate systems, it is not at all difficult. So, let us do it for the cylindrical polar coordinate system and I would not spend too much time doing this, I will show the result to you in a hurry. Because what you all you do is write the expression for the gradient in the cylindrical polar coordinates and this we have done in great details, we found how to get this expression you do not have to by heart this expression all, you have to do is to use the idea of a directional derivative connected to the idea of a gradient and from this connection you can easily get the expression for the gradient in any coordinate system. So, you do not have to by heart this formula, please do not do it try to get it from first principles by using this idea of a directional derivative and how it connects to a gradient.

So, this gradient operator in the cylindrical polar coordinate system is given by e rho del by del rho plus e phi 1 over rho del by del phi and then you must add the e z del over del z and then take the cross product of this vector now, represented in polar coordinates. So, you have got e rho A rho plus e phi A phi plus e z A z but then the vector A is a point function, so each component could be a function of the particular point represented by three degrees of freedom in cylindrical polar coordinates these are the rho phi and z coordinates.

So, this is the quantity you now have to determine, all I have done is to separate out these three terms. So, you have got e rho del by del rho cross this whole vector then you have got the second term and then you have got the last term.

(Refer Slide Time: 18:10)

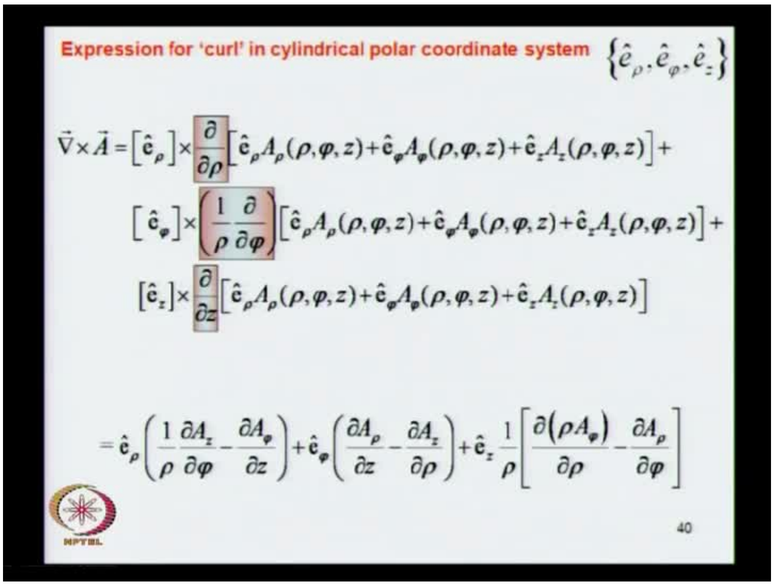


So, we will carry this to the top of this slide and now, these three terms again we separate out one by one. So, we write it explicitly for e rho del by del rho and construct this cross product with this these three components. Likewise, take the middle term over here and construct the superposition of these three components and then you take the last term over here and again take the cross product with these three terms.

Now, this is not in ordinary cross product of two vectors, because this side what is in this rectangular bracket is of course a vector, but over here is not a vector, this is a vector operator. So, it must satisfy the vector algebra as well as the operator algebra, the operator under consideration is a differential operator. You take the derivative with respect to rho in the first term, you take the derivative with respect to phi in the second term and you take the derivative with respect to z in the third term.

So, these are differential operators and what you must therefore do is before you construct the cross product take that derivatives do the calculus first and then the vector algebra next. So, find out the derivatives on this side the reason you have to do it is because unlike the Cartesian unit vectors which are constants, this cylindrical polar unit vectors e rho and e phi they are not constants, they do change from point to point.

(Refer Slide Time: 20:37)



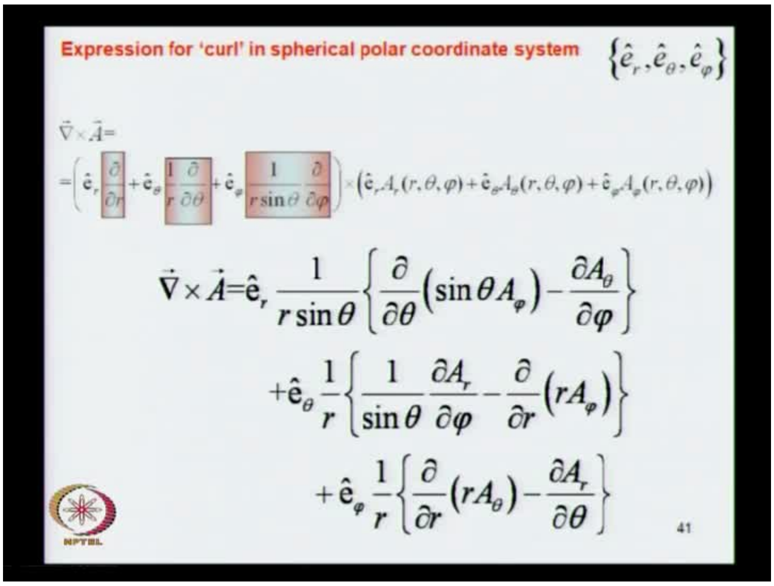
So, they do not in fact change with respect to rho, but they do change with respect to phi. So, they do change from point to point with respect to the azimuthal angle and this we have done in considerable detail when we did unit 3. So, you already know how to do this very elementary derivative operation. So, you can take the derivative with respect to rho of e rho A rho plus e phi A phi plus e z A z and then plug in the results. So, this is the differential operator that you must operate first and you carryout these differential operators and then once you have the results collect all the terms in e rho collect all the terms in e phi and collect all the terms in e z.

So, some of the intermediate steps I will skip, you can easily work them out in your note books or if you are cleverer than that please do it in your mind, but you will have the same result quickly enough.

So, this is your end result and we now have the complete expression for the curl of a vector in the cylindrical polar coordinates. Every parameter on the right hand side over here is in terms of the polar coordinates, they involve the polar coordinates of the vector A and the derivatives with respect to the polar coordinates rho phi and z and the directions are also given in terms of the unit vectors of the cylindrical polar coordinate system.

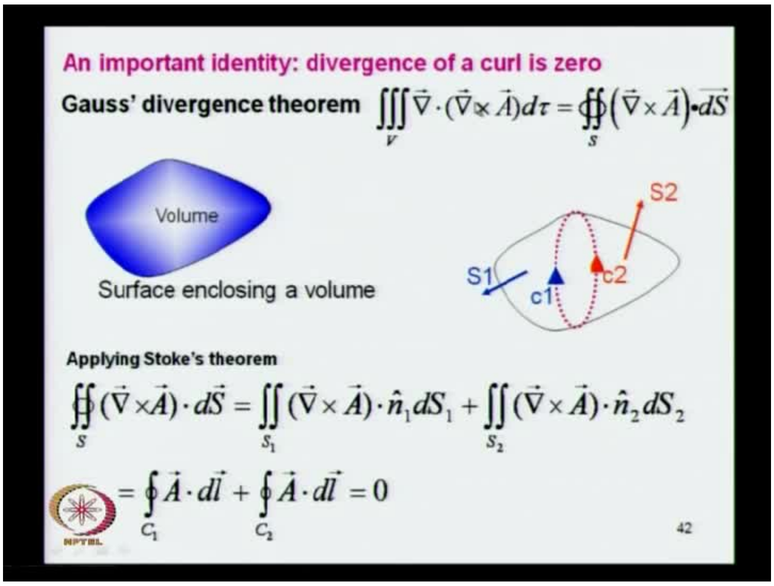
Now, you know the technique of how to get expression for the curl in any polar coordinate system. You can also get this in spherical polar coordinate systems; so that will be our next task.

(Refer Slide Time: 21:45)



So, again we have written the expression for the gradient operator in the spherical polar coordinate system; we have done this in detail in unit 3 again and you follow exactly the same procedure as we did for the cylindrical polar coordinates. Get the differential operators to operate first and finally do the cross product at the end and if you just follow that procedures step by step, you will get the net result. Do not forget the scaling factors in the denominator and once you do that you will get the end result which let you work out for yourself. So, this is the expression for the curl of the vector in the spherical polar coordinate system.

(Refer Slide Time: 22:35)



Now, let me suggest to you that there is an important identity which is very often used in vector calculus that if you have the curl of a vector and you would subsequently take its divergence, because you could always take the divergence of a vector. So, if you take the divergence of curl of any vector then the result is always 0 no matter what this curl is of, no matter which vector this curl is of, the divergence of that curl is always 0. And you can see this result very easily we will prove it using Gauss's divergence theorem which connects the volume integral of divergence of a vector with the surface integral of that vector with the difference that the vector under consideration is the curl of a vector.

But it is a vector nevertheless the curl of a vector defines the vector point function and you can apply the Gauss's divergence theorem for this curl of the vector which is exactly what is done in this equation in front of you. And this volume integral is then equal to the surface integral over the surface which encloses the corresponding volume of the dot product of that vector with the surface element and for the closed surface there is no ambiguity defining the direction of the surface element it is always the output normal so there is no ambiguity here.

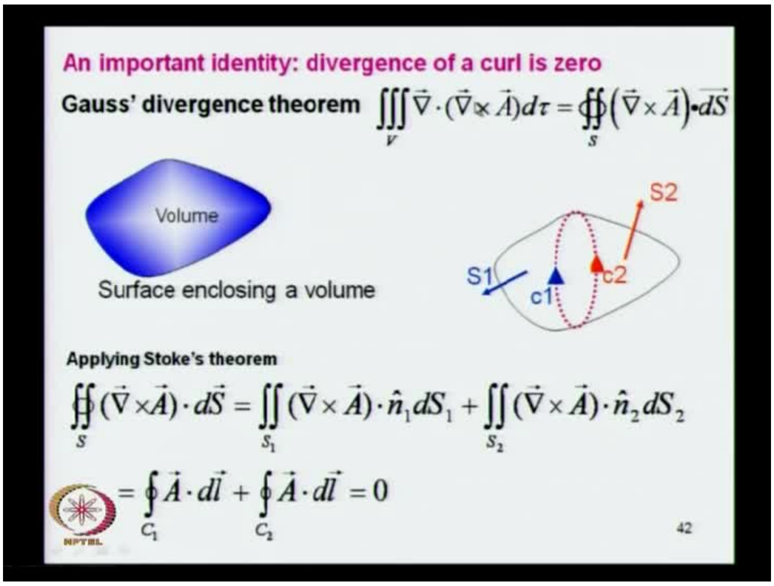
So, this is the divergence theorem and if we apply it to any region of space no matter what the shape of this region is, it really does not matter, it does not have to be any regular shape and the proof for this particular identity comes in a very simple manner. If you just construct some path about this volume region, on this volume it does not have to be regular path, which looks like an ellipse in this figure it could be any crooked path, it does have to be a closed path. So, you take some closed path, it does not have to be in between it can be anywhere, because for an irregular shape body there is nothing that you define as in between, because there is no center of symmetry of any kind, there is no reflection symmetry. So, there is no real symmetry of any kind; so it does not matter where you take it, all you do is to just construct one closed path somewhere along the surface. So, if it is this bottle which is a closed thing, you can take a thread and run it along like this, then run it down over here, then run it over here, then run it over here, and then bring it back over here and you have got a closed path. So, there is nothing regular about the path that we are talking about.

All you do is you divide this closed surface into two pieces, one which is on one side of this path and the other which is on the other side of the path and these are not two equal regions, they are not equal in any sense, they are not symmetrical in any sense, but they are two.

What they do share is a common boundary which is this and what you can do is, if you think of a path, which is going along the direction shown here, so on this leg it is going from bottom to the top on this leg, but you can also construct another path which is going from bottom to the top from the other side, so it will end up when you take this path over the closed loop these two paths are in opposite directions.

So, these two paths are in opposite direction and with reference to these two paths you can now define the direction of the surface element. So, this surface element with reference to this blue arrow is an arrow pointing to the left and with reference to this red arrow, it is an arrow pointing in this direction that will be the direction suggested by the right hand screw rule.

(Refer Slide Time: 22:35)



The shape of the surface you mind you it like a butterfly net, you can pinch it upward and downward so no big deal about it, so it could be any irregular shape. Now, if you apply the Stokes theorem to these two surfaces S 1 and S 2. S 1 is this surface which is on one side of this path S 2 on the other side, then the net surface integral over this closed path is equal to the sum of the surface integrals over the surface S 1 and S 2, because the surfaces S 1 and S 2 is precisely what together generate the closed surface.

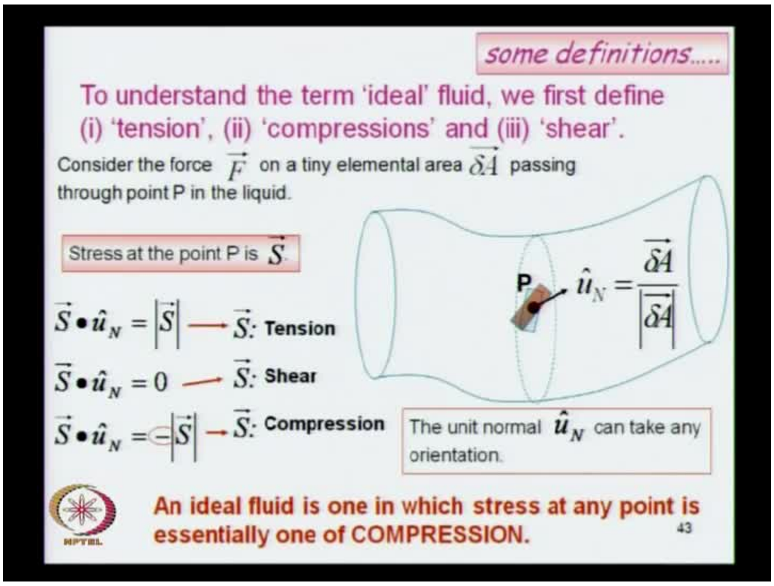
So, the closed surface is made up of two pieces; one is the piece S 1 the other is the piece S 2. So, this is a complete identity and now these surface integrals over S 1 and S 2, you can define in terms of the circulations using the Stokes theorem. So, here is the result, so the surface integral over S 1 is the circulation A dot dl over the path c 1 and this surface integral over the surface S 2 is the circulation of A dot dl over the path c 2 with the difference that c 1 and c 2 are opposite paths.

And they being opposite paths the direction of this elemental increment dl on that path this one and this one will always be opposite to each other and the corresponding pieces in the two integrals will cancel each other then net result will be 0.

So, what does it tell us that this surface integral vanishes; this is coming over here on this right hand side. Now, if you go back and look at this equation you have a right hand side which is identically 0 no matter which volume integral you are looking at.

So, the corresponding integrand must be identically 0 this particular identity is going to hold good no matter, what shape you have in mind no matter, what volume you have considered and if it is as general as that then it guaranties that the integrand itself must be 0 which is the identity that the divergence of the curl of a vector, this is the divergence of the curl of the vector, it must be identically 0. So, this is a very useful identity which one employs quite the lot in vector calculus and in electrodynamics and many other applications. So, this is your result that the divergence of a curl of a vector is always 0.

(Refer Slide Time: 29:51)

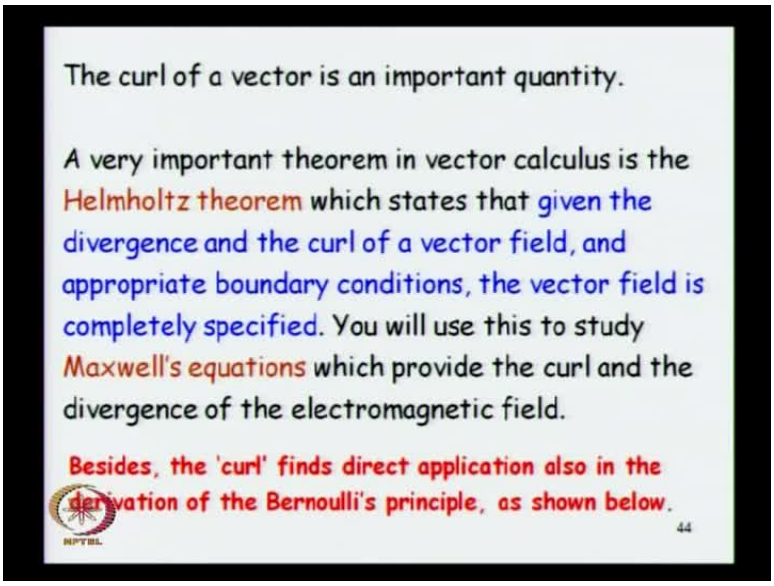


We also considered some of these definitions earlier in unit 8, but I will quickly remind you these definitions. Because we will be dealing with ideal fluids when we get to the discussion on the Bernoulli's principle and to define the ideal fluid we need to have these three definitions what is tension, what is compression, and what is shear.

So, if you consider the force on any tiny elemental area delta A and you can orient this strip any which way it could be like this trip or it could be this tilted strip and accordingly it would have a direction which is normal to that particular surface element. And the definitions are provided over here that if you consider the stress at any point p and if you consider the component of this stress in the direction of the unit vector which is in the direction of an elemental area of a surface elemental area, then if it is equal to the magnitude of the stress then you say that the stress is tension.

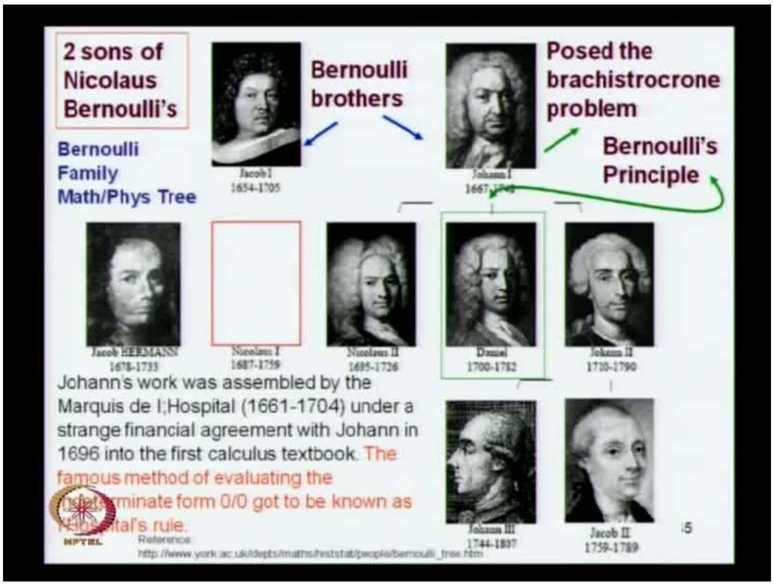
If it is orthogonal to it, if it is 0 then we say it is shear and if it is equal to the negative of the magnitude then you say that the stress is one of compression. And an ideal fluid is one, this is the definition of an ideal fluid that it is one in which the stress at any point is essentially one of compression. So, this definition we are used in unit 8 and we need it one more time and hence the quick recapitulation over here.

(Refer Slide Time: 31:44)



So, we will be using the curl of the vector I have already stated that it is an extremely important quantity and in fact there is a very important theorem in vector calculus which is known as the Helmholtz theorem. What it tells us that to define a vector field, you need both the divergence of that vector as well as the curl of the vector knowing only one over the other is not enough you need both of them and further more you also need some boundary conditions but that is a matter of detail. So, these boundary conditions are also required, but most certainly you do need the divergence of the vector, you also need the curl of the vector and both together will define what a vector field is, which is why when you want to ask yourself what is an electromagnetic field in a certain region of space then it is given in terms of the divergence of the vector as well as the curl of the vector and these are provided by Maxwell's equation. So, we are preparing the background to introduce the Maxwell's equation in the next unit.

(Refer Slide Time: 33:16)



So, right now we will first consider an application of the curl in the derivation of the Bernoulli's principle and that is what we will discuss here and I like to share this with you. Because there are several Bernoulli's, who are brilliant mathematicians and brilliant physicists so many of them and they were all related to each other they all came from the same family, what a distinguish family this must have been.

So, there were these two brothers Jacob 1 Bernoulli and Johann 1 and then Johann had children, these three Nicolaus, Denial and Johann 2 this is Johann 1, this is Johann 1 and the children of Johann 2 were Johann 3 and Jacob 2.

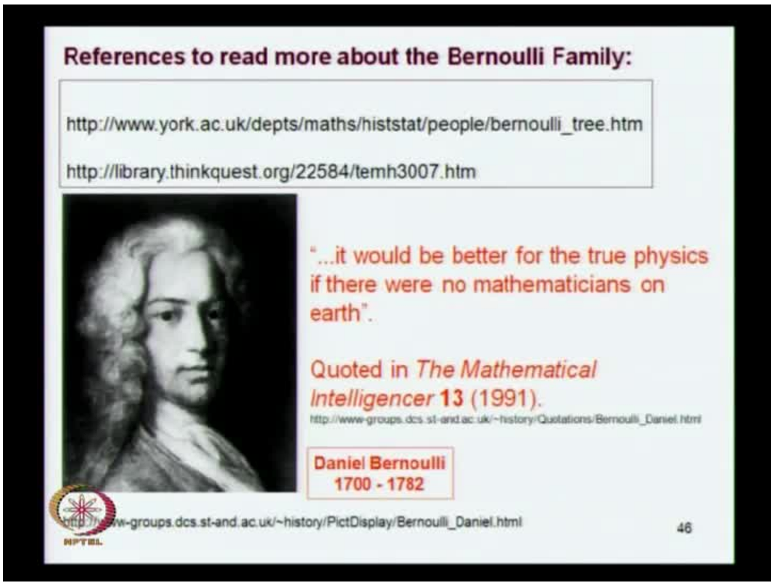
So, they use the same name, so you have to be very careful about it naturally which Bernoulli, but you also have to keep track of which Johann and which Jacob so the so that several Bernoulli's few Jacobs and a few Johanns and the Johann 1 is the one. This is the Bernoulli he pose the famous brachistochrone problem to Isaac Newton and Newton was of course able to solve that.

So, he was brilliant enough to be able to challenge Newton with a problem, but the Bernoulli's principle that we are talking about is after Denial Bernoulli who was the son of Johann 1.

Now, Johann 1 was brilliant not only because he pose the brachistochrone problem, but he also came up with this rule which we recognize as the l hospitals rule I do not know how this is read. Marquis de l hospitals rule this is a very famous theorem in differential algebra when you take the derivatives, if you have an indeterminate quantity you can take the derivative of the numerator and take divided by the derivative of the denominator and then from that ratio you can get the indeterminate quantity which originally you believed was indeterminate.

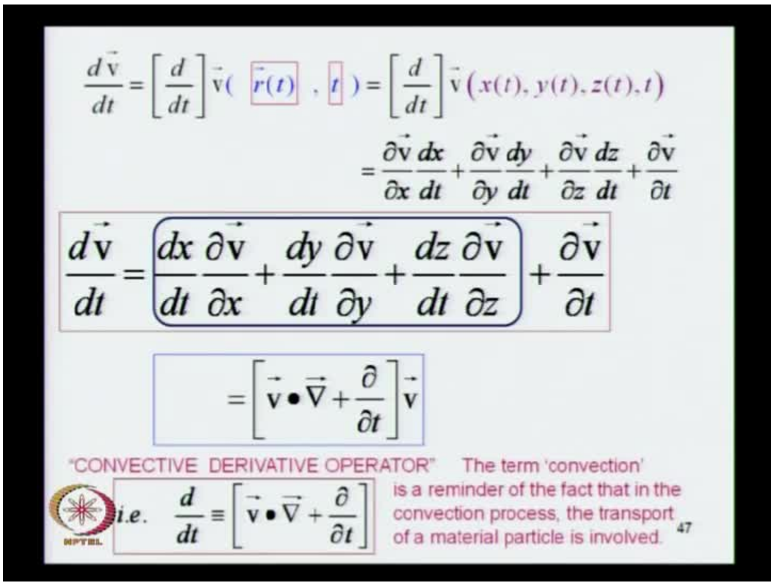
So, this is known as the L hospitals rule, but this rule was in fact invented by Johann 1 and he entered some sort of a financial arrangement with L hospital entered a financial arrangement with Johann Bernoulli and the result is that although it was formulated by Johann Bernoulli it got to be described as L hospitals rule. So, you could buy theorems in those days and sell them under your own name. So, it was a different business strategy all together.

(Refer Slide Time: 35:58)



So, the Bernoulli we are talking about is Denial Bernoulli who is known to I have stated that it would be much better for true physics, no not much better you just said it was better. It would be better for true physics if there were no mathematicians on earth. So, I code this with de apologies to mathematician amongst you nothing against I have nothing against mathematics, I love mathematics and I think it is an integral part I do not like to distinguish it from physics. Galileo told us that it is the language of physics and you would not really separate one from the other, but Denial Bernoulli had his own ideas about mathematicians and he did not want them for true physics.

(Refer Slide Time: 36:44)



So, anyhow, so let me use another little piece of differential calculus over here and here I consider the time derivative of the velocity this of course is the acceleration. This is the acceleration which goes into the cos effect relationship of Newtonian dynamics. This is the acceleration which goes into the linear response formalism of Newtonian dynamics.

Now, if this velocity of a liquid is a function of time, it could be a function of time on two counts; one is because it would be a function of the position of a drop of that liquid which is in motion and therefore its position is a function of time. This is what makes the velocity and implicit function of time, but it can also be an explicit function of time, because this velocity could also change not only with the drop of the liquid which is in motion, but also from time to time it could change from season to season. For example, in the river, in the monsoon season here in India this water speeds would be quite high when the water is flooded, where does whereas in summer it could be low in fact, there could be no water at all if it is a bad summer. So, there would be no velocity to talk about it, so obviously it is a function of time it would be an explicit function of time.

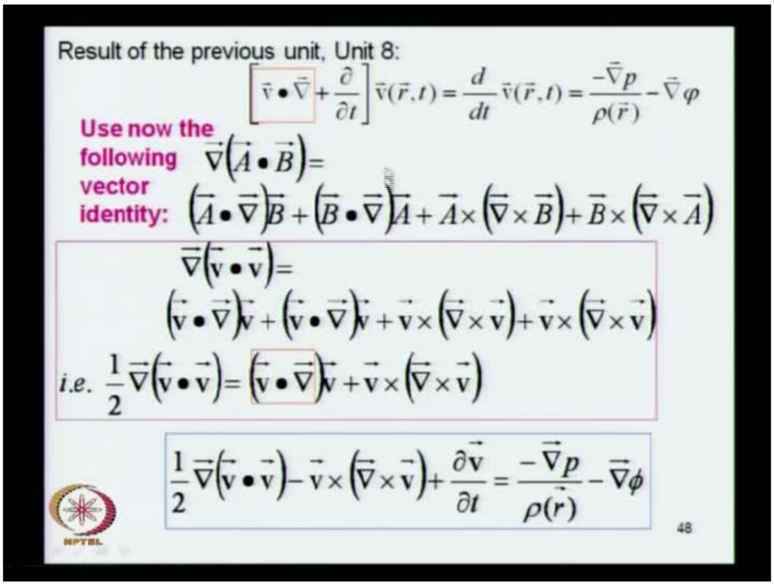
So, here you are talking about a quantity which has got both an explicit dependence on time from this parameter and an implicit dependence on time from here and this position vector can be written as a function of 3 degrees of freedom, which an a Cartesian geometry would be x y and z. And now, if you take its time derivative then you have to take it term by term use that chain rule, you first take the partial derivative of velocity with respect to x and then the derivative of x with respect to time, likewise you take the partial derivative of this velocity with respect to y del v by del y and then the derivative of y with respect to t because the drop of water itself could be in motion and then you have got the third term coming from del v by del z and then dz by dt and the last term is coming from the explicit dependence of the velocity on time.

So, here you have these one, two, three and four terms which give you the acceleration of the liquid and if you look at what is inside this loop here, if you just look at these three terms then you can write this in a rather simple form which is v dot del operating on the velocity and the 4th term is the partial derivative operator del by del t which is here operating on the velocity.

In other wards words, the differential operator d by dt is completely equivalent to the operator, which is inside this rectangular box, which is v dot del plus del by del t. So, this is known as the convective derivative. The reason it is called as convective derivative, the term convection will remind us that there is a certain flow which is involved in convection currents heat is conveyed, it is conveyed physically by the transport of the gas molecules or the liquid molecules. So that is how, the convection currents are set up. And because it is that kind of motion which is coming into the picture, it is coming from this dx by dt dy by dt dz by dt and it is because of this that this is known as the convective derivative.

So, d by dt this differential operator is completely equivalent to this v dot del by del t and now, we have got all the tools with us all the algebra, all the differential calculus with us all the vector calculus with us and all we need to do is to do a little bit of very simple manipulation of these terms and what will come out of it is the Denial Daniel Bernoulli's theorem.

(Refer Slide Time: 41:40)



So, we will use the result which we derived in unit 8 that the convective derivative was given by the ratio of this gradient of the pressure divided by density with a minus sign and the gradient of some external potential which in our case was the gravity. So, this result we have derived explicitly in unit 8 and I will just remind you of that result and use it along with another vector identity which I will not prove.

But if you determine this quantity in any coordinate system, you will find that the result follows and you already know how to take the gradient in Cartesian coordinate system, you can do it in spherical polar coordinate system, you can do it in cylindrical polar coordinate system and it does not matter.

The result is completely independent of the coordinate system and the result I want to use is this identity. That if you take the gradient of a scalar function which is expressible as a scalar product of two vector functions; so A is a vector function B is another vector function and you take their in a product or the scalar product or the dot product, these are all equivalent terms they are all synonymous in our context.

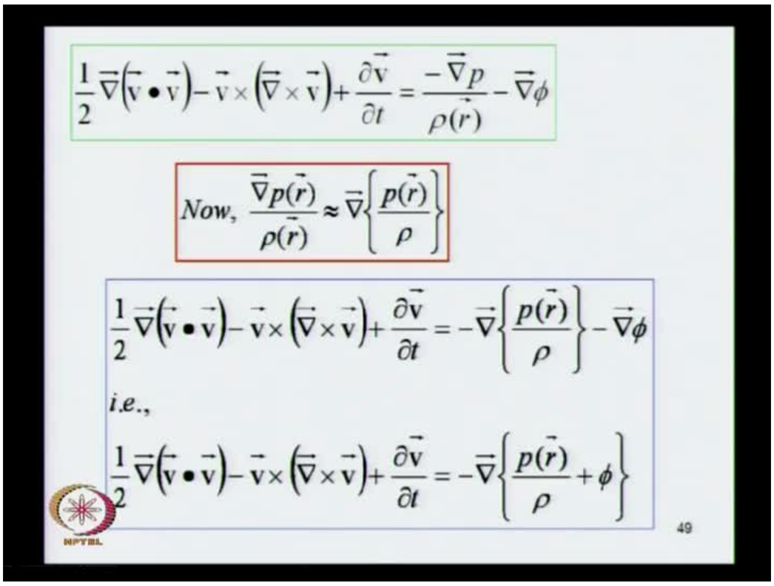
Then it is given by this identity you can write it as a sum of four terms A dot del operating on B, B dot del operating on A, A dot curl of B and B cross curl of A, this is A cross curl of B and B cross curl of A. So, these are the four terms that come out of this identity and we will use this identity to the special case when both A and B are the same vectors namely the velocity.

So, this is quite general, no matter what A function is and what the B function is. So, might as well applied to the special case when both A and B are the same, not only they are the same but they represent the velocity of the fluid because that is the quantity we are interested.

So, the left hand side of this identity will be the gradient of v dot v and this is equal to just I replace A by v, so v dot del and this B is also replaced by v likewise I replace every A and B over here by v and now, I have these two terms to be the same, because both are v dot del operating on v and here these two terms are also the same. So, if I divide both sides by half, I get half of the gradient v dot v which is equal to v dot del operating on v plus v cross curl of v. So, this is an identity that I will make use of.

And where do we want to use it? We will replace this operator v dot del over here, so v dot del operating on v is given by this term which will be half of this minus v cross curl of v. So that is what I have got over here, this is half of the gradient of v dot v minus v cross curl of v plus del v by del t which is this term and the right hand side is given by this ratio negative gradient of the pressure divided by the density minus the gradient of an external potential which we have chosen to be gravity.

(Refer Slide Time: 45:57)

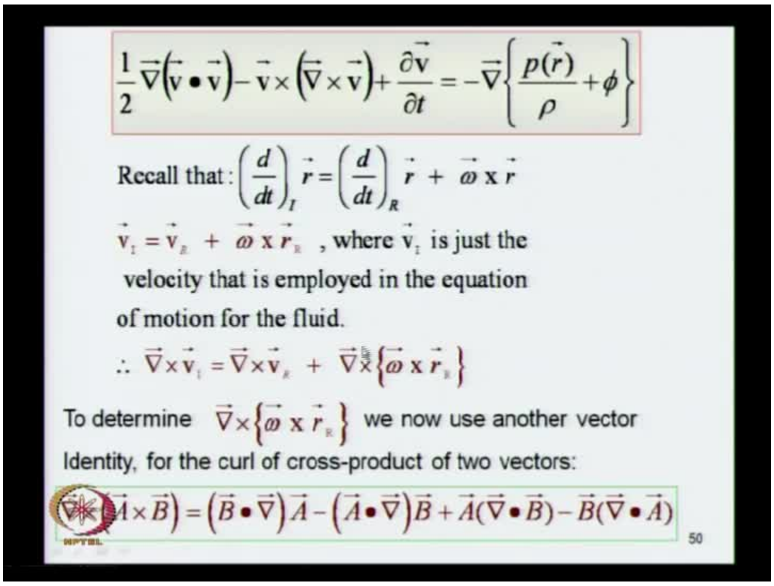


So, this result is easily derivable and we are getting closer to the Bernoulli's principle in which I have use this identity and I have used the convective derivative. And the right hand side comes from the equation of motion for the fluid, which we had established in the previous unit. So, it is the same relation which I have brought to the top of this slide, so that we do not lose any term in our analysis.

Now, as would be obvious to you, if the density of the fluid is not changing very much from point to point then this ratio of the gradient of the pressure to the density can also be written as the gradient of the pressure divided by the density. Because if the density is not changing from point to point it is like a constant. So that makes it easy to write this term that this half gradient of v dot v is here, minus v cross curl of v is here the del v by del t is here and this term del p over rho is written as gradient of p over rho with this minus sign which is brought forward from here and the last term is minus del phi which of course survives. We better carry it forward.

And these two terms on the right hand side, both are gradients of appropriate scalar functions. So, we can combine these two terms and write this collectively as the negative gradient of this term plus the scalar function phi, which is the external field, which is the gravitational potential in our case. So, all I have done is to combine these two terms.

(Refer Slide Time: 47:41)

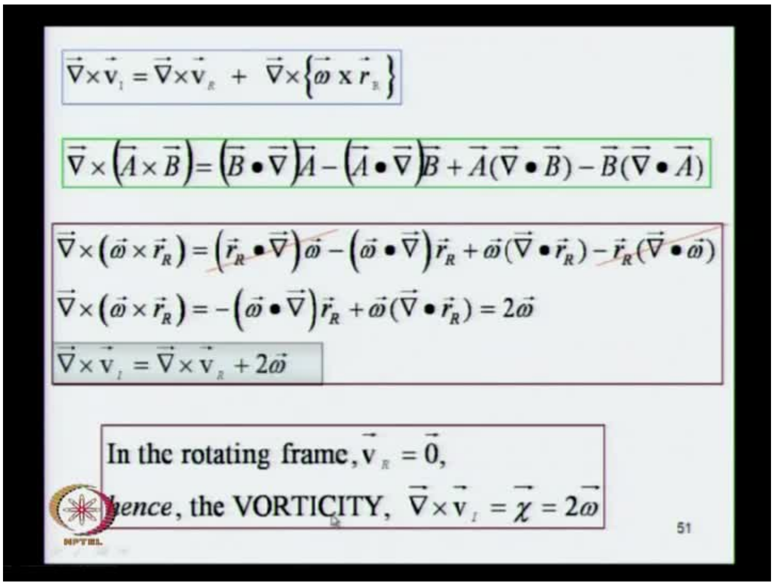


Now, let us again carry this result to the top of the next slide, so that we do not lose any term which is all which has been done on this slide. Let me quickly remind you that in the rotating frame of reference, the time derivative operator when it operates on a position vector gives you the velocity in a rotating frame of reference and to this if you add the cross product of the angular velocity of the rotating frame with the position vector, you get the corresponding velocity inertial in the frames of reference.

This is a result which we discussed in our previous class as well; we derived it at great length in unit 5. And this is a quick reminder of the fact that when you do this you got the velocity in the inertial frame of reference to be given by the velocity in the rotating frame of reference plus this omega cross r term and now, we will take this special case in which this velocity in the rotating frame of reference disappears, but before we do that let us take the curl of this vector, you have got an identity over here.

In which you need the curl of a vector, which itself is described as a cross product of two vectors. So, we should know how to take cross product of two vectors. Now, how do we do that? We make use of an identity which again I will not prove, but I have already discussed how to get the curl of a vector in any coordinate system. So, you can define the cross product of these two vectors in any coordinate system and take its curl in the corresponding coordinate system and the right hand side will be completely independent of the coordinate system, so the details are left as an exercise. The result is the following that the curl of a cross product of two vectors is given by B dot del operating on A minus A dot del operating on B plus A times the divergence of B minus B times the divergence of A.

(Refer Slide Time: 50:12)



So, this result, this identity with A replaced by omega and B replaced by the position vector in the rotating frame of reference can be used to get this result. So, let us do that. So, all we had to do is to replace A and B respectively by omega and r which is the position vector in the rotating frame of reference and when you do this substitution, you get the curl of omega cross r A is replaced by omega, B is replaced by the position vector, and then B dot del becomes r dot del A is replaced by omega. So, you get r dot del omega minus omega dot del r plus omega times the divergence of r minus r times the divergence of omega.

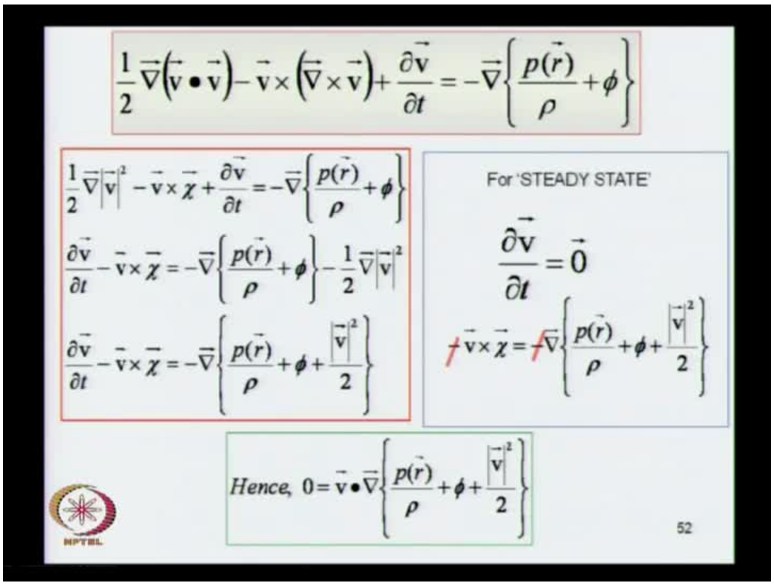
Now, this term has got the derivatives with respect to space of the angular velocity, being which being constant those derivatives will vanish. So, when you have a differential operator which is taking derivatives with respect to space of a constant vector, you will get a 0. So, this term vanishes, because you will have the derivative operators which are sitting inside this gradient operator operate on constant vector, so those derivatives will vanish. Likewise the differential operators with respect to space coordinates when they operates on a constant angular velocity this term will also vanish.

So, the first term and the 4th term cancel out and the remaining two terms the divergence of this will give you 3. So, this is 3 omega this will give you minus omega and the result is 2 omega.

So, now, you have the result that the curl of the velocity in the inertial frame of reference is given by the curl in of the velocity in the rotating frame of reference plus twice the angular velocity. And when this v R is 0, when the velocity in the rotating frame is 0 that curl of a velocity in the inertial frame of reference is equal to twice omega. So, once again we find that from a linear velocity, you get an angular velocity.

So, this again suggest to you as to why the operator curl del cross is called as a curl. Because it takes a linear property and curls, it takes the linear velocity and generates an angular velocity. We have seen this happened before, which is why the term curl is used it is like a piece of hair which you can take stretch it out, but also it curl it. So that is the reason this is called as a curl and it is when you take the curl of the velocity field it is called as a vorticity, because that is how the vertex current in liquids or are formed.

(Refer Slide Time: 53:30)



So, now, we have got all the machinery of the calculus with us, we have got this essential result which we obtained earlier. You remember how we got this special term, under the this density term in the denominator under the pressure for the case when the density is not changing from point to point. So, we have already discuss this term earlier in some details.

And now, let us simplify this because you have got the half gradient of v dot v which is nothing but the square of the modulus of the velocity right v dot v is just v square minus v of the curl of the velocity which is vorticity for which I have use the symbol chi, chi is the vorticity. So, this is minus v cross the curl of the velocity which is the vorticity then I have got the partial derivative of the velocity with respect to time and then on the right hand side I have got the negative gradient of these two terms which appears as the sum.

Now, I have got a gradient term on the right side, I also have got a gradient term on the left side. So, let me take all the gradient terms to the right, so I move this term to the right and when I do so it comes with a minus sign over here and now I combine all the gradient terms this one as well as the other one and write this as a negative gradient of sum of three terms which is p by rho plus phi and this minus half v square by 2 becomes plus v square by 2 because this minus sign I have taken as common outside.

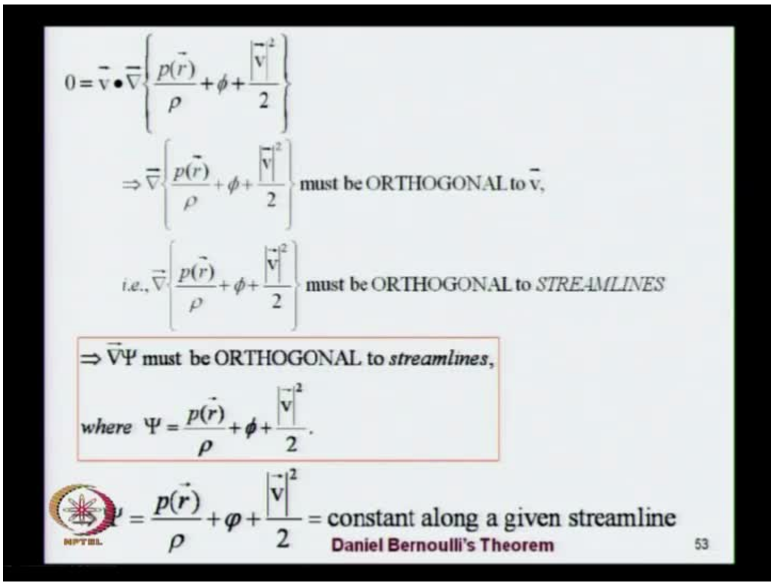
So, on the left side I have got del v by del t minus v cross chi equal to the negative gradient of this term. So, now, we can do some special cases, if we take a steady state a steady state is one in which the partial derivatives with respect to time vanish that is the signature of a steady state.

So, if you take a steady state the partial derivative of velocity with respect to time vanishes which means that minus v cross chi will be equal to the negative gradient of this sum of these three terms which appeared in this beautiful bracket.

So, you have got minus v cross chi equal to minus gradient of the sum of these three terms in the beautiful bracket, the minus sign is now completely redundant; so you just drop it, it appears on both sides of the equation.

So, you can knock it off and you got v cross the vorticity, this is the cross product of the velocity with the vorticity. And now, if you can take the dot product of this equation both sides of the equation with velocity what you get? On the right hand side, you get v dot the gradient of this term, on the left side you get v dot v cross chi, which is the scalar triple product of three vectors v, v and chi two of which are the same therefore the box product the scalar triple product goes to 0. So, on the left hand side you have a 0 and on the right hand side you have got the scalar product or the inner product or the gradient of these three terms, these three scalar functions with the velocity which is an identity. Is it always an identity? No, it is so only for steady state because we have used that condition.

(Refer Slide Time: 57:42)



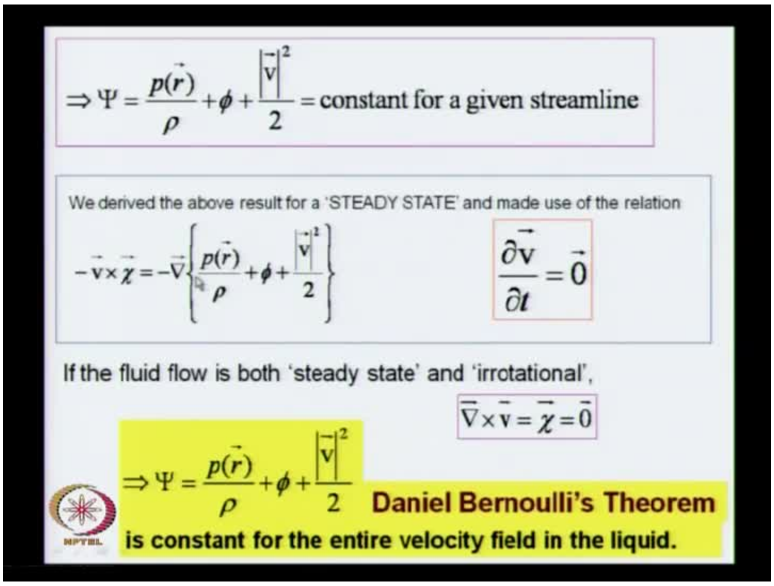
So, if and when the velocity flow is one of steady state which is when you have nice stream lines formed. So, you have got a steady state there is no turbulence of any kind, nice stream lines are formed and what you essentially get is the dot product with velocity of the gradient of these three terms vanishes and with the velocity field not being 0. And this not being a null function, you can guaranty that the gradient of this beautiful bracket is orthogonal to the velocity, because that is when the corresponding scalar product will vanishes cosine of 90 degrees will vanish,

which means that the gradient of the sum of these three functions is orthogonal to the velocity and the stream lines are along the velocity field which means that the gradient of this function is orthogonal to the streamlines and we know this idea of the gradient was coming from the idea of the directional derivative.

Directional derivative is the measure of the rate of change of a function in a given direction and this gradient is orthogonal, which means that along the streamline there is no change in that function. Because whatever change is there is necessarily orthogonal to the velocity field, it is necessarily orthogonal to the streamlines, because the streamlines are along the velocity field, when the fluid flow is a steady state flow.

So, for this case, the gradient of this quantity in the beautiful bracket the sum of these three terms I write as a function psi. It is the sum of these three terms which is written as a function of psi, our conclusion is that the gradient of psi is orthogonal to the streamlines which means that psi itself is a constant for a given streamlines. The sum of these three terms is a constant for a given streamlines.

(Refer Slide Time: 60:26)



Now, this is a part of the Bernoulli's theorem that I will give you the fuller form. This is after Denial Bernoulli as I may remind you there is a fuller form this part tells us that the function psi which is the sum of these three terms is the constant for a given streamline.

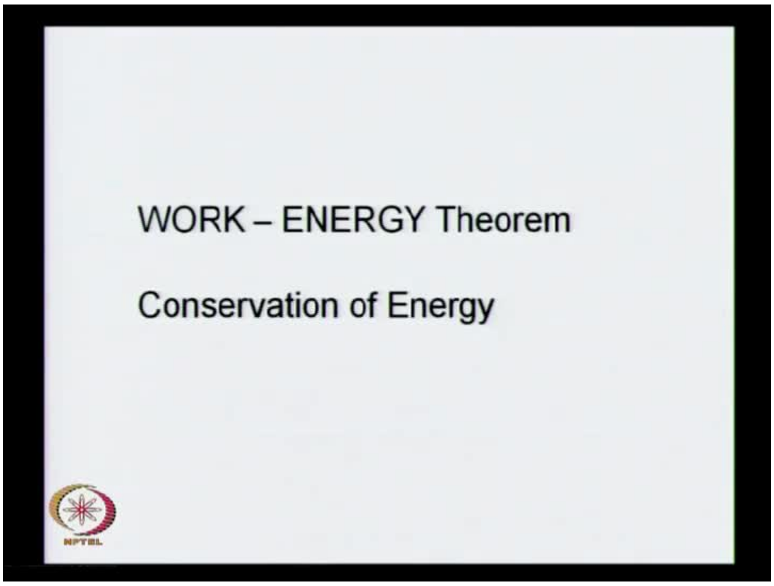
This came from the fact that we consider the fluid flow to be a steady state flow when the cross product of vorticity with the velocity was given by this expression. This was the condition that we analyzed, rest of it was simple algebra all we did was to take the scalar product with the velocity field.

Now, we take a further condition, a further consideration that if the fluid flow is steady state, so the del v by del t is equal to 0, but it is not only a steady state but also if the velocity flow velocity field is irrotational what does it mean? If the velocity field is irrotational, an irrotational vector is 1 whose curl vanishes.

So, the vorticity would vanish and if the vorticity vanishes then the left hand side vanishes identically. So, when the vorticity is equal to 0 which means that the curl of the velocity is 0, the flow is not only steady state flow, but it is also irrotational. Then the left hand side becomes identically 0, because the vorticity is now a null vector. So, the right hand side becomes a null vector; so the gradient of this function becomes 0 which means that the function itself is a constant for the entire velocity field, not just for a given streamlines. It is a constant for a given streamline, if the fluid flow is only steady state, but it may have vorticity.

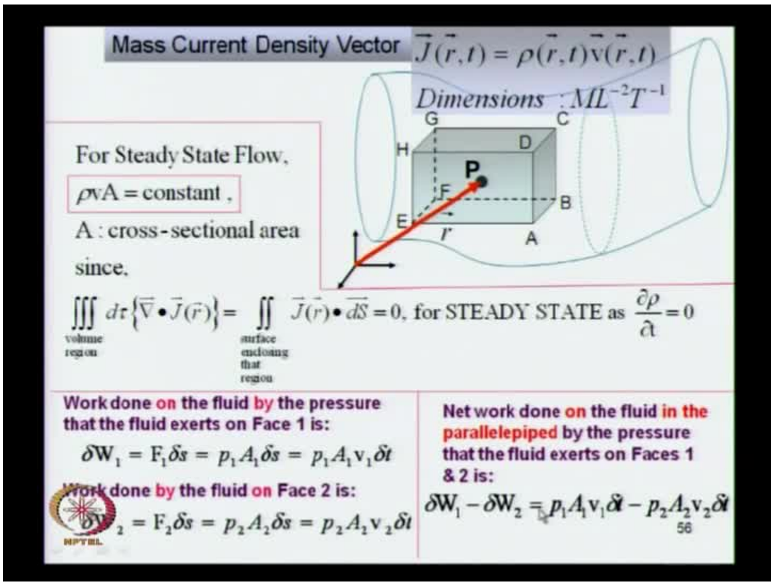
But if it is also free of vorticity, if the vorticity chi is 0, if the velocity field is irrotational then you have got the entire function psi to be constant for the entire field of the velocity not just for a given streamline.

(Refer Slide Time: 62:59)



So, this is Denial Daniel Bernoulli's theorem and we have obtained it using vector calculus and this connects to the work energy theorem in which the Bernoulli's principles is very often interpreted in high school books.

(Refer Slide Time: 63:12)



In ordinary school books this is often interpreted in terms of the work energy theorem. Because what you do is if you consider a rectangular parallelepiped in a medium through which fluid is passing and then you apply the divergence theorem that the divergence of a vector if you take the volume integral of the divergence of a vector, then it is equal to the surface integral of the vector taken over a closed path closed surface, which encloses the corresponding region.

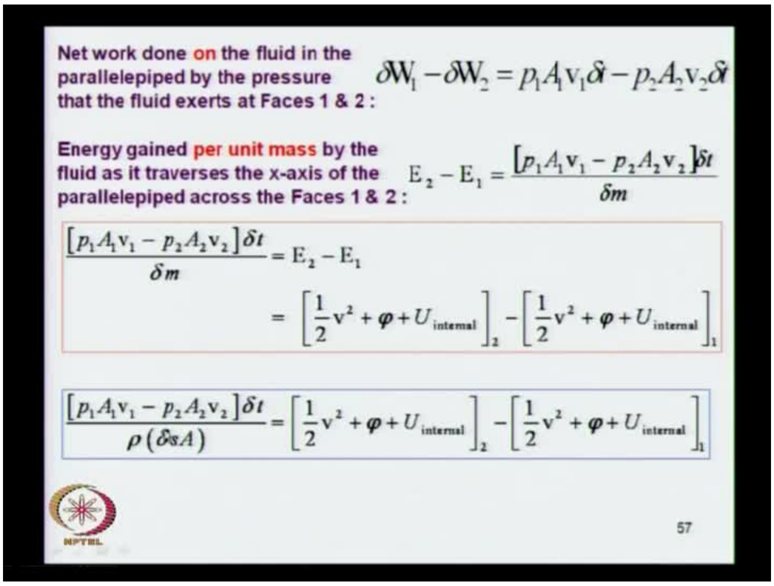
And this is really the equation of continuity that we have discussed in unit 8. We have discussed this already in unit 8 and we will use that result coming from the equation of continuity for the steady state and find the work done on the fluid by the pressure that the fluid is exerts on face one. So, there are these two faces E F G H is face one and A B C D this is the face 2.

So, the work done on the fluid by the pressure of the fluid, which exerts this pressure on the face 1 will be force times displacement that is the work done is given by force times displacement force itself is pressure times area of the face 1, times the displacement and the displacement itself is velocity multiplied by the time taken to cross that segment. So, the segment we are talking about is this linear segment from face 1 to face 2 this is the segment.

So, v 1 so it takes let us say delta t, a small time interval delta t to cross this segment. So, there is work done on the fluid by the pressure that the fluid exerts on face 1 is p 1 A 1 v 1 delta t, whereas the done by the fluid on the face 2 is given by an exactly similar consideration which is F 2 delta S which will be p 2 A 2 v 2 delta t its exactly the same consideration.

And that tells us what is the net work done on the fluid in the parallelepiped work done by what by the pressure of the fluid, which the fluid exerts on face 1 as well as 2; so that must be obtain by the difference of these two works. So, you subtract the work done, this delta W 2 from delta W 1 and this difference is given by the difference between these two quantities. So, p 1 A 1 v 1 delta t minus p 2 A 2 v 2 delta t is this difference.

(Refer Slide Time: 66:19)



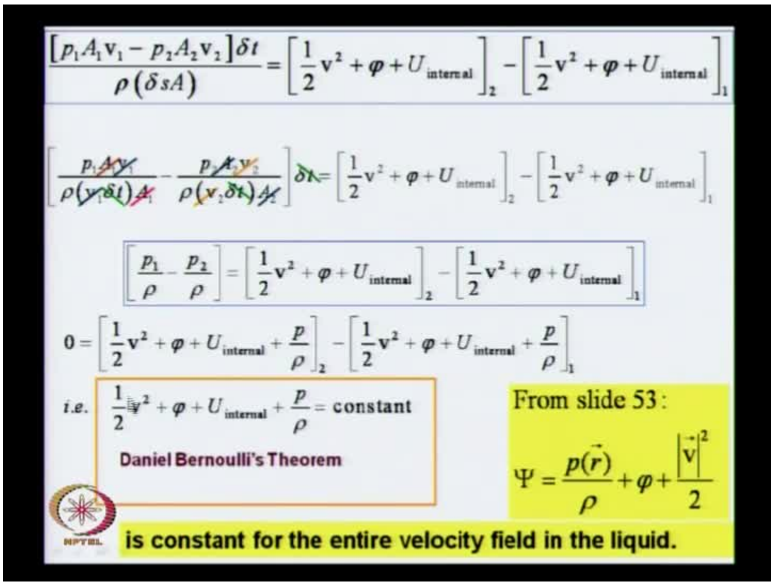
So, let us interpret this particular difference. We will now, consider that a certain amount of work done on the fluid is what we have determined, where is this work going to? It will go as the energy gained per unit mass by the fluid as it traverses that energy can neither be destroyed or create. Now that some amount of work has been done on the fluid, it has to manifest through conservation of energy as something. So, it will manifest as the energy gained per unit mass by the fluid as it traverses that particular distance. So that energy difference is that energy gained is E 2 minus E 1 and that will be this difference divided by delta m, which is the energy per unit mass.

So, all I have done is to write this expression now, in terms of the E 2 minus E 1, what is the net energy over here? This is the kinetic energy per unit mass, half v square is the kinetic energy, but we are determining the quantities per unit mass which is why the m does not appear. So, this is the kinetic energy, this is the potential energy plus there is some internal energy which the liquid has.

So, this is the total energy E on the face 2 minus the corresponding energy on the face 1. So, this is the difference E 2 minus E 1 and this is what appears as the energy gained per unit mass by the fluid as it traverses the x axis when the liquid passes from left to right from face 1 to face 2.

So, here we go we have just written that relationship. So, this left hand side p 1 A 1 v 1 minus p 2 A 2 v 2 this whole thing multiplied by delta t divided by this mass element and this is volume into density. So, the density is rho, the volume is the length segment times area. So, delta S times area is the volume and at the right hand side you have got the difference of these two energies. So, let us write this result at the top of the next slide which is here.

(Refer Slide Time: 68:57)



And now, you can cancel the common terms very easily. So, I have only separated these two terms now. So, I have got p 1 A 1 v 1 times delta t divided by this rho v 1 delta t delta S is nothing but the velocity times the time interval. This is the length delta S that is the distance which is traversed and that distance is velocity times the time interval. So, this is the velocity v 1 times the time interval and from the second term you have got from the second term over here, again this is the velocity times the time interval.

And now you can cancel some other common terms, because there is an area A 1 here there is an A 1 area over here. So, these two terms cancels then these two v 1s cancels, likewise these two A 2 elements cancels, the corresponding velocity is cancel and now this delta t cancels the delta t in the denominator over here and there is a delta t in the denominator in the here as well. So, the delta t also cancels and that simplifies the expression quite a bit. So, you have got p 1 over rho minus p 2 over rho equal to half v square plus phi plus U internal for the face 2 minus half v square plus phi plus u internal for the face 1 and what are the subscripts 1 and 2 they correspond to the two faces the face 1 and the face 2.

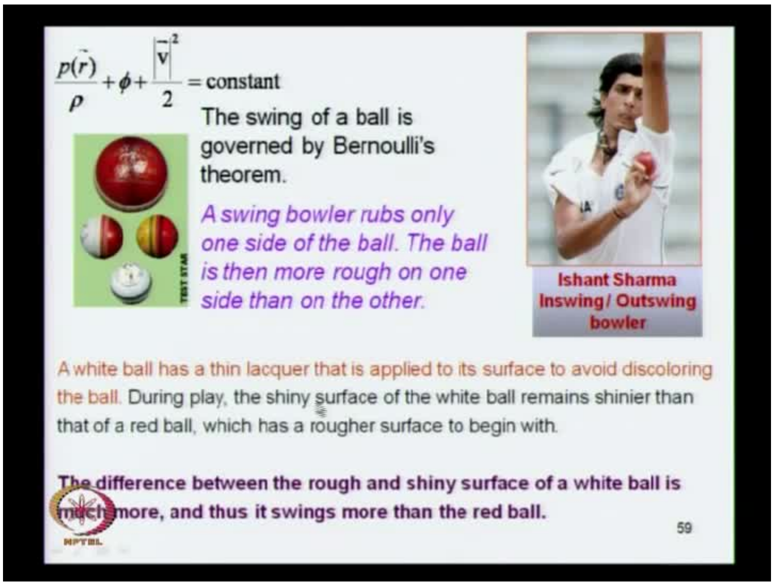
And whatever value you have, if you combine all the terms for the face 2 and here you have got the corresponding terms for the face 1. So, you move these terms p 1 by rho and p 2 by rho also to the right side, but when you do that you must change their signs. So, p 1 will go with a minus sign, so p at the face one which is p 1 goes with a minus sign but p 2 when it moves to the right has got a minus sign over here. So, when it moves to the right it goes with the plus sign, so you have got the pressure term over here on the face 2 and it has comes with a plus sign.

And what you find is that these two brackets must be equal, they have got opposite signs and therefore they must be equal and therefore they do not depend on the subscript 1 and 2. No matter what the indices are, whether you take it on face 1 or face 2 or face 3 or face 4 no matter what you do, the quantity inside this bracket must therefore be constant, because they are independent of the location on the x axis.

It is the independent; it is the location on the x axis which is indexed by the subscript 1 and 2 which is exactly the Bernoulli’s principle, which we got from vector calculus. We showed that the pressure over density plus phi plus v square by 2 which is what we have got over here, the p by rho is here, the phi is here and the kinetic energy is here, this is the kinetic energy per unit mass the only thing which is in addition over here is the internal energy.

So, this is basically a conservation of energy statement, which is how it is interpreted in high school books. And we have found that this is the constant for the entire velocity field in a liquid, then the velocity flow is a steady state flow and also when it is irrotational, but if it is only steady state flow, but if the vorticity is not 0 then we know that this is the constant only along a given streamline.

(Refer Slide Time: 73:23)



So, this is an extremely important theorem in fluid dynamics and this has got application in various kinds of things and I think what youngsters like to see is its applications in sports. And the question which you might want to ask is if there is a cricket match and you can have all kinds of cricket matches these days, you can have a 5 day cricket match a test match or you can have it over a single day a 50 overs match a 50 50, a one day test as you call it or you can have a smaller version that 20 20.

And what kind of a cricket ball would you use, would you use a red ball or would you use a red white ball or would you use one which is half red and half white or half yellow and half red there are all kinds of cricket balls which are manufactured. And you might have notice that people use different kind of cricket balls for different kind of matches and the reason of course is the fact that in a shorter version of the match it is the swing bowlers who play a big role, I am sure you would have notice this that each team tries to pack its bowling squad with seem with in swing and out swing bowlers. Because what they do is they rub the ball only on one side and then what it does is that the ball is move rough on one side than on the other so that the rougher side carries the velocity with a drag. So, there is a velocity difference on the two sides of the ball.

And if there is a velocity difference on two sides of the ball then the sum of these three quantities been constant if the velocity term is different than the pressure will have to be different. So, if the velocity is more on one side then the pressure on that side will be less, if the velocity is more on one side then the pressure on that side will be less.

So, there will be a pressure gradient across the ball and is this pressure gradient which will swing the ball when it is moving through the fluid. So, this is what the swing bowlers exploit and they prefer to use the white ball because it has got a thin lacquer that is applied to its surface to avoid discoloring and then during the play what they can do is the shiny surface you know remains shinier and the other ones starts becoming rough. So, what they are trying to do is to exaggerate this difference between the smooth surface and the shiny surface or the smooth surface. Because the more the exaggerate this difference, the more velocity difference they can generate and the more velocity difference they generate, the more pressure gradient they can generate and the more the ball would swing.

(Refer Slide Time: 76:43)

Now, you know how to swing the ball it does not mean that you will be able to swing it. So, if Ishant loses his place in the cricket team does not guarantee your chance but anyhow this is fun and there are some people this Rabindra Mehta for example, who was a graduate student somewhere in the late 80s and he studied all these dynamics of sports balls in all kinds of situations the cricket ball, the football, the soccer ball and the tennis ball and he did a lot of this survey. So, lot of this is available at his website and there are some very interesting papers on the swing of a cricket ball in flight and this is published in you know major journals like this is published in the proceedings of the royal society of London, then the effect of humidity on the swing of cricket balls in international journal of mechanical sciences and may be one of you wants to do a PhD on this because there is enough to work on.

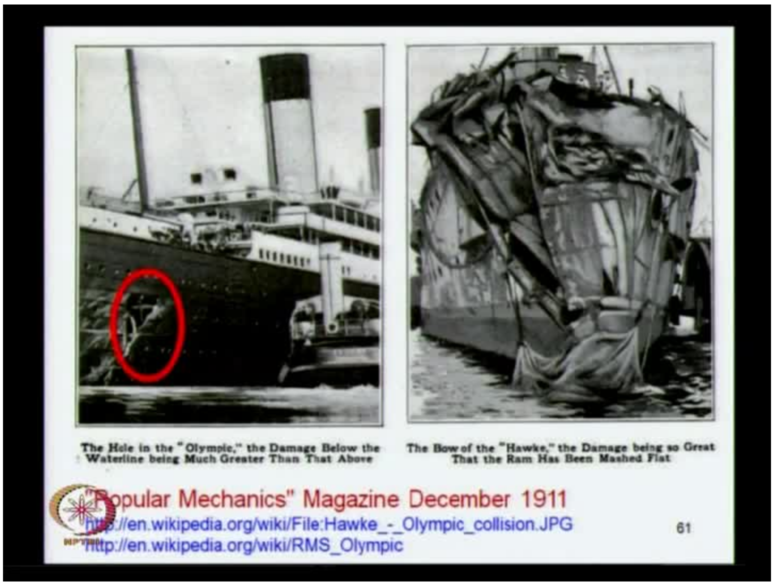
This is connected with the Magnus effect and this Rabindra Mehta if you go to his website you will find many more references. So, I let you go to that and enjoy it. So, as I mentioned, there are the number of very interesting papers on the dynamics of sports balls and cricket players, swing bowlers, tennis players, soccer players they all exploit this, when they are playing their respective games and they can make the ball which they hit.

And make it swing in a particular direction depending on how they control the movement of the ball and the Magnus effect is a very fascinating effect. It involves the drag and the actual dynamics is quite complicated, but the Bernoulli's principle in its very basic form does play a very important role in the dynamics of the ball swing.

Now, let me invite Atanu to show you a very simple experiment. So, Atanu will you just come up with those sheets of papers which you have with you and just come up to this place and what Atanu is going to do for us is, he is holding two sheets of paper and he is going to blow air in between. So, what do we expect? We think that the papers are going to fly off from each other. So, Atanu would just blow air in between, so what do we see there? We go just do it one more time very good.

So, what we see is that the paper sheets really do not fly away from each other, but instead they get rack toward each other, they get closer and it is once again because of the conservation of energy that we have seen in the Bernoulli's expression what is happening Atanu is that when the speed the velocity of the air in between the two sheets is reducing then the velocity of the air in between the sheets goes up. Because that is where you are blowing the air and if the velocity goes up the pressure drops and then the pressure outside the sheets is larger; so that generates a pressure gradient which pushes the sheets toward each other and this is the reason that the paper sheets actually get sucked in toward each other.

(Refer Slide Time: 80:47)

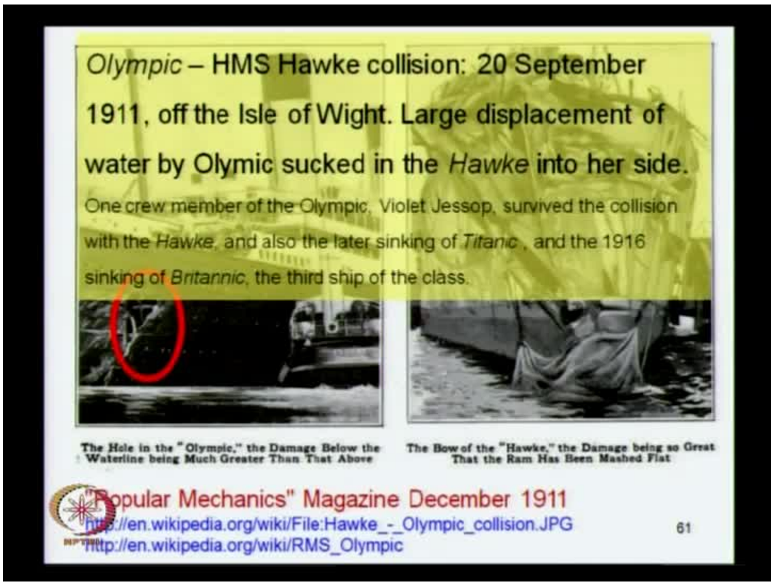


Now, this was a relatively straight forward experiment, it did not cause any damage to the pieces of paper when they came closer to each other and hit each other the paper sheets remain intact. But let me show you some damage which is there on the slide that these are the two ships, one is the ship on the on the left is the ship by name Olympic and the one on right is Hawke and what had happen is that the displacement of water when these two ships were in the sea, this was I believe in 1911 September 20th if I remember.

And what happened on this sweat full day is that the water which was displaced by these ships which were not too far from each other, but the displacement of water generated velocity currents in such a way that the velocity of the water in between the two ships increased just the way, the velocity of air between the two sheets of paper in which Atanu was blowing increased and then the pressure outside push the two ships toward each other and look at the damage that is caused.

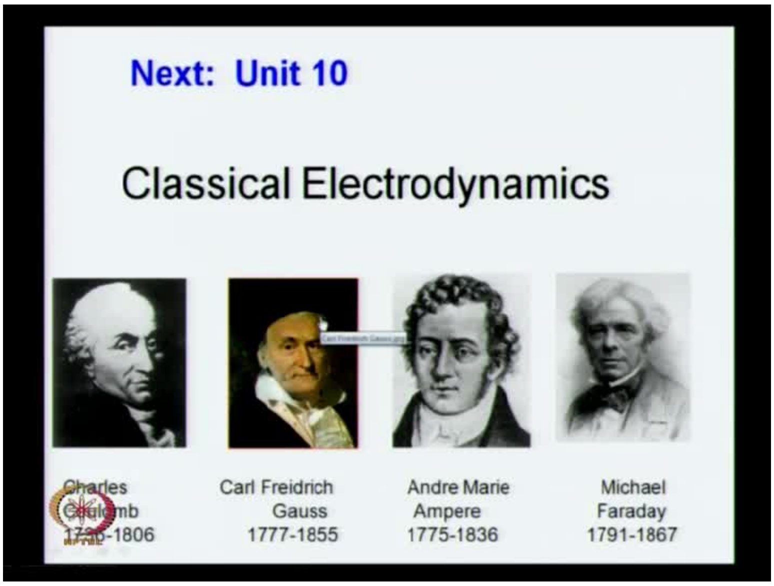
This is the ship Hawke, this is enormously damaged there is a hole over here and if you are able to read this note, the damage below the water line was actually much greater than the damage about and these ships belong to the family of ships like the titanic and the accidence actually took place around the same time.

(Refer Slide Time: 82:25)



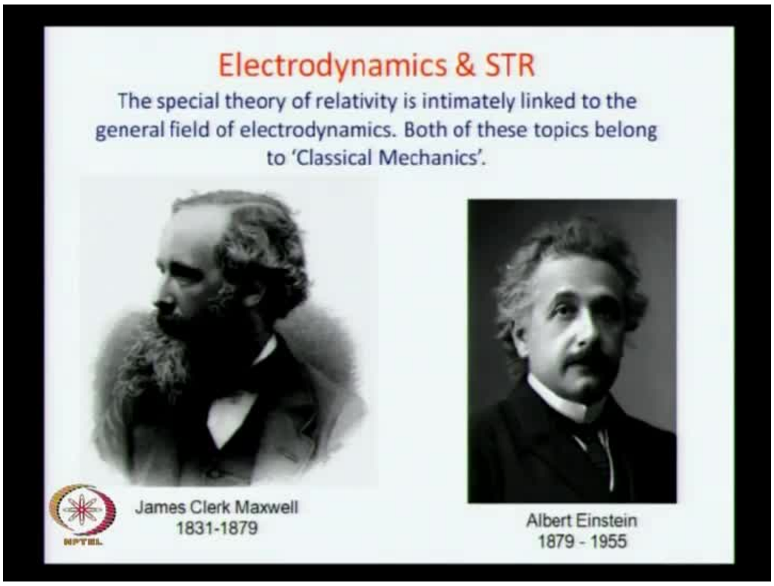
So, in fact this is the Olympic Hawke collision yes, indeed it is twentieth September nineteen eleven and this is what caused the damage of the Olympic which was sucked into the Hawke on the side and that caused a good bit of damage over. One crew member of the Olympic by name violet Jessop who survived the collision with the Hawke and he also survive the collision in which the titanic sank which some of you would have seen in that very famous movie and subsequently there was another ship which sank which was as Britannic and our man violet Jessop survived that as well. So, the Bernoulli's principle appears in some very fascinating stories in voyagers through the sea so here is the damage that you can see.

(Refer Slide Time: 83:26)



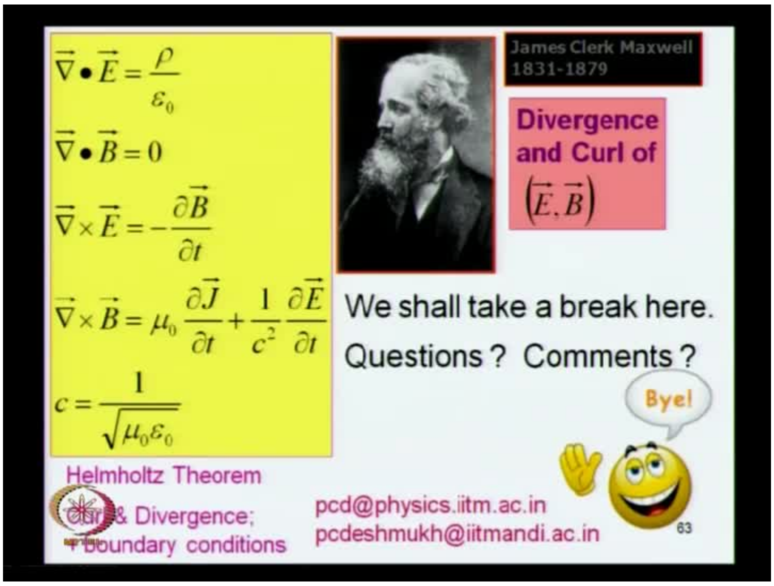
And in the next unit we will apply vector calculus to classical electrodynamics, which is an essential part of classical mechanics. So, classical mechanics distinguishes itself from quantum mechanics, but in the entire domain of classical mechanics when you applied to electromagnetic phenomena as well you must invoke the special theory of relativity, because the special theory of relativity is intimately connected with classical electrodynamics

(Refer Slide Time: 84:12)



So, this will be the subject of our discussion in the next unit, but at this point I guess we could take a little break. So, the next unit we will deal with electrodynamics and the special theory of relativity, the electrodynamics of course comes from James Clerk Maxwell and the special theory of relativity from Albert Einstein.

(Refer Slide Time: 84:30)



So, at this point I will ask for a few questions, so that you can ask some questions and then we will of course study the Maxwell's equation. You notice that you have got the divergence in the curl and as I mentioned through the Helmholtz theorem, you do need both of them to specify a vector field. So to specify the electromagnetic field you need both the divergence of the vector field E and the curl of the vector field E and also the same for the magnetic fields.

So, any questions, comments?