Select/Special Topics in Classical Mechanics

Prof. P. C. Deshmukh

Department of Physics

Indian Institute of Technology, Madras

Module No. # 11

Lecture No. # 36

Chaotic Dynamical System (ii)

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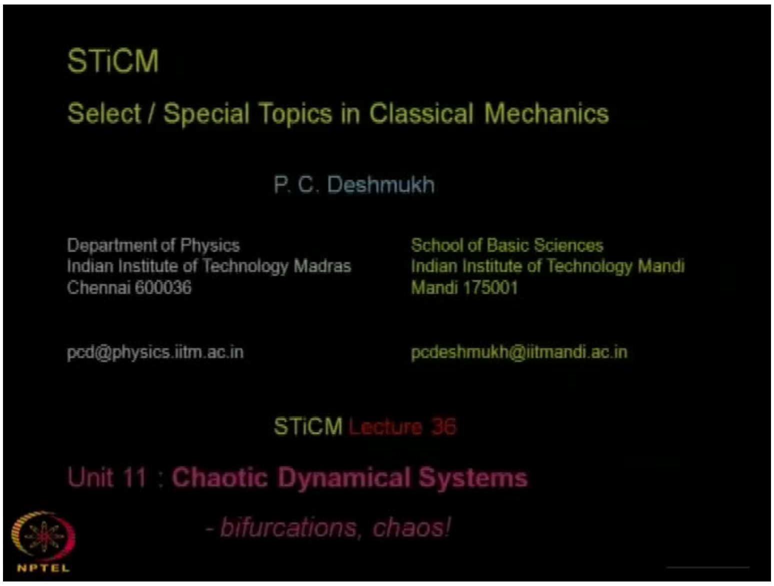

Greetings, we will continue our discussion on chaotic dynamical systems. And in this class, we will meet new ideas called bifurcations and we will discuss some of these things.

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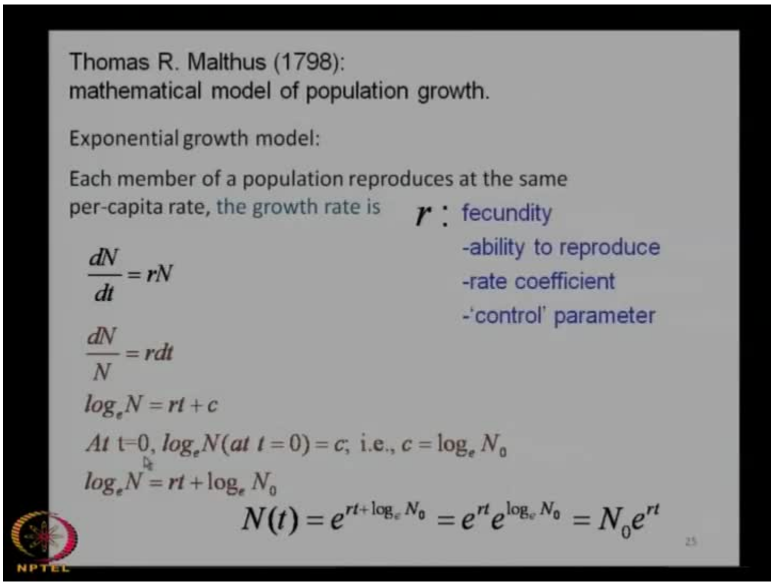

So, we were talking about our ability to make predictions, studying temporal evolution of whatever be the system, it could be the physical state of a system; it could be the physical state of any object whose dynamics is predictable in the Newtonian or Hamiltonian sense; it could be predicting how much money you will get after, if you were to put so much money in a bank at a certain rate; it could be predicting what will be the population of a certain society after a certain number of years. And the biological model is an interesting one; it is one, which is discussed quite extensively in Chaos theory. So, I will introduce you to that.

This was developed by Thomas Malthus in the end of the eighteenth century and this was a model on population growth and this is a well-known model, means, we physicist know this very well, because it proposes that the rate at which the population changes, which is dN by dt is proportional to the population itself at that instant of time. So, this is the kind of phenomenon that we see in a large number of physical situations all the time.

So, there is a growth rate, which is symbolized by the letter r in this equation and this is sometimes called as the fecundity; it indicates an ability to reproduce; it is a control parameter. And this particular model, which has applications in various different branches in physics, it could be the rate at which a capacitor gets discharged, for example, or radioactivity, and you know different physical situations at the rate at which water flows out of a container, if it had a hole at the bottom. And if you do elementary math with this, what you get is this relationship, and on which essentially gives you the well-known exponential law, which is what you find in a large number of physical situations, whether it is the rate at which a capacitor gets discharged. So, many examples are there in physics. So, Nt the population at a time t is given by N 0 times e to the rt, where r is this control parameter. Now, this is the solution to the Malthus problem of population growth, it is an exponential law that you get out of it, which is a well-known function.

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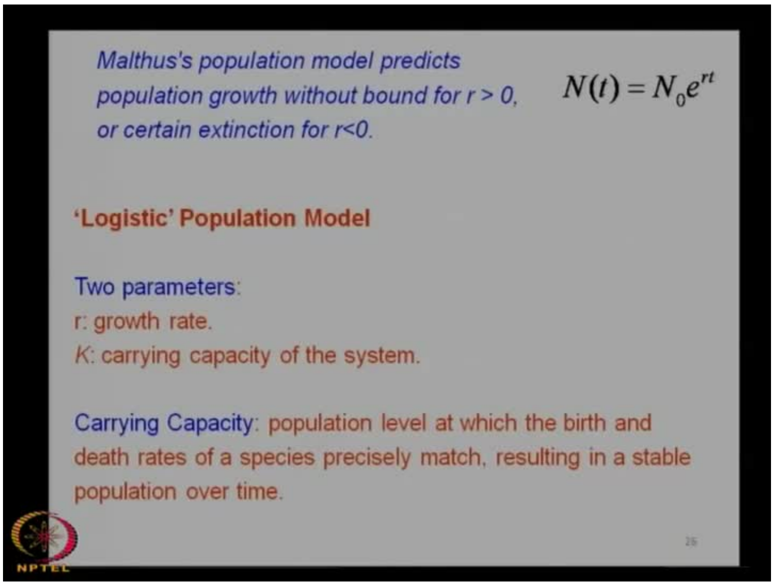

Now, what this population model tells us is that, if this coefficient r is less than 0, then the population will definitely become extinct, it is guaranteed; if it is greater than 0, then the population will grow without any bound. Now, this is guaranteed and this is the essence of the Malthus population model.

Now, this model can be improvised, because you do know that a population cannot really grow without any bounds, because there will be you know as the population grows, then you run short of water, you are not able to maintain hygiene in the society, there are diseases, there are various other things. So, there is some feedback, some control parameter and due to some other reason, there will be some check on the population growth, it will happen automatically.

So, there is another model, which is an improvement on the Malthus model. Malthus model has a single parameter, which is the parameter r; you introduce another parameter, which is labeled as K in our discussion, it is called as a carrying capacity of the system, because the system cannot continue to grow without any bounds; so there is a certain carrying capacity, which will limit the growth. And what it tells us is that, at a certain you know depending on various circumstances it does not have to be a single parameter, but it is symbolized by a single parameter in this; so, this is the second parameter. At a certain situation, the death rate and the rate at which the population is increasing the birth rate will match and there will be a balance; so that you do not have to worry about uncontrolled population growth. So, this is the two parameter model.

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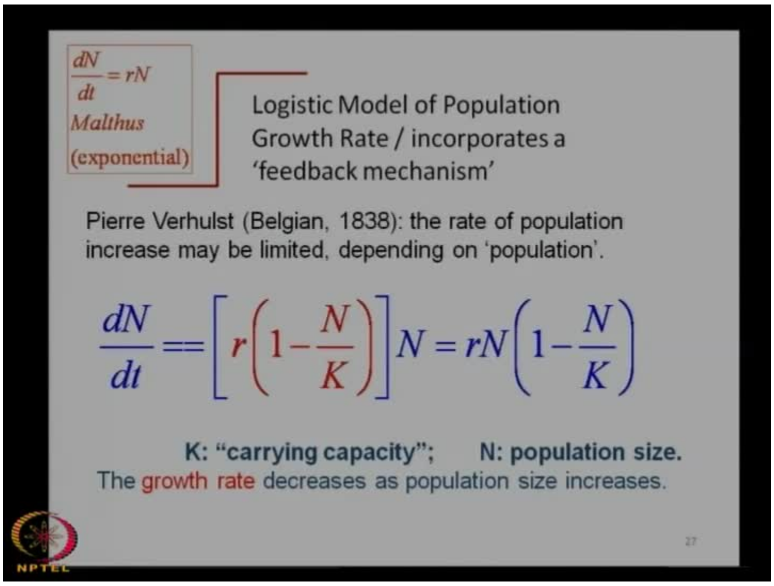

The second parameter is K and you can introduce this and what this will do, is to provide some sort of a limit on the rate at which a population would grow if the control parameter r were positive. So, this model is called as the logistic model of population; it incorporates a feedback mechanism. This was proposed by a Belgian biologist Verhulst, and he proposed that the rate of population increase would be limited and this would also depend on how much population there is, because the more there are the people, the more would it be likely that there are diseases. So, this will also depend on the population in the certain way.

So, you set up this equation; so, instead of just r times N as you have in the Malthus model, this is dN by dt the rate at which N changes with time is proportional to N; this is r times N; this is the exponential model that we considered. This r is modulated by a factor, which is 1 minus N over K. So that, when N is equal to K, you get 1 minus 1 and dN by dt will vanish; so the population will stabilize. So, notice that the growth rate will be checked and this feedback term also has this N over this has got the N over K term, where N is the population size itself and the presence of this term makes this equation non-linear. Because this term, the first term is linear in N; the second term will have N square; so, it becomes quadratic in N. So, you introduce non-linearity in your system of equations.

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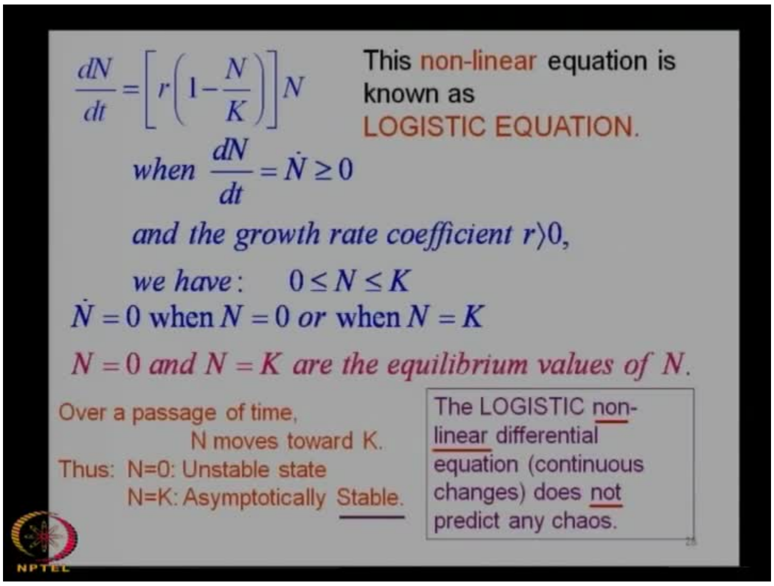

So, this is called as a logistic equation; it is a non-linear equation as you can see. Notice however, that dN by dt, which is N dot when it is greater than 0, when the growth rate coefficient is r, is this fecundity coefficient is greater than 0; otherwise, N will be less than or equal to K, because when N becomes equal to K, dN by dt stops changing. Because when N is equal to K, 1 minus N over K vanishes and N will not change any further with time. Therefore, N equal to 0 and N equal to K are equilibrium values of N; they will remain stable. These are equilibrium points with the difference that N equal to 0 is an unstable state, because from there the population will keep growing because the rate coefficient, growth rate coefficient, the fecundity coefficient r is greater than 0; so it will grow. So, N equal to 0 is an unstable point, but N equal to K is asymptotically stable; there is no Chaos, there is nothing unpredictable; you are actually led to stability, because N equal to K is an asymptotically stable point; at N equal to K, dN by dt becomes 0.

So, this is a non-linear equation. The point I want to emphasize here is that the signature of Chaos is not nonlinearity that whenever you have nonlinearity it does not automatically guarantee that there will be Chaos; there may be, there may not be. In this case, you see that there is no Chaos. So, this does not predict any Chaos; it is a non-linear equation; this is known as the logistic equation. It is a differential equation and differential equation of course assumes that the function is continuously differentiable that the function is analytical. So, the function has to be continuous and only then you can define the derivative of the function that is a necessary condition for a function to be differentiated; the function must be continuous, otherwise you cannot get the derivative.

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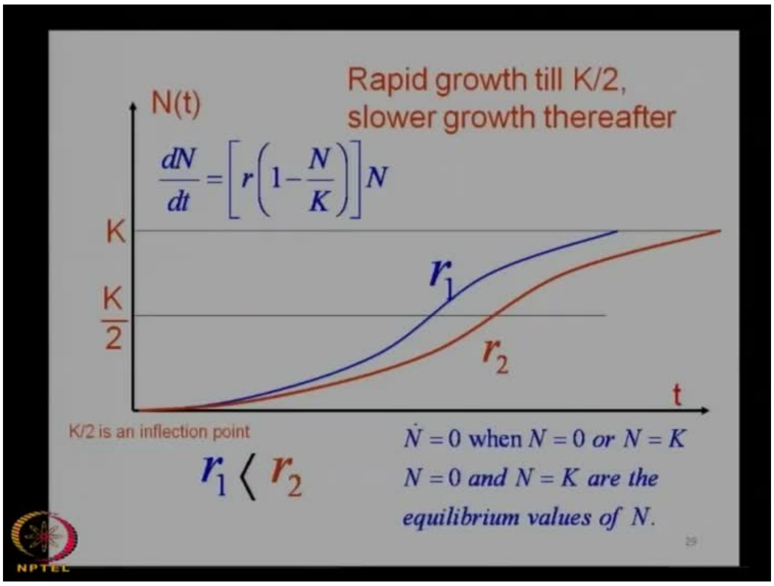

So, if you plot the function N as the function of time, so N t versus time, then there will be rapid growth till you get to K by 2 and then it will sort of stabilize, at N when N becomes equal to K. This graph is a little bit unrealistic, because this is this should become asymptotically horizontal; so that its slope must vanish, but that is not very easy to show in this diagram, but you know what I mean hopefully. So, this is you have to understand that this growth rate will slow down and it will actually asymptotically become horizontal so that dN by dt will actually vanish.

Now, if you have a different fecundity coefficient r 2, which is larger than r 1, then you will have a different curve, but the general form of the curve will remain the same and N equal to 0 and N equal to K are the equilibrium values; N equal to 0 being unstable and N equal to K being stable.

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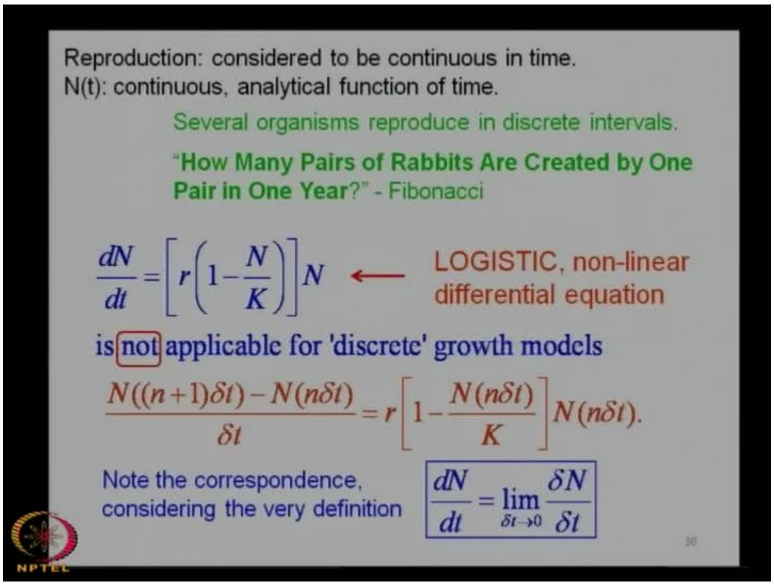

The assumption here is that, N is a function of t is the continuous function and therefore, a derivative can be constructed, which is what the logistic equation is about. Now, this is not always have biology functions, remember the problem, which Fibonacci addressed- how many pairs of rabbits are created by one pair in one year. And the biology the population was growing not continuously, but from month to month there was a periodicity. So, the shortest time interval that was of concern to us in that problem was 1 month. When you take a derivative dN by dt, you derive delta; you divide delta N by delta t and take the limit delta t going to 0. Now, that is not the limit, which is applicable in a problem like how many pairs of rabbits are created by one pair of rabbits in one year. So, biology operates in a different way and certain organisms they reproduce in discrete intervals, rather than, continuous intervals. So, you cannot use the logistic differential equation; you have to change it.

So, this logistic equation which is a 2 parameter equation - if 1 parameter is r, the other is this carrying capacity K. This is not applicable when you deal with discrete growths, what you need to do is to construct a relationship for discrete steps. And now, we consider a time interval delta t, but not take the limit delta t going to 0, delta t will remain a finite interval; it will be the discrete time interval, which in the case of the rabbits that we talked about was one month. So, delta t is that particular time interval and you construct the equation, which is very similar to the logistic equation, but it is a discrete growth model. So, you take the difference, when you take the left hand side it is delta N, which is the difference at the n plus oneth step from the nth step. So, the numerator is the difference of N at n plus oneth step minus the N at nth step divided by delta t, but the limit delta t going to 0 is not applicable you have to deal with this ratio for what it is and delta t will have a certain size. The right hand side has got the same structure as you have in the logistic differential equation, which is a non-linear differential equation; it has exactly the same form. So you have got the fecundity coefficient r and then, you have this 1 minus n over K term multiplied by this term which is what makes the equation non-linear. So, this is also a non-linear equation, but unlike the logistic equation this is the difference equation, rather than, a differential equation. So, you are now dealing with a difference equation not a differential.

Sir what does that small n stands for?

Small n is the number of steps. So, when you go from 1 generation to the next generation. So, n in the case of our we talked about the discrete steps in which the rabbits multiply they their population grows. So, when you go from the first month to the second month to the third month to the fourth month and see how the populations grows, and watch the number of pairs of rabbits. So, n was the generation index; the first generation, the second generation, the third generation and it is a discrete counter; it is an integer. It grows in steps and it is a generation index so that is the small n. The capital N is the actual size of the population after small number n small n number of that time interval. So, you can then ask what will be the population after two months, after three months and so on.

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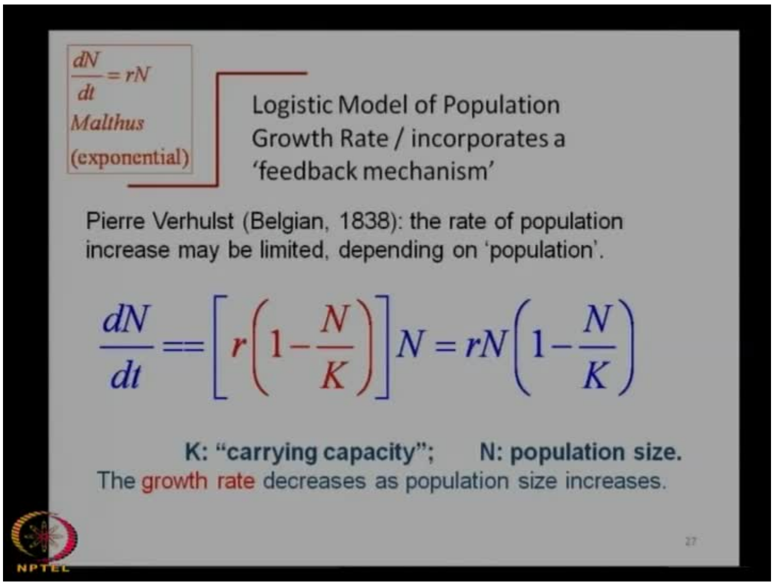

So, this is a difference equation. So, Verhulst model, which was an improvement over the Malthus exponential growth model, which took into account the feedback mechanism and included the concept of carrying capacity. This is a differential equation; a non-linear differential equation.

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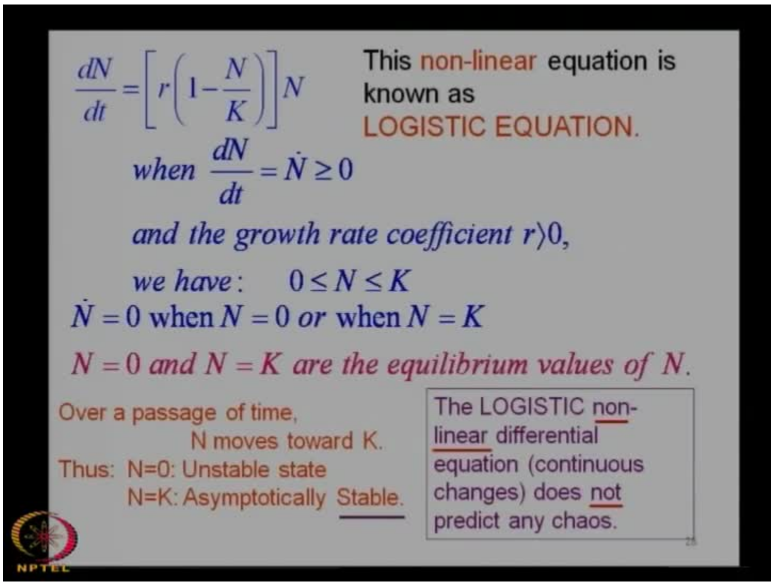

But what we are now dealing with is the fact that, whereas this logistic equation did not lead to any Chaos in fact, it led to stability for N equal to 0 and N equal to K were the stable points for this particular model for the logistic non-linear differential equation.

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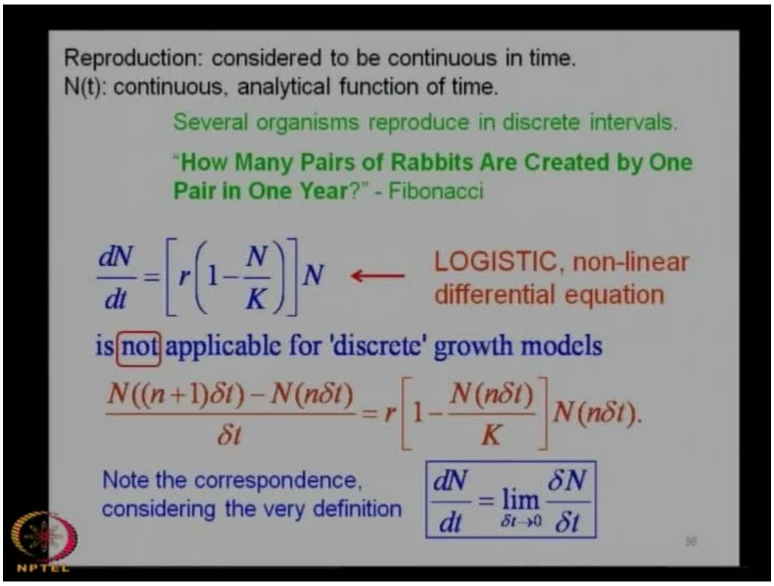

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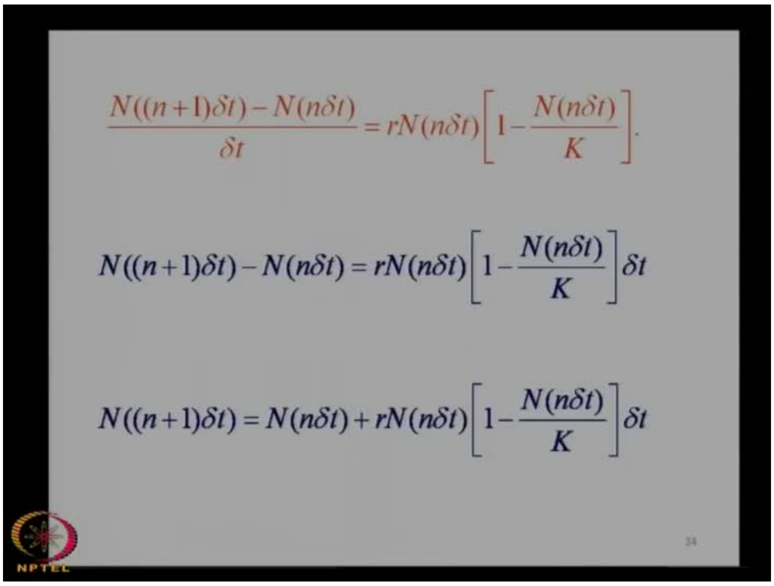

For the case of the discrete growth model, in this case we are going to have we will not have a differential equation, but a difference equation and this difference equation you can write in some equivalent ways. So, this is how we wrote it on the last slide. You can multiply both sides by delta t and you can bring this term to the right. So, you can write the population at n plus oneth generation in terms of what it was at the previous generation and it will be the that number plus that number times the fecundity coefficient plus this feedback mechanism times the time interval. So, that is the statement of this equation.

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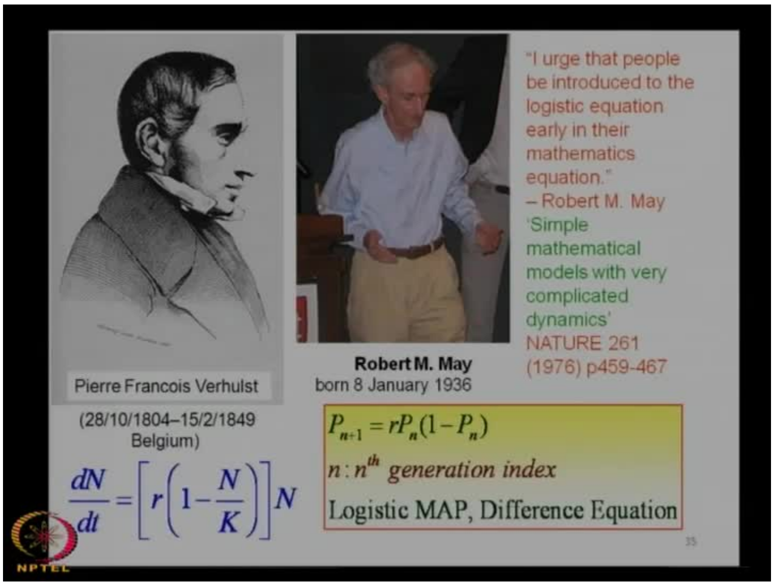

Now, this brings us to Robert May and he must be given a huge amount of credit for drawing our attention to these discrete equations - discrete step equations. So, Verhlust model made use of this differential equation; Robert May emphasized these discrete difference equations and a classic example of this is, this relationship that P n plus 1 is r times P n times 1 minus P n. So, this summarizes the essential idea of having this difference equation. So, this is the essential idea in the difference equation and this is called as a logistic map, it is a difference equation. So, as opposed to a differential equation, this is the difference equation and I will like to quote from Robert May's article in nature that he suggest that "I urge that people be introduced to the logistic equation early in their mathematics equations" and he wrote this article that on simple mathematical models with very complicated dynamics. So, notice that the equations we are dealing with are not at all complicated; they are very simple differential equations. And now, we have considered discrete step population growth and we are dealing with a difference equation. It has got exactly the same structure as the differential equation we dealt with; with the difference that, this is the difference it is called as the logistic map, rather than, a differential equation, it is a map.

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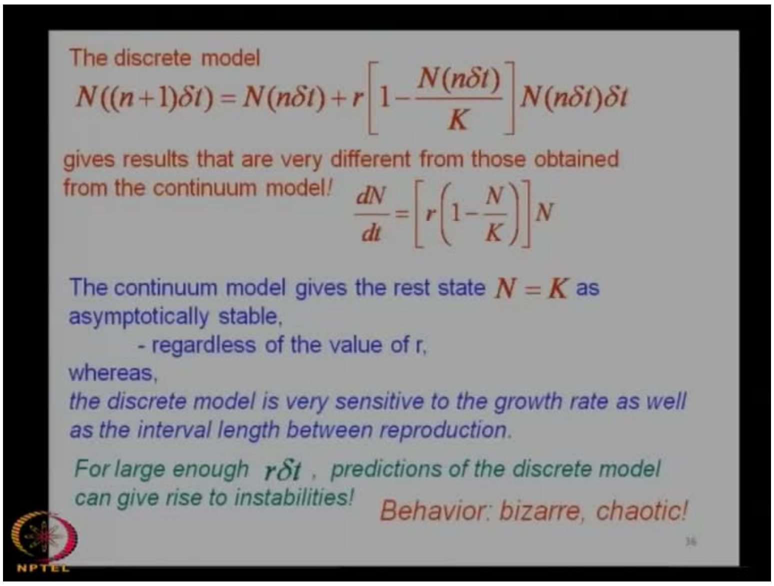

It is a difference equation and what it does is, it produces results which are very different compare to those that you get from the continuum model; the results are completely unpredictable; in some cases, you get very bizarre behavior. So, the continuum model gave us stability, N equal to K was a stable point. But the discrete model is extremely sensitive to the growth rate as well as to the initial conditions and to the interval and so on, and it gives rise to instabilities, it leads to Chaos; it comes from this difference equation. So, let us see how that works.

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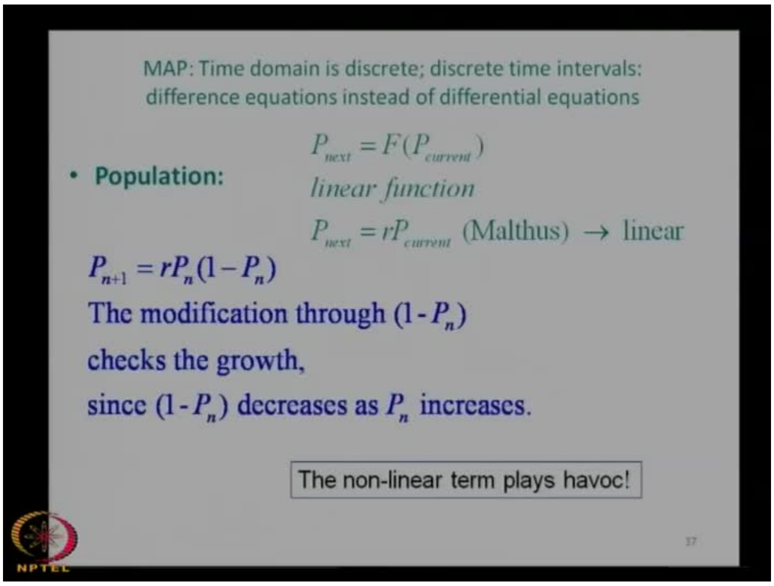

So, we consider this particular map. The important idea over here is that when you consider a map, the time domain is discrete; it is not continuous. You deal with difference equations; you do not have differential equations, you deal with difference equations; you do not take the time interval going to 0.

So, now, we do know that the population of the next generation will be some function of the population of the current generation. You can construct a linear relationship just the way you had in Malthus model, but you can also incorporate the idea of the feedback mechanism that you had in the Verhlurst model and have a difference equation of this kind, which will provide a check on the population growth. So, this is the difference equation, we will study and we will see what consequences we are led to.

There is of course a non-linear term, because the first term is r times P n, the second term is r into P n multiplied by this P n. So, you get a P n square, you get a quadratic term; you get a non-linear term and this is what leads to Chaos; it did not the non-linear term did not lead to Chaos in the differential equation, but when you deal with the difference equation, you are led to chaos.

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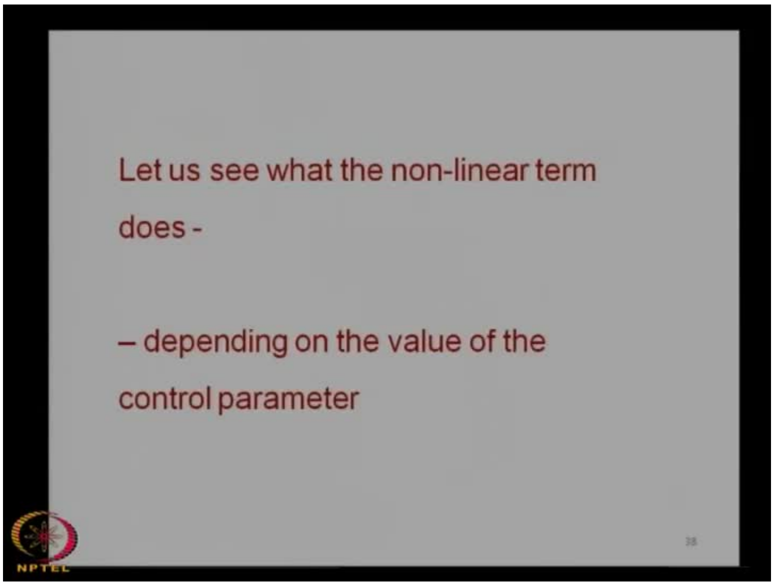

And this of course, does not always happen, it depends on certain situations; it depends on what is the value of the control parameter. The control parameter in this case we are talking about is r. So, depending on certain values of the control parameter, you are led to Chaos not necessarily, because if the value of the control parameter is something else you may not be led to Chaos. So, it depends on what the value of the control parameters, which is why sometimes it is called as deterministic chaos.

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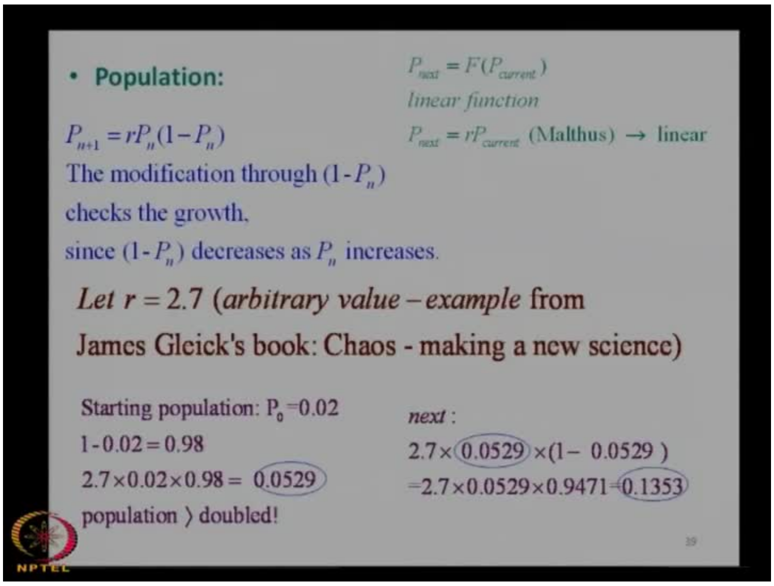

So, let us study this. We will take a particular case and this example is from Gleick's book; it is a wonderful book and I strongly urge all of you to read it; you will enjoy it. We consider a case in which r is equal to 2.7, and we consider a starting population P 0 equal to 0.02. Now, we have everything that we need; we have the starting population P 0 equal to 0.02 and we have got the value of r. So, we can use this equation and predict what the population will be after n number of generations, can we? So, let us do that.

So, let us begin at the zeroth population, which is 0.02; here, you need to determine this 1 minus P n. So, we find 1 minus 0.02, which is 0.098. So, you can do this arithmetic in your mind or use calculators or whatever you need, but you have to do this arithmetic with me as I you know, as we discuss this further, because these numbers are important; these numbers are absolutely important. And then, you multiply 1 minus P n by P n and then multiplied by r. So, that is what we do over here. So, r is 2.7, P 0 was 0.02 1 minus 0.02 is 0.98, you multiply these three together and you get 0.0529 and you find that the population has doubled in the first generation.

Now, what is going to happen in the next time? So, there is some discrete time interval. So, you go to the next generation again, you do the same thing, 2.7 times the previous generation which is 0.0529 times 1 minus the previous generation. So, do this arithmetic in your mind, use your calculators and maybe your brains are faster than the calculator, but you must get this final number, which is 0.1353 and this number is important; it is going to be a play an important role in our discussion. So, now, you know what we are doing.

We started with; remember that title of May's work. I will like to go back to just to show you quickly over here, simple mathematical models with complicated dynamics. The mathematical equation we are dealing with is a very simple one; nothing unusual about it, but see what kind of complex dynamics results out of it. So, we have got this number after the second generation. Now, do it for the third; do the same thing exactly, 2.7 times the previous generation times 1 minus the previous generation done?

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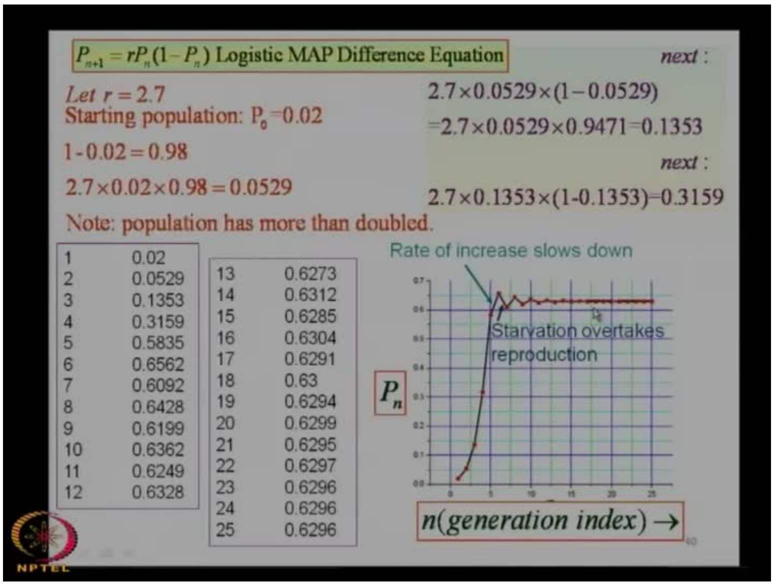

Now, you get 0.3159. So, you began with 0.02; then you got 0.0529; you do it the next time; you get 0.3159. Now, do it for the next; are you quicker than me. So, do it for the next and then for the next and then for the next and if you have done it all, these are the numbers you are going to get.

So, after the class is over please sit down with your calculators or do it in your mind, but workout this arithmetic in detail; step by step for a number of generations and these are the numbers you will get. So, you do not even have to write them down, because you are going to generate these numbers by doing this arithmetic on your own. So, do not even bother about writing down these numbers. You have to get this you cannot get anything else; it is just simple arithmetic. Now, look at these numbers from 0.02, we got to 0.0529, then we got to 0.1353 we saw how this was developing. Now, you see it goes to 0.3159 watch these numbers very carefully, then it goes to 5835, 0.6562, 0.06.

Do you see that the difference is shrinking? We started out by doubling now, it is not doubling anymore; subsequent numbers are very close to each other. So, 0.6562, 0.6092 0.6428, 6199, 0.6362 they remain close; you go further 13, 14, 15 and so on and you find that they remain close to 0.6. None of them depart from 0.6 and if you go to the next 4 places of decimal, they sort of converge to 0.6296 over here by the twenty fifth generations.

Now, if you plot it, P n versus the generation index, this is how the graph will look. That it increases and then the rate of increase slows down, then you also have a reversal. The slope actually becomes changes the sign; instead of a positive slope, you get a negative slope, that is when starvation overtakes reproduction, but then again it shoots up right and then it sort of oscillates a little bit, but then settles down to this 0.6296. So, if you see on this graph this sort of stabilizers, this is 0.6; this is 0.7 and this is about 0.6296 or 0.63 there about and that is where this graph has settle down. There is no Chaos; it is predictability.

So, as the generation increases you know where it is going to stabilize. Now, this is where it is important to keep track of the value of the control parameter. This is the result we got for a control parameter when r was equal to 2.7. What if r is something else? Will we still get predictability? We do not know; we have to check it out. So, let us do it for some other value of r.

So, again I will like to use this idea of an attractor, because when it stabilizes, here we are talking about populations; we are certainly not talking about a point in phase space or something, but I can still use the idea that where it stabilizes, you are led to an attractor. So, over a passage of time, the system which in this case is the size of the population it settles down to an attractor. You get the idea? So, I am not defining the attractor as yet, but I am letting you develop the meaning of an attractor in your minds. It will become clearer when we discuss this further; where I hope that the idea is already taking some shape in your mind.

So, an attractor is a region in the configuration space or a phase space that is invariant under time evolution and attracts nearby configuration. So, if you look at this one - this figure here, this number 0.6296; this size of the population in a certain sense is an attractor. These values around it, which was 0.6291 or 0.6304 over here or 0.6295 these are some other values in the vicinity of this attractor and these values which are in the vicinity of that attractor, they all finally converge to that attractor and what converges to that attractor is from a range of values, which you call as the basin of the attractor. So, that is a meaning of a basin of an attractor.

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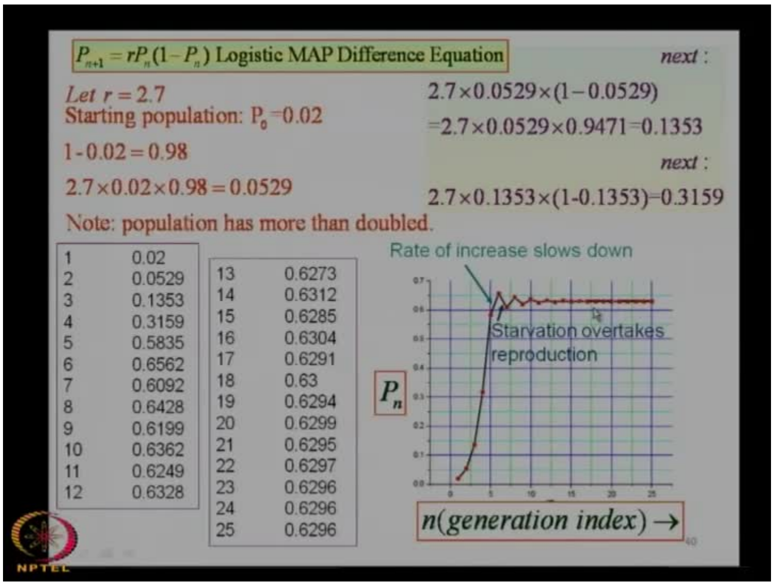

We are not talking about a phase space as such, but this idea can also be used for a phase space. Because if you described a system - a mechanical system - by a point in the phase space and track its system evolution by this in the phase space, then where ever it finally will reach, is what you will call as an attractor. And the principle of least action tells you that if you considered the variation in any neighborhood, it will vanish; the action integral will vanish; that change in action will vanish, integral l d t will be an extremum.

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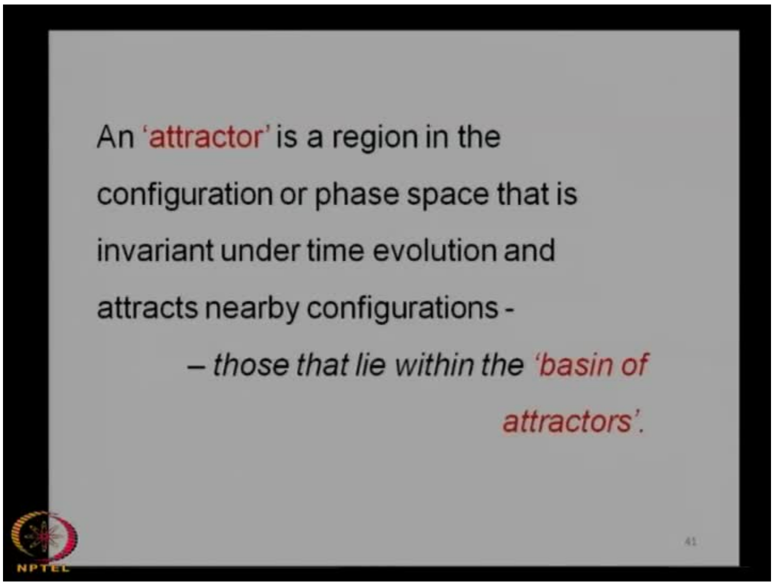

So, a change in this action will vanish, which means that it will attract all neighboring paths to that is a principle of extremum. So that is idea of an attractor. You can use it in a phase space, you can use it over here in the case of a population dynamics and you can use it in some other abstract mathematical sense as well. That any system which is changing with a certain passage of time, if it settles down to some value, then what it settles down to is what you can call as an attractor, and it will attract neighboring values which are in the basin of the attractor. So, that is the meaning of the word basin of attractor.

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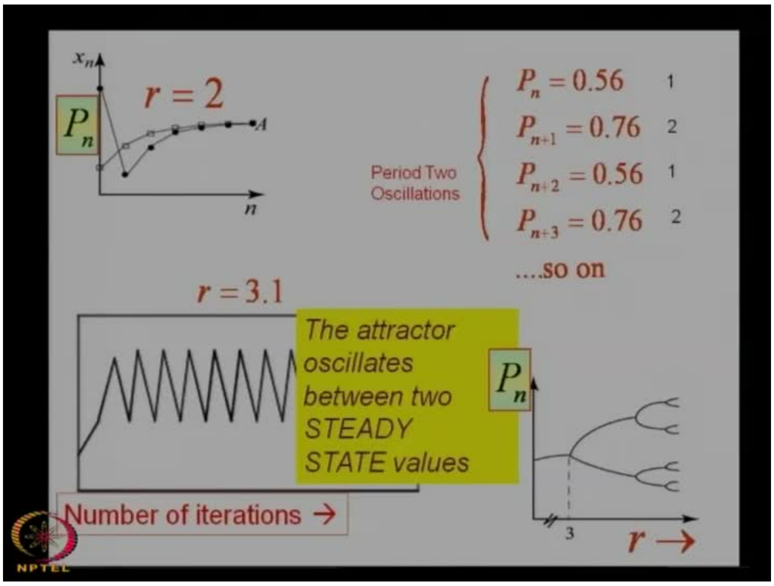

Now, we take some other value. We have taken 2.7; let us take r equal to 2; if r equal to 2, this is the kind of graph that you get, this is something you can work out these numbers yourself. You take some other values of r, take r equal to 3.1. What are we doing? We are changing the control parameter. You take r equal to 3.1 and then you get P n after sufficient number of generations, you get 0.56, and then next one becomes 0.76. So, it is not converged, but the one after that is a repeat of the previous one. So, you get 0.56 again, the one after this is again a repeat of the previous one.

So, you get one value here; next value here; then again the previous value over here and again the previous value over here. So, which means that the attractor is not a single value, it is a double value and the population sort of oscillates between these 2 values. It goes from one to the other to the back to the first, then to the next, then again back to the first. So, this is called as period two oscillations. Now, this is what happens, if r is equal to 3.1.

Now, let us take some other values. So, this, if you plot this population as a function of the generation index, the number of iterations, then you see that this graph oscillates between these 2 values which are 0.56 and 0.76. So, it goes zigzag. There is no meaning to this curve, which joins this point to the next point, it is because those points are not of interest to us; we have already agreed that we are talk about discrete steps. This is how the graph is drawn, where does not mean that there is any sense to these oblique lines;

the only points of interest are the values of the population after n generations, which is a discrete index; which is the generation index. So, for r equal to 3.1, what we get is an attractor which oscillates between two steady state values; that attractor is not a single value and the attractor does not have to be a single value. Now, that is a new idea. So, please keep in mind it does not have to be a single value, but it has to be something; so that the population stays within a closed proximity of the attractor.

So, this is what is happened, that up to 3 you had a single value attractor; beyond 3 you have a double value attractor. So, for 3.1 you have a period of two oscillations and you have these two values and r keeps increasing, you continue to a period to oscillations, but then when so this is 3.1, 3.11, 3.12, 3.2 and so on.

So, the value as r values of r increases; you keep having period two oscillations. The value of these stable points is not 0.56 and 0.76, it changes and what you see is this bifurcation, this is called as the bifurcation diagram. You get a bifurcation, but you keep getting oscillations of period two till r gets to this value, I will tell you what this value is in just a movement, but once r exceeds this then this branch again bifurcates into 2 and you get period four oscillations. So, there is a period doubling and then there is a period doubling again over here.

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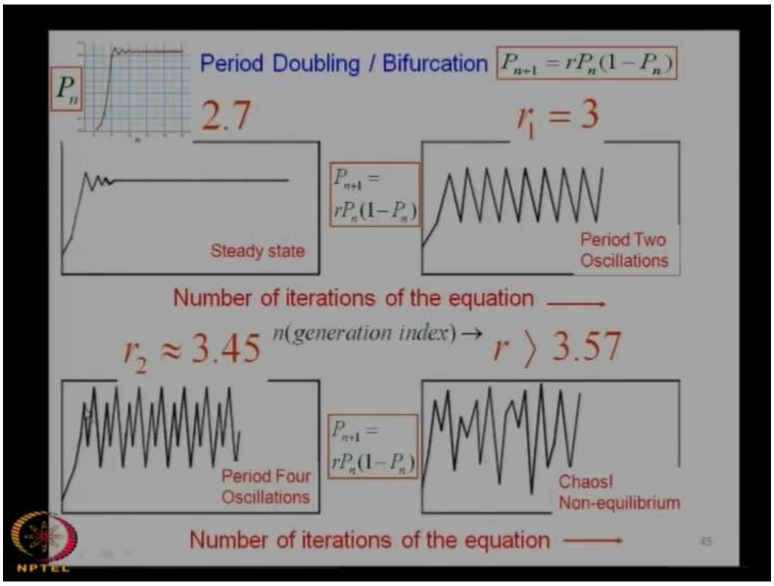

So, there are specific values of r beyond which there is a period doubling. You get the idea? So, let us summarize this; so we have seen what happened for r equal to 2.7, this is the first case that we studied; then we saw that for r 1 equal to 3 you have period two oscillations; then for 3.45 what happens is that you get period four oscillations; this is the bifurcation again, this is nearly 3.45. And now, if you increase the value of r further, if you let it exceed 3.57, look what you get. What you get is Chaos. Until this point, things were predictable, they did not have to be unique, but they were predictable; when they were not unique, they were two values, but they were predictable two values. Then there was bifurcation when you had period two oscillations, then there was bifurcation again, but it was still predictable. You had four values it is one of those four values. Here, see this pattern is exactly identical with a periodicity four. So, you have got this point, this value repeats itself after 1, 2 and 3 steps, the fourth point is back to this and it carries on and on and on and on. So, you have regular periodicity, but it is predictable; it is a period four oscillation.

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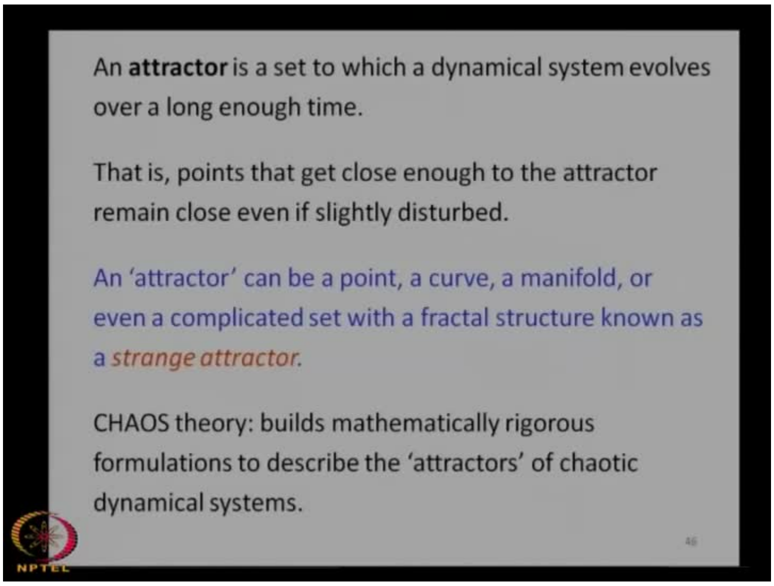

So, now, you are introduce to the idea of what bifurcation is and you know that as you change the value of the control parameter, then you cross a certain value which you can actually predict and beyond this value over here, you have complete irregularity and this is Chaos. This is the onset of Chaos which occurs at a specific value of r, which is 3.57. It is slightly close to 3.57. I will tell you exactly what it is.

And what is happening is that when you have Chaos, then the system is no longer converging to any 1 or 2 or 4 points. We gave already agreed that the attractor does not have to be a single value. So, depending on what you are talking about it can be a point, it can be a curve, it can be a manifold or it can have some complicated fractal structure in which case it is called as the strange attractor. And I have already used two terms, which are perhaps new to many of you or at least to some of you, which is the idea of a fractal and the idea of a strange attractor. So, I will define both of them. Keep in mind nevertheless that an attractor can be a point, it can be a set up points, it can be a curve, it can be a closed path, it can be a manifold or it can have some other structure that we will learn to call as a fractal structure and we will then understand what is called as the strange attractor. So, what the Chaos theory is that it builds mathematically rigorous formulations to describe these attractors of chaotic dynamical systems. A lot of research in Chaos is aimed at developing these formulations to describe these attractors.

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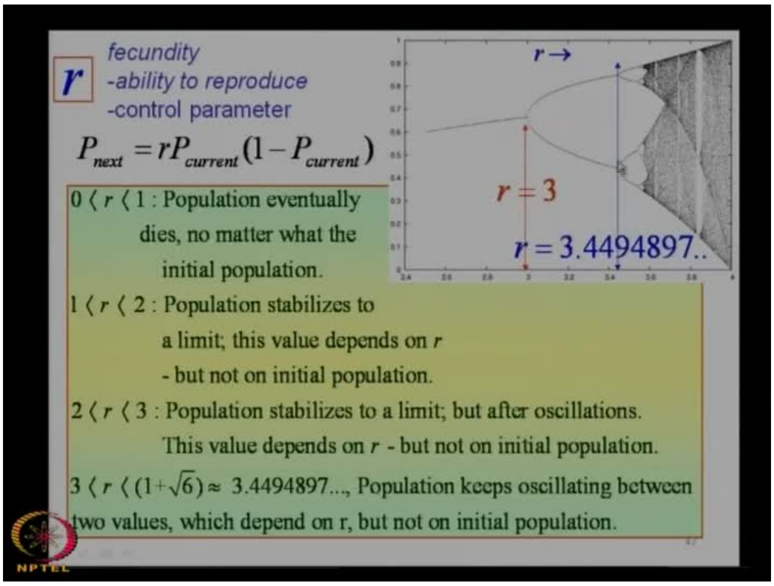

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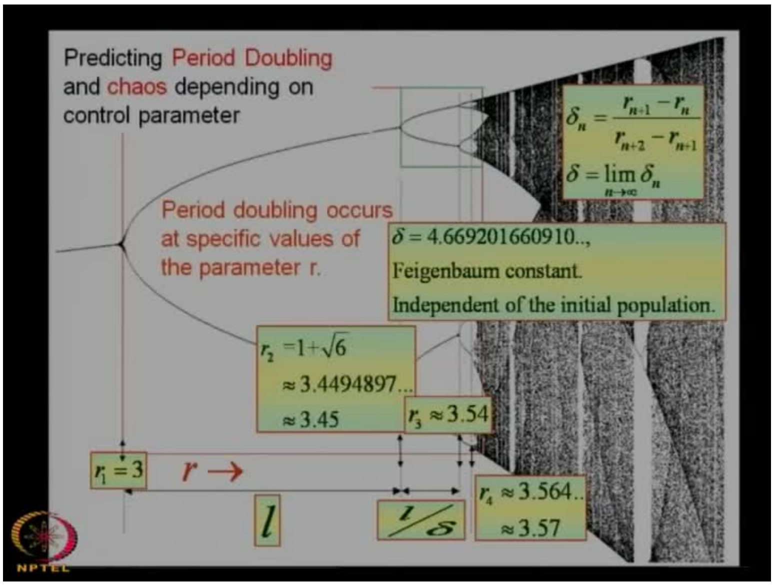

And now, let us look at this bifurcation diagram. We have already seen that at r equal to 3, there is a bifurcation. The next level of bifurcation I had rounded off this number at 3.45 more correctly it is somewhere close to this up to so many places of decimal; these are more accurate numbers then what I showed you earlier. The earliest were rounded after the second place of decimal. Over here you have bifurcation; you will notice that if you see where this pointer is showing on this figure over here, there is again a bifurcation of every branch; every branch under goes a bifurcation. Like if over here this unique branch bifurcates. Now, you have two branches, one going up and the other going down.

Now, this branch going up again bifurcates the same thing happens to the lower branch and this happens again over here. So, bifurcation keeps on repeating itself. So, you have period two oscillations and period four oscillations and period eight oscillations, because every time there is a bifurcation, the number doubles; the number the so the attractor is not a single point, it could be a set of points.

So, what happens is that if the control parameter r is less than 1, then we know that the population eventually dies that is all right. When the control parameter is between 1 and 2, the population stabilizes to a certain limit, which depends on the value of r, but it does not depend on the initial population it will always settle down to that. And then as you study the variation of the control parameter further, if you increase it further between 2 and 3, the populations stabilizes to a limit, but after oscillations this value depends on r, but not on initial population. We have already seen this for the case of 2.7, there were these oscillations, but then it settled down we know that, and then we know that when r exceeded 3 we considered r equal to 3.1 for which we already saw the bifurcation and then it oscillates between 2 values till you get to 3.449 when it bifurcates further.

So, this is how the bifurcation diagram looks. Now, we have understood the lower part of this diagram; this part of the diagram till it get till r is equal to 3, till it r gets to 3.449 and then we know that the period doubling is taking place at specific values of r. You see these vertical lines on this figure and these vertical lines are drawn at specific values of r. So, period doubling always takes place at these very special values of r. So, the first value of r, where the bifurcation takes place is r 1 is equal to 3, the next one is 3.449 it is actually 1 plus root 6 to whatever place of decimal you want to get it. The one after that takes place at 3.54, then at 3.564 it is getting close to 3.57 now and you see that beyond this you are entering the chaotic region. Now, what is interesting is, if you look at these intervals, the interval from between the neighboring vertical lines, these vertical lines are drawn at specific values of r where bifurcation takes place. Then look at the vertical lines; vertical lines are drawn at specific values of r, where bifurcation takes place and if you take adjacent intervals, so interval between this vertical line and this one and this vertical line and the next one.

So, if this delta r, this change in the value of r is a, then next one you can divide it is smaller than a, so you can divide it by some number which is a by d. You know that they get closer and closer and if you take this ratio of r n plus 1 minus r n and divide it by r n plus 2 minus r n plus 1. So, you are taking the ratios of two adjacent interval segments, n plus 1 minus r n or n plus 1 minus r n divided by r n plus 2 minus r n plus 1 and if you take this in the limit n going to infinity, then this number converges to a particular number. This is the ratio of these 2 intervals and this is called as the Feigenbaum constant. I mentioned earlier there are the some numbers from which you can learn. You can learn from e, you can learn from pi, you can learn from the golden ratio of phi, you can learn something from the Feigenbaum constant, it is a number.

It is independent of the initial population, it is about 4.669201.You can go on writing it to as many places of decimal as you have patience for. So, I guess I am going to stop here for this class, I will be happy to take a few questions and then we will continue from this point in the next class.

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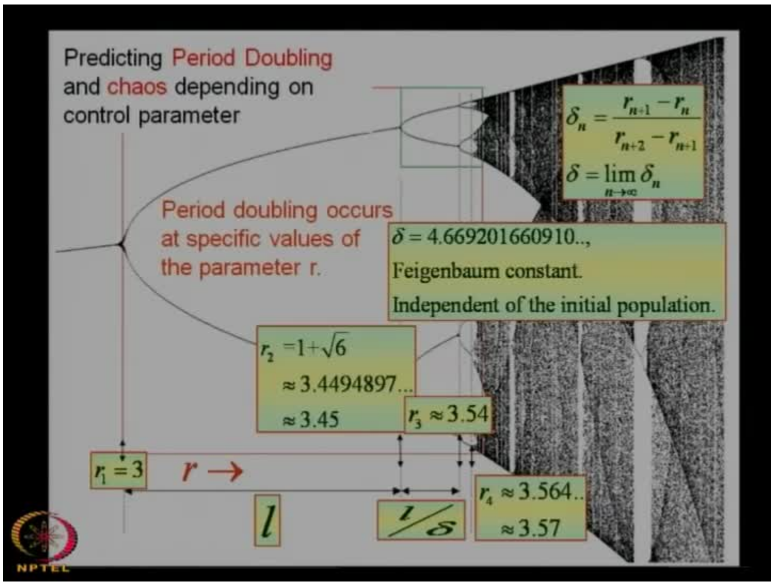

So, if there any questions I will be happy to take it. It is a comfortable point to take a break and I hope that you go home with some ideas about what bifurcation is, you go home with some ideas about what an attractor is, you go home with some idea about what a basin of attractor is, that you recognize that the attractor does not have to be a single point. An attractor can be the size of a population or certainly you do not put the size of a population on a phase diagram of position and momentum. So, it does not necessarily have to be a part of the phase space, but it can be a part of the phase space as well why not.

So, an attractor is anything to which a system would grow to and converge there and if the system evolves to a point somewhere in the vicinity of that attractor, so it can be some phase space or it can be some mathematical space or it can be a space of some physical parameters and if the system evolves to some point within the basin of that attractor, then it will fall into the attractor. So, the attractor can be quite a complex creature it may still have some simplicity amidst this complexity and you will see this in our next class when we discuss the strange attractor. So, I will take a break here.

Yes

Almost every of these does not depend on the initial population and it is rather depend upon the carrying capacity, is it also independent of the carrying capacity?

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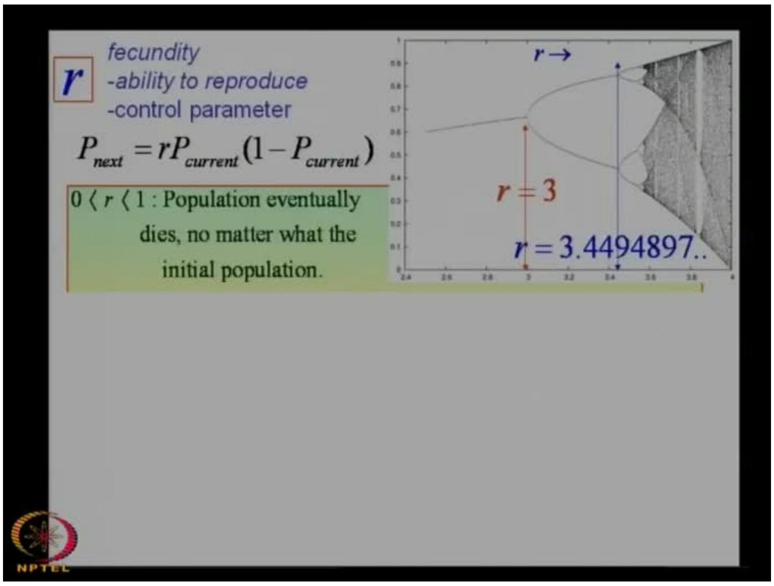

What we have done is we have introduced a feedback mechanism over here, means look at, this is still a single parameter theory; the only parameter over here is r. We introduce the carrying capacity in the differential equation model in the logistic differential equation model and there we dealt with two parameters r, which is a fecundity and then the carrying capacity K. Here we are dealing with a single parameter and this is the point I have emphasized earlier also, that you can get Chaos even when you are dealing with a single parameter. So, it is not because there are many parameters which is what you expect in statistical models.

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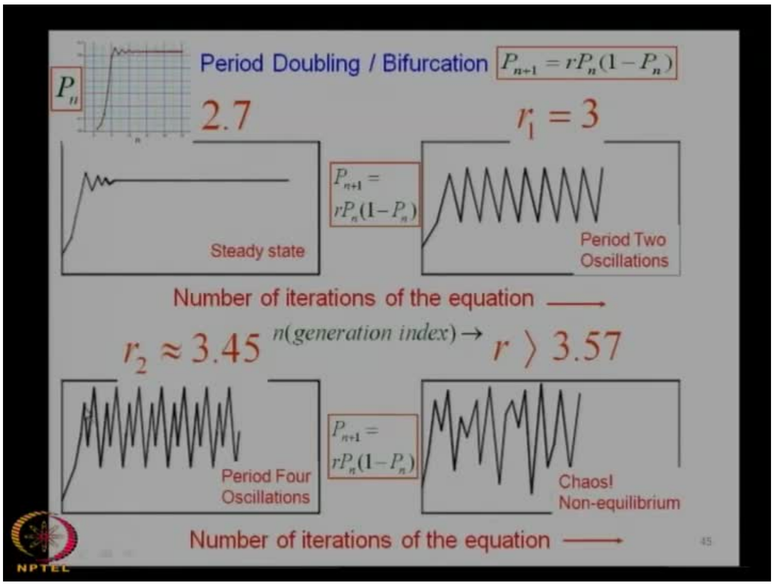

So, you get unpredictability, you get this chaotic behavior even when you are dealing with a single parameter and here look at this figure over here, the 1 in the lower right corner here. You have a single parameter this is the equation you are dealing with; it is a very simple equation. This is what Robert May was telling us, complex consequences of simple equations. This equation has got a non-linear term. We had a non-linear term in the differential equation also, we were not led to any Chaos, but here we have a non-linear term in a difference equation in a map. And in this case, it is not that we are always led to Chaos, we are not led to Chaos when r is equal to 2.7.

We are not led to Chaos when r 1 is equal to 3; we are not led to Chaos when r is equal to 3.45. We are led to a population which is not unique, but it changes from one to the next to the next to the next to the next and then back to the first. Then again it changes to the next to the next to the next and again back to the first. So, there is a period four oscillation, but if it exceeds 3.57 then no pattern emerges; there is no pattern that is Chaos. There is no quantum theory, there is no principle of uncertainty, there is no probabilistic distribution like a quantum wave function or anything like that, this is not quantum mechanics this is classical and you are led to Chaos. There is a single parameter and that single parameter, but you can predictive the value of this parameter; if it exceeds this 3.57, you have Chaos not otherwise. Then you have bifurcation and so on. You can means for smaller values of r, here you it converges to a single value, here it converges to two alternate values, in this case to four alternate values, we have seen in the bifurcation diagram that it can oscillate to eight alternative values.

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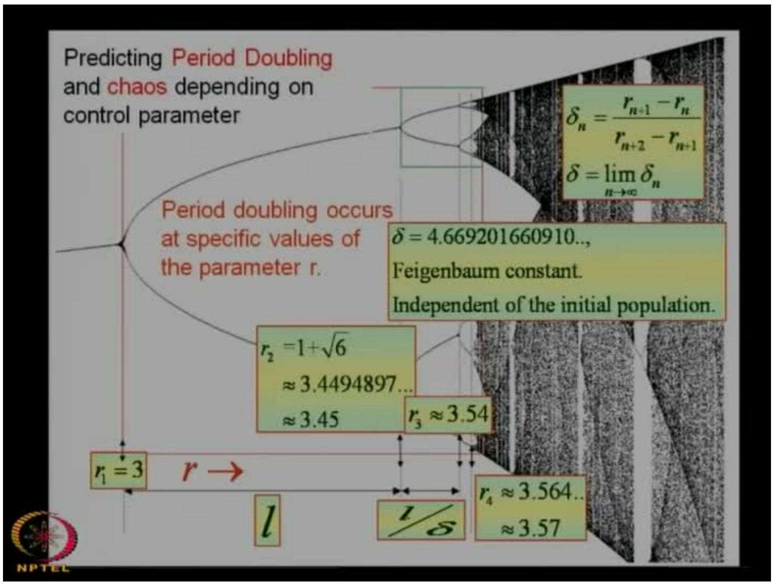

So, there is a period doubling, every time there is a bifurcation and bifurcation takes place at specific intervals. Now, what is very exciting is that if you take the ratios of these intervals - adjacent intervals - then the ratio converges over here, this r n plus 1 minus r n if you divide by r n plus 2, which is the corresponding interval for the next segment, then it converges to a very specific value which is called as the Feigunbaum constant.

So, thank you very much, we will continue from this point in our next class.