Select/Special Topics in Classical Mechanics
Prof. P. C. Deshmukh
Department of Physics
Indian Institute of Technology, Madras

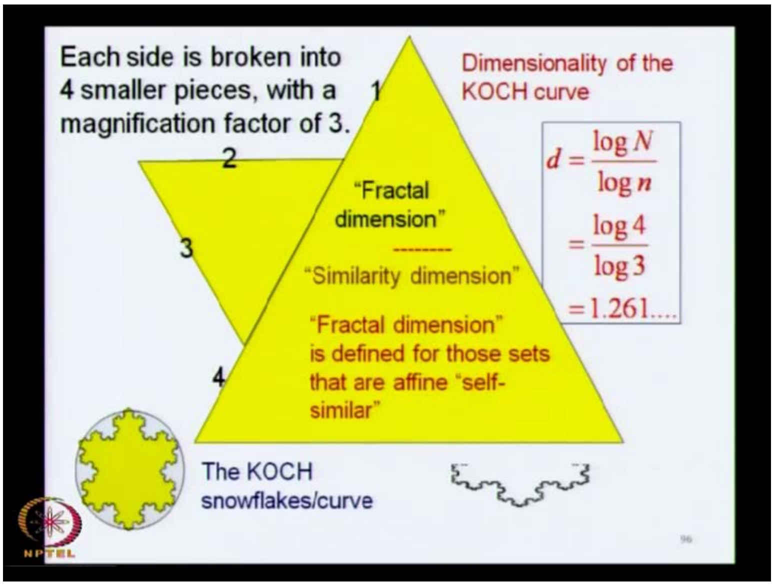
Module No. # 11
Lecture No. # 39

Chaotic Dynamical Systems (v)

Greetings, today, we will take a couple of steps, beyond what we did yesterday in discussing chaotic dynamical systems. We have already had some exposure to bifurcations and chaos and we met Benoit Mandelbrot in our previous class.

And today, we will talk about Mandelbrot sets, which are named after him. And these are very fascinating objects and you will love the pictures.

(Refer Slide Time: 00:57)



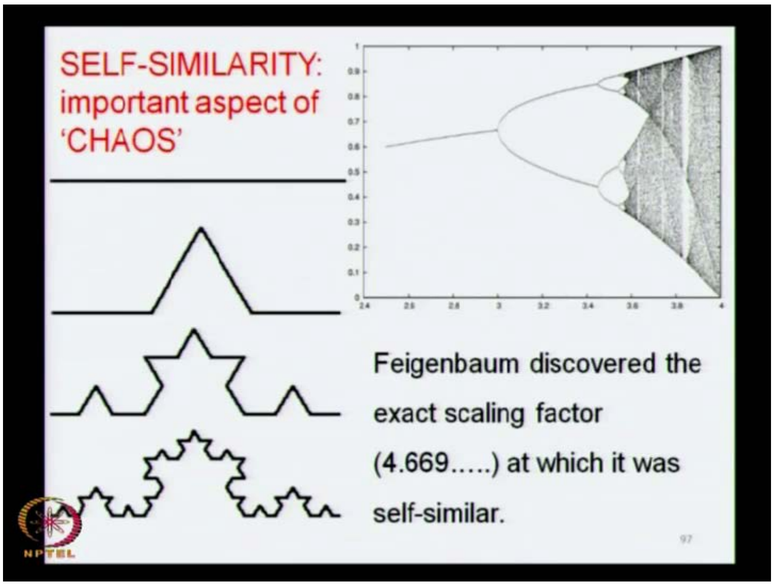
Now, the essential idea that we discussed, had a close connection with self-similarity. So, here you see triangles, on triangles, on triangles, on triangles. So, when you construct a triangle, on the middle edge. On the middle one third, you are left with two one third pieces on the wings - on the sides. And those are then the smaller segments to which the first larger segment is reduced.

So, you have the first segment, which has got this length, which is original triangle then, you construct a triangle on the middle. And then, you are left with four sides, four segments, each of one third the size and then on each of these segments.

Each of these four segments, the original segment has been reduced and on each of these segments, you again carry out the same process. So, there are two elements over there, one is self-similarity, the second is iteration. So, both of these have a combined effect, which leads to these fractal dimensions that we discussed in our previous class and the Koch curve is a great example of perfect infinite degree of self-similarity. It is completely self-similar, it is not like the coastal line of Great Britain or coastal line of India or any such place, which has got a physical boundary, because the physical boundary of any real physical object, any material object, it will have some similarity to a larger extent but it will not be completely mathematically infinitely self-similar.

So, the Koch curve is the example, that we discussed and using this, we defined the fractal dimension, as the ratio of these two logarithmic numbers, and we recognized the fact that this ratio certainly does not have to be an integer.

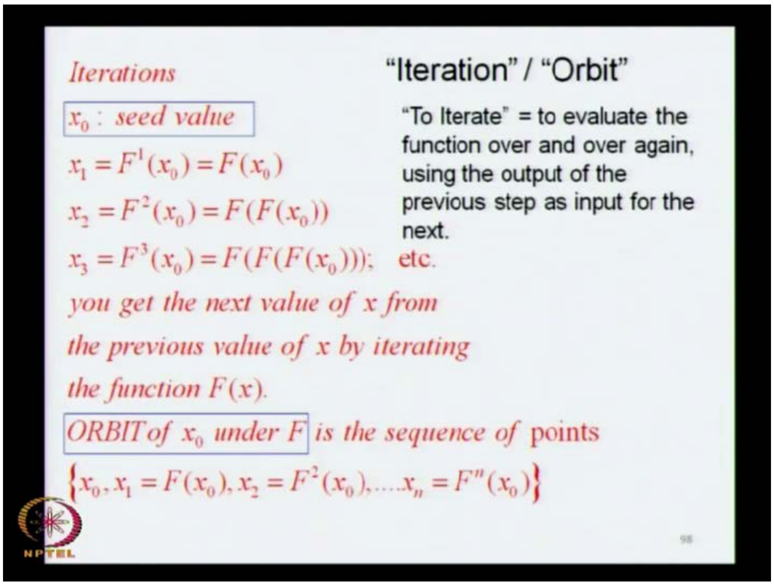
(Refer Slide Time: 03:17)



So, (Reference slide time 03:24) this is the generation of the Koch curve. This is an important aspect of chaos. And we have met self-similarity earlier in this logistic path. Where we met this bifurcation and then you have a repeat of these bifurcations. So, what you have is an oscillation of period 2 it doubles up to an oscillation of period 4 and then it branches again. So, there is yet another bifaurcation.

And when you see this cascade of bifurcations, you notice that once again, you have one common element which is self-similarity. So, what Feigenbaum really discovered is the exact scaling factor at which this self-similar behaviour manifests itself. It was the ratio of these consecutive intervals of r, if you remember the definition of Feigenbaum constant and it is in fact a scaling factor at which this self-similarity appears.

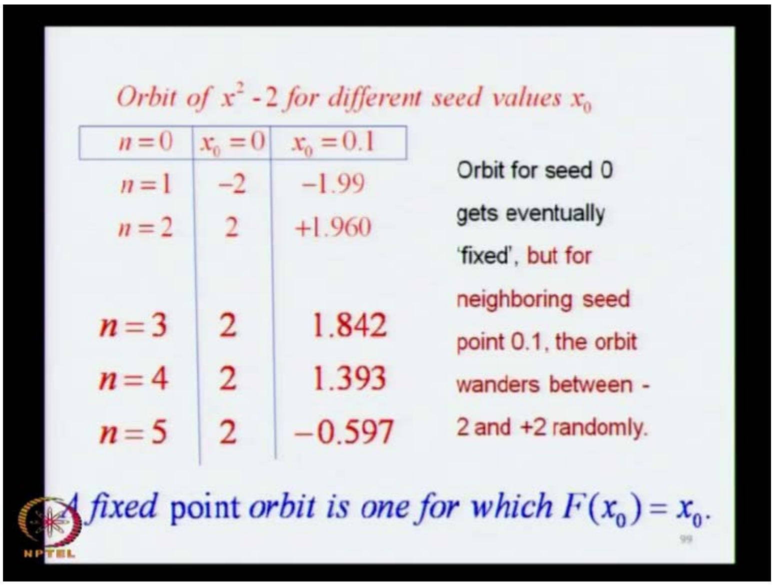
(Refer Slide Time: 04:44)



We also define, what an orbit is, you begin with a certain seed value, you defined a function on it. And then you get a new value, which becomes the seed value on which you operate by the same function and you get the next value of that particular parameter.

So, it keeps changing, you again you have an iterative process and this will generate a sequence of numbers, which is an orbit it may settle down to some fixed value, it might settle down to a set of values, which repeat itself, themselves in some sequence like the periodic orbit of an oscillator in the phase space. And the orbit can be either a fixed point or it can be a steady state fixed point or it can be a closed orbit which repeats itself.

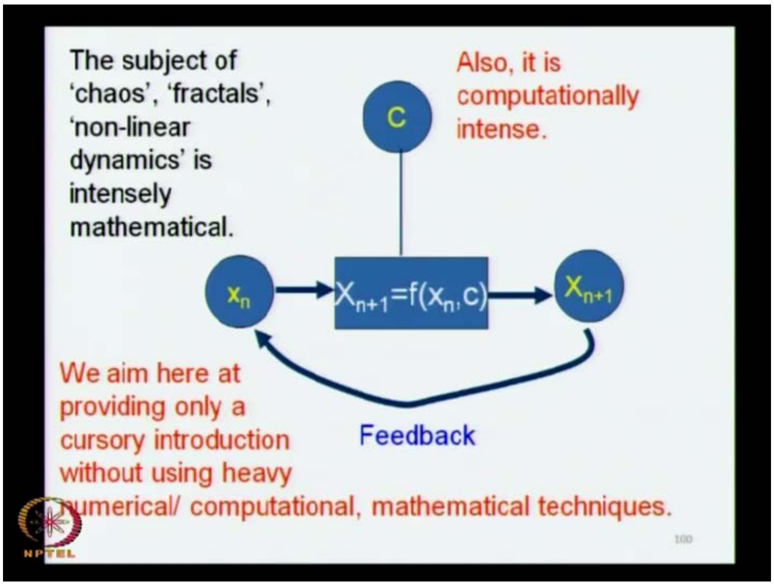
(Refer Slide Time: 05:47)



And we also considered, some numerical examples in which we showed that if you have a function like defined by x square minus 2, then its orbit is a fixed point, if the seed value is 0, but if the seed value is even slightly different from it if it is a small departure from 0 like 0.1 then you get completely random numbers.

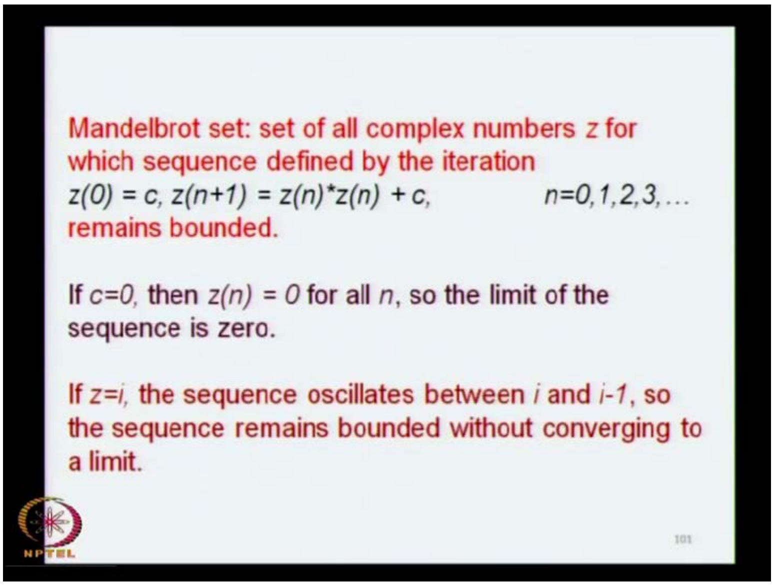
You get numbers, we do not show any kind of regularity, but you do not get the same kind of an orbit, if you have slightly different seed value. They do remain within the confines of plus 2 and minus 2, but within this range it is not easy to predict as to what the value will turn out to be. Whereas, if the seed value was only 0, then you know exactly that the function will settle down to the value 2 and it will remain fixed.

(Refer Slide Time: 06:54)



So, what we have is you begin with a certain function, you get a new function, you from the seed value. So, you take the seed value over here and on the seed value, you would operate by a certain function you get a new value. And then this new value can become the seed value and you can iterate. So, this is some sort of an iterative procedure and this subject of chaotic dynamical systems is an intensely mathematical subject, you need very high speed computers to get to operate these iterative, you know steps to carry out this iterative mathematics. And you know some people make use of super computers and spend a lot of time getting doing this number crunching. So, it is mathematically and computationally very intensive subject.

(Refer Slide Time: 07:56)



Now, there is a particular orbit which is of great interest in this subject, which is called as the Mandelbrot set. This is named after Benoit Mandelbrot. What it is, it again it can be defined in a very simple way and this is we have carried from the very beginning that you deal with very simple systems, but they lead to extremely complex behavior.

So, this is the complex behavior of very simple systems, the equation that you have in front of you is an extremely simple one, you begin with a seed value, this is the complex number c, this is the seed value. So, this is the value at the zeroth iteration.

And then the next value is defined by this relationship. That the next value is given by the square of the previous value plus this constant, which is the seed value that you began. It is a very simple equation. Right now, you will get as a result of this an orbit, which is a sequence of complex numbers and these complex numbers can be identified by putting a point in the complex plane, which is made up of a real axis and an imaginary axis. And the orbit will go from one complex number to the next, to the next, to the next and generate a sequence of complex numbers.

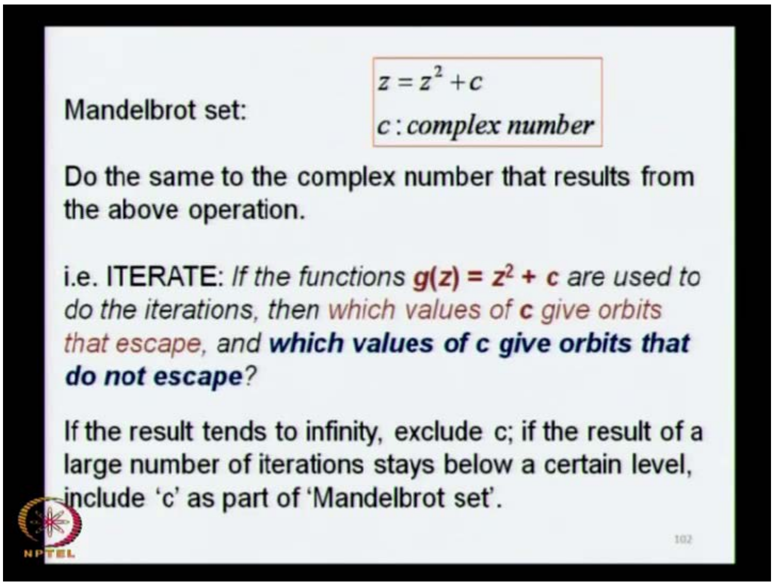
Now, the corresponding point in the complex plane will therefore travel in the complex plane. And the question, you can ask is will it remain in a bounded space of the complex plane or will it escape to infinity.

Now, if it does remain bounded, then the number c which was the seed value is considered to be a member of the Mandelbrot set. So, the Mendalbrot set is a set of all complex numbers z for which the sequence defined by the iteration z n plus 1 equal to z n square plus c remains bounded; so, very simple definition.

Now, you will see what kind of amazing results this leads. And you will then see that it has something to do with bifurcations that we talked about, it has something to do with the logistic map that we talked about. It has something to do with the self-similarity that we talked about, it has something to do with chaos that we talked about.

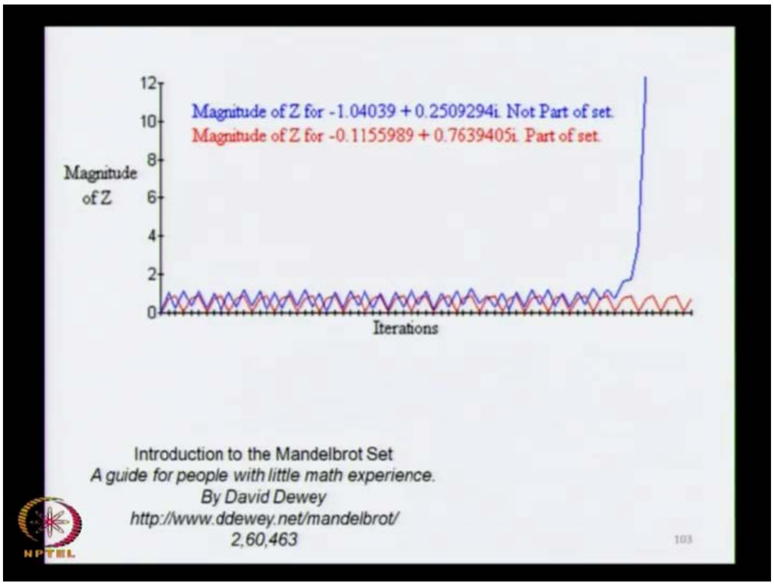
Notice that, if c is equal to 0, then z n is equal to 0 for all n and the sequence is just the complex number 0. If z is equal to i, then the sequence oscillates between i and i minus 1 i square root of minus 1. So, the sequence remains bounded but it does not converge to a limit it oscillates between i and i minus 1.

(Refer Slide Time: 11:06)



Now, let us start doing it, you could take a calculator and work this out. You know how to take the square of a complex number. So, that is a very simple process but just to get the complete Mandelbrot set, you really need powerful computers because the amount of number crunching, you have to do is huge. And you will see that although you are dealing with a very simple expression. So, I am going to show you, some of those results. So, what you are going to do is to iterate, get the next value and you have to keep carrying out these iterations, how many times we will figure out. But you have to carry out these iterations and at a certain point if you find that the value either will blow up or else it will remain bound in a certain part of the complex plane and if that happens, you would say that it belong to the Mandelbrot set. So, you are looking at the sequence, which comes out of it, which is the orbit and you are asking this question, which values of the complex number c give orbits; which do not escape that is the question you are raising. If it escapes then you exclude that complex number, if it does not, you include it as a part of the Mandelbrot set because that is what defines the Mandelbrot set.

(Refer Slide Time: 12:47)



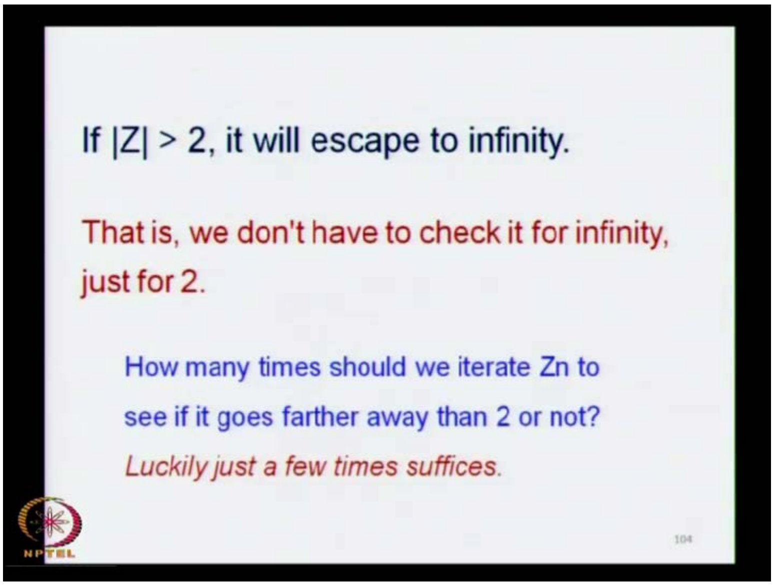
Now, let us take two examples of wave here. There are two complex numbers, one this is minus 0.04039 this is the real part, and the imaginary part is this which is 0.2509294 times i, which is the square root of minus 1.

The second complex number, I am dealing with is the one shown in red, which is minus 0.1155989, this is the real part and the imaginary part i is sitting over here is i times this part which is 0.763940, which is the imaginary part of this. Now, if you work with this previous expression, z square plus c, where c is either the blue number or the red number and carry out this iterative process, over and, over and over and over again. And as you carry out this iteration, you get new complex numbers. So, you get this at the first iteration. This is at the second, this at the third, this at the fourth and so on and you have to carry out these iterations n number of times.

Now, notice that this sequence of numbers that you get it sort of oscillates for both the red curve and the blue curve. The red curve is the one which describes the orbit that you get, if the seed value is this, the blue curve is what you get if the seed value is this upper complex number. But then what happens is if you carry out these iterations further the one which had the blue seed blows up it does not remain bounded, whereas the one which is given by the red curve it remains in a bounded space.

So, which means that the blue number. The blue seed that you begin with is not a part of the Mandelbrot set, but the red one is a part of the Mandelbrot set. Now, you know how to get the Mandelbrot set.

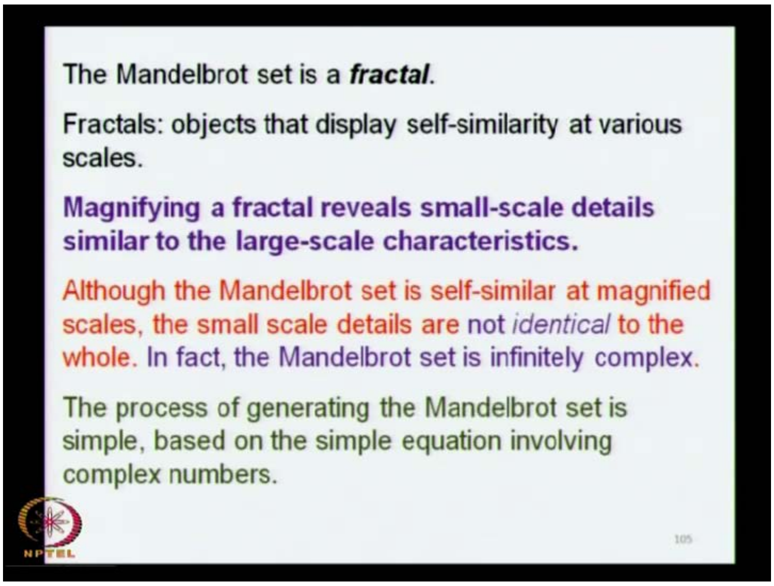
(Refer Slide Time: 15:18)



Now, what happens is that in case the modulus of the complex number z happens to be greater than 2. Now, the modulus is the real number. It is the square root of the sum of the squares of the real part and the imaginary part, if this modulus of the complex number happens to be greater than 2 then, it will always escape to infinity. This is small exercise, which can be worked out as homework problem, I will not discuss the proof, but if it exceeds 2 then it will escape to infinity, which means that this simplifies your effort, because all you have to check is that you can carry out a certain number of iterations. And if you ever in a finite number of iterations, you find that the complex number has got a modulus, which is larger than 2 then you can already exclude it.

Because, then it is guaranteed that it will escape to infinity. So, you do not have to proceed with that iterative procedure any further. So, that simplifies your search for those seed values whose orbits will generate the Mandelbrot set. So, you do not have to wait till it really goes to infinity you just have to see, if it ever goes to such a value whose modulus would exceed 2. Now, how many times do you need to iterate to check this out usually, this number is not very large as it turns out and this comes from practical experience, it is not that you have to do it infinite times, it is not that you have to do it a 1000 times or a 100 times you have to do it a few times. And then you are able to quickly check, out that if it is going to have a modulus whose value will exceed 2 and then you can just throw it out, because that is not the number you are looking for the Mandelbrot set.

(Refer Slide Time: 17:28)

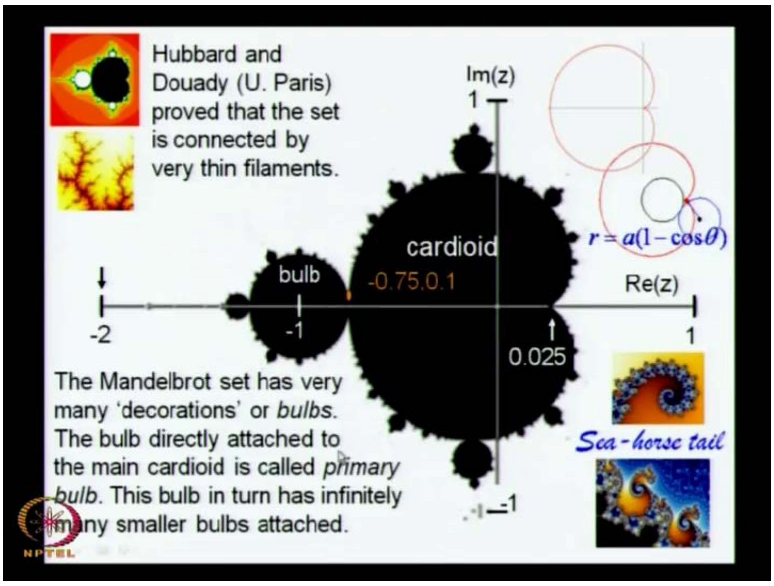


Now, the Mandelbrot set in fact is a fractal. You know what a fractal is it displays self-similarity at various scales. We have already met this and when you magnify this, you get structures, which look very similar to what you had seen on a larger scale that is a very characteristic feature of a fractal and a Mandelbrot set in fact does have that feature.

But unlike the Koch curve, whose self-similarity is complete it is a mathematical curve. So, this is but there is a little difference. In this case, this small scale details are not completely identical to the large scale details, you will again see it, when I show you some examples.

So, there are differences which really makes the Mandelbrot set extremely complex in fact, infinitely complex.

(Refer Slide Time: 18:45)



So, this is the process is simple all we are doing is multiplying complex numbers and this is what you get for the Mandelbrot set. All points that you see which are shown and it is a dense picture notice that everything remains confined to a compact space. All points have got a modulus which remains confined.

These complex numbers remain in a bounded space, the first major shape is this cardioid and you know what is a cardioids, it is defined by this function, you can get it by following the trajectory of this circle on a circle, which is rotating like this that is what generates a cardioid it has, it can be described by this relationship between r and theta.

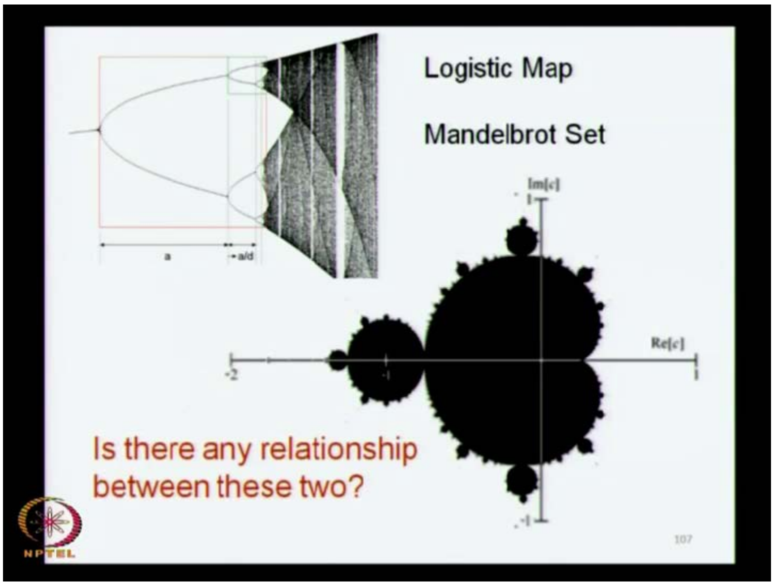
So, this is the cardioids, and you see that there is this major cardioids, but then outside this cardioids, there are a number of decorations they are in fact called as decorations. So, there are these bulbs and then there are other bulbs which are attached to these bulbs and now you are going to start looking at it in detail zoom into this and as you zoom into this just the way as you would zoom into a Koch curve you would see more and more self-similarity. But in this case, you will see more and more similarity, yet there will be some difference, so in that sense it is somewhat different. So, this is a Mandelbrot set it has got this amazing shape.

Now, let us see some further detail, now what happens is that if you see closely at these boundaries, you will see that there are these nerves kind of thing or fibers and there are the serpentine figures, which come out of it. So, there are the small things, which come out of these exterior buns and then you have some more complex behaviour and if you zoom in on any one of these you are going to see more structure.

So, what you will start beginning to see that there are, these serpentine shapes, then the some of the shapes look like seahorse tail and then if you start looking into these details, you will see more structure and now you can see, how beautiful shapes you get out of this it is really amazing.

Now, Hubbard and Douady, they proved that the set is connected by very thin filaments to the eye it might appear, as if some of the points are not connected, not so, they are all connected by thin filaments. They may not show up on a certain scale, but if you magnify, if you zoom in you will see and in fact there is a theorem proved by Hubbard and Douady that the set is actually connected by very thin filaments. I already mentioned that there are these decorations or bulbs. The main cardioid is the primary bulb and then what you have is a number of these secondary bulbs, which are attached to that which are known as decorations.

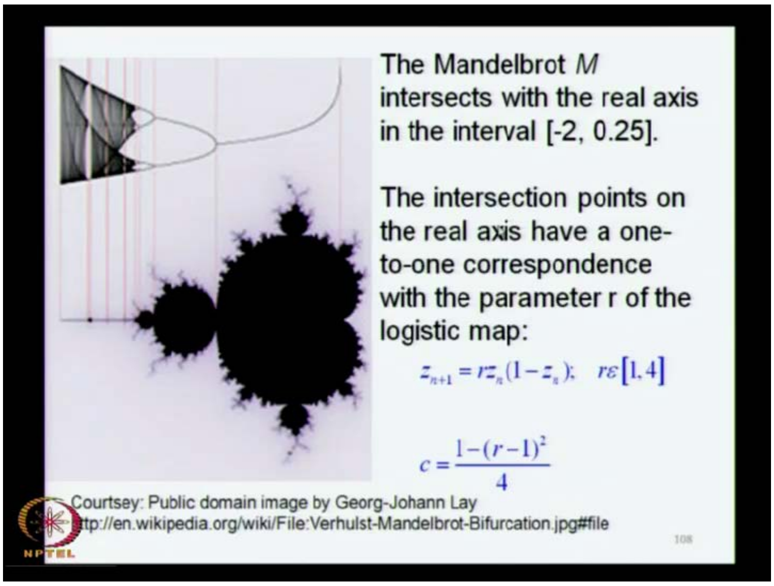
(Refer Slide Time: 23:02)



Now, we have studied the logistic map, in one of our earlier classes and now we have described what are Mandelbrot sets? There are a large number of Mandelbrot set, beautiful pictures they are in the public domain, you can download them from Wikipedia if you like or you can download them, if you just Google Mandelbrot set you will be led to god knows, how many websites and you know people enjoy doing arithmetic and number crunching with this simple complex number algebra.

All you have to do is to multiply the complex numbers and if you write a program, then it is just a one-step trying to generate the iterative process, but of course the programming becomes very challenging, because then you what you do next is to add colours to these bulbs. They will bring out the detailed structure, because otherwise everything is in black and white and you do not really see the structure, say if you then attach different colour to different pixels then, you will start to see the details of the structure. The question we are asking over here is that we studied the logistic map, we have also studied the Mandelbrot set is there any relationship between these two.

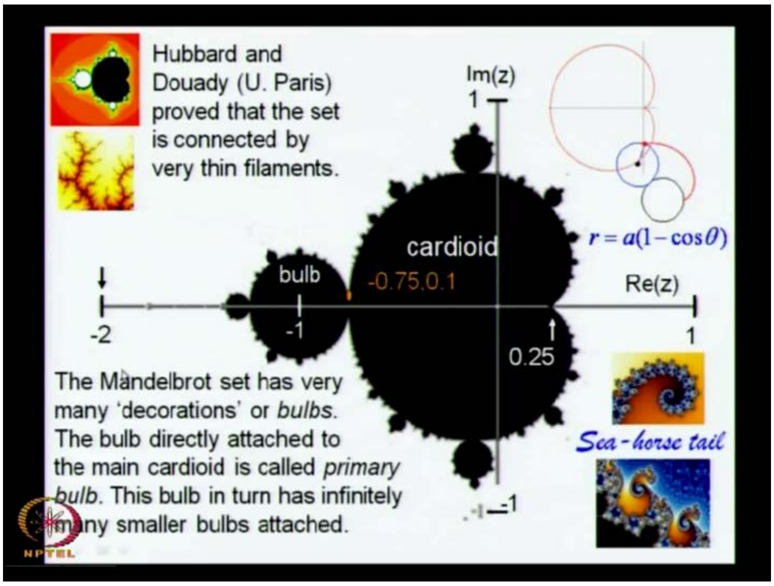
(Refer Slide Time: 24:56)



It turns out that there is a relation and a relationship, which is very simple, very straight forward and it follows the Feigenbaum constant here. It is if you just place these figures on top of each other, you know that these vertical lines, I have shown that this bifurcation diagram moving from right to left rather than from left to right as we discussed earlier that is the only thing we have done.

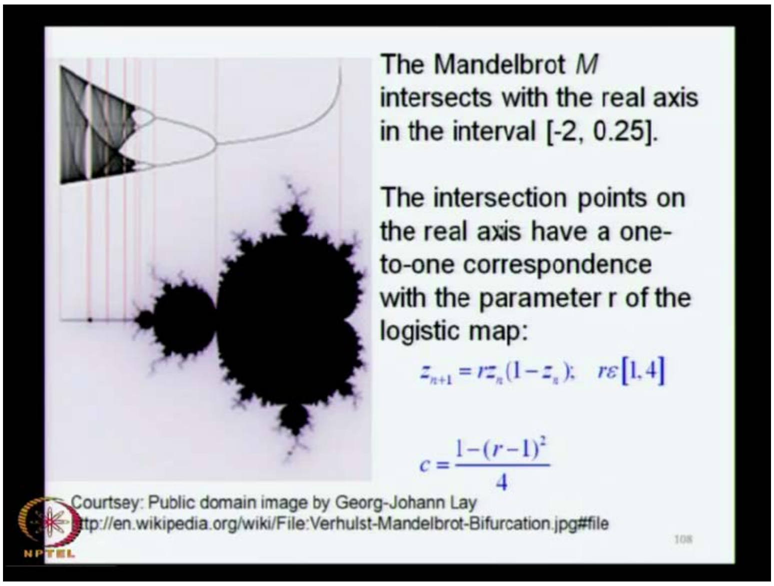
But in scaling factor the mathematics remains invariant and you notice that if you look at the major intercepts of the Mandelbrot set with the real axis. They fall along the same segments of these vertical lines, whose intervals in successive segments generate a ratio which is the Feigenbaum constant. Now, that is how close they are related. So, the Mandelbrot set actually intersects with the real axis in the interval minus 2 and 0.25.

(Refer Slide Time: 26:05)



This is much clearer in the previous figure, which I have taken from Wikipedia this one the intercept here is minus 2 on the real axis the intercept here on the real axis is 0.025, no sorry it is 0.25 naught 0.025, there this needs to be corrected. This is 0.25 this is 1 and a quarter of this is where it intersects.

(Refer Slide Time: 26:44)

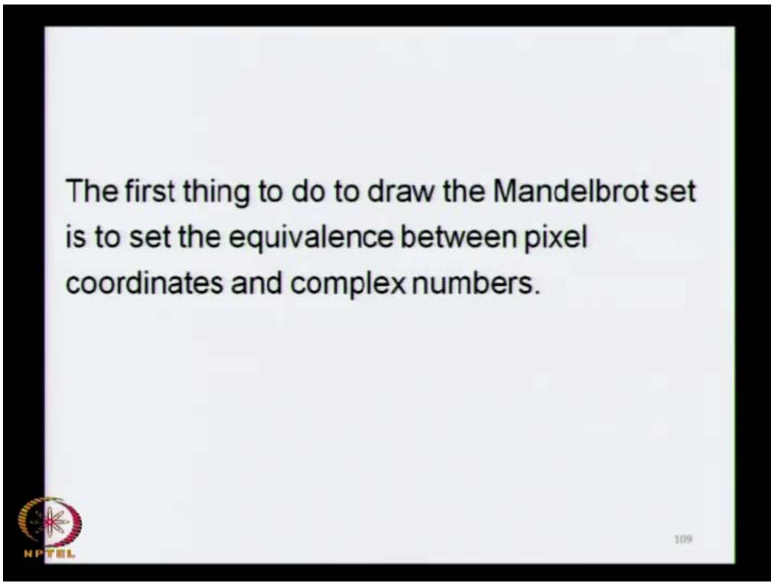


So, this remains, these are the intercepts on the real axis and if you now see these intersects with the real axis, these I already showed you minus 2 and 0.25 and the intersection points of the real axis, they have a one to one correspondence as you can see from this figure with the parameter r of the logistic map. We generated the logistic map by changing the value of r and there is a direct relationship between the intersection points on the real axis with the parameter r and that relationship is given over here.

You do not really have to write it down, if you just Google these things you will get it very easily on the internet and there are some very fascinating books on Mandelbrot sets and chaos and I will give you some of the references toward the end of this class, when we can have those references as well.

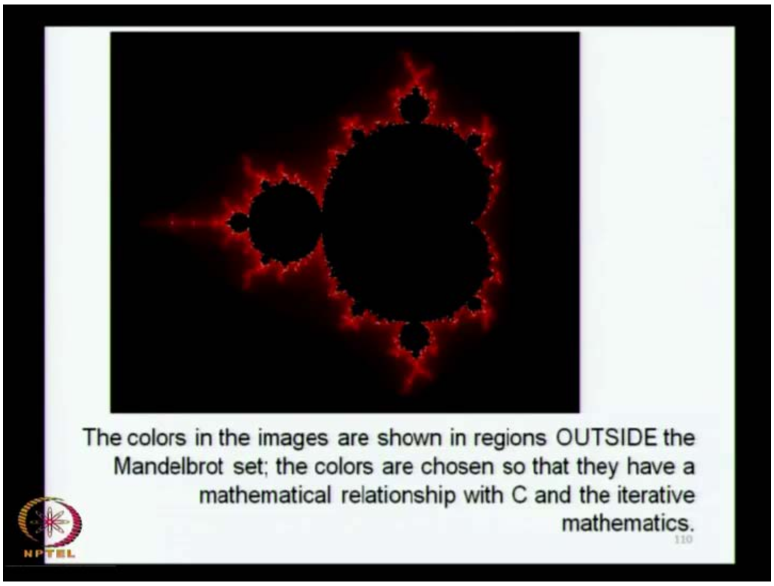
But you notice that the relationship is so simple, right, the relationship between the complex number c and r is given by what you see on the screen and it is a very simple relationship connected with the intersection points on the real axis and that is the reason why you get the same kind of scaling factor in the which is defined by the Feigenbaum constant.

(Refer Slide Time: 28:11)



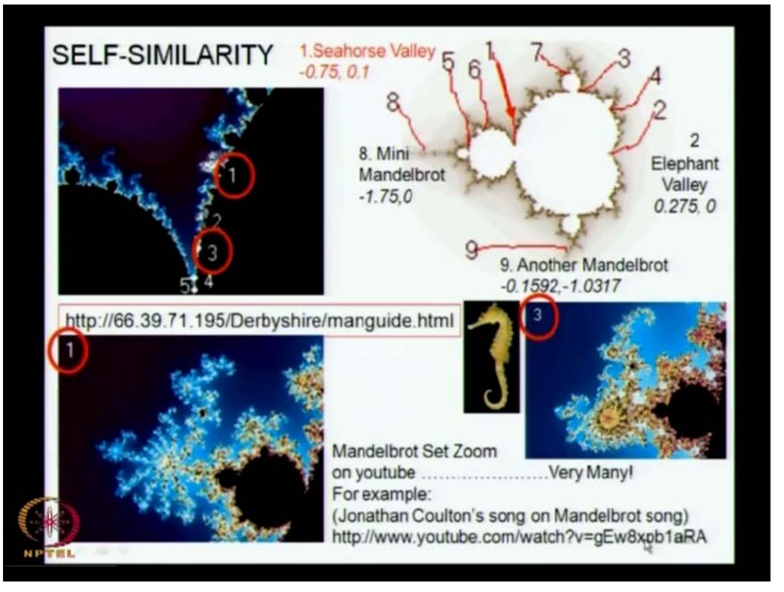
So, how do you generate the Mandelbrot set, the first thing to do is to set up equivalence, between pixel coordinates and complex numbers, because this is very intense computer graphics is used to generate these plots.

(Refer Slide Time: 28:37)



And then have a colour code, which is linked to this. Now, can you see this beautiful figure. So, to highlight in the boundaries you know this particular programmer, this is one of the very many that you will find already, but you can generate your own and notice how beautiful it looks, because the boundaries are very nicely highlighted by choosing a colour for that corresponding pixel. But then there are details in every part of this figure every boundary will have more detail, because you are going to see this self-similarity as you zoom in just the way you found it in the Koch curve.

(Refer Slide Time: 29:28)

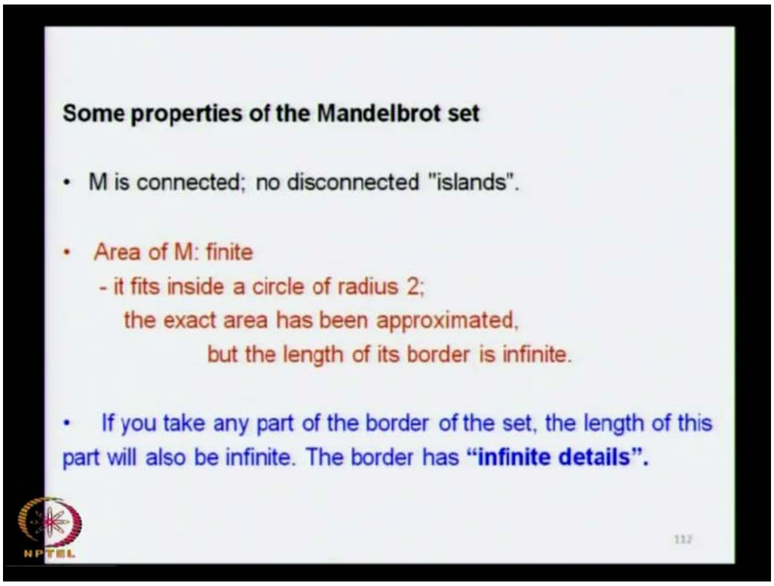


So, let us see how it looks like. Now, this is the Mandelbrot set, there are various points and they have got specific names. So, for example there is a point over here which is called as the seahorse valley and these are the coordinates in the complex plane for that then there is a point number 2, which is called as the elephant valley whose coordinates are 0.275 and 0.And then, there are other points, which have got specific values these are just the real part and the imaginary parts. So, those coordinates are given and then if you start looking at some details. So, if you zoom in on a seahorse valley for examples, in this part then you can get a detailed structure and then you zoom in further, you get some more detail, you zoom in further and you see that inside this. There is more structure and this seems like endless it is really amazing.

In fact, there is a very nice song which is composed by Jonathan Coulton, you will get it on the YouTube. So, if you just Google Coulton's song on Mondelbrot set, you do not have to write down this link, you will get it easily and enjoy it, there are a number of Mandelbrot, you know figures which are available on the internet it is a great source for you know just developing your basic interest and intuition about the Mandelbrot sets.

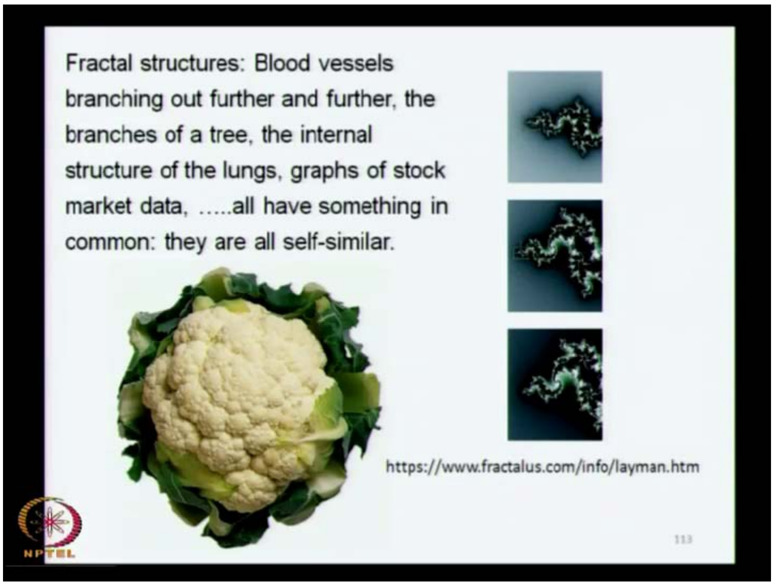
But of course internet is always a source that you should be used with a lot of caution; so, that is a different story.

(Refer Slide Time: 31:20)



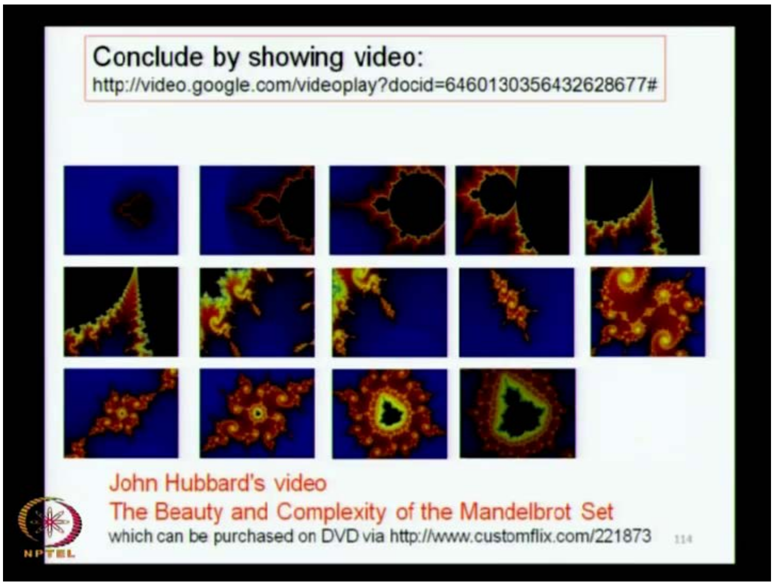
Let me highlight a few properties of the Mandelbrot set. It is connected I already mention this there are no disconnected islands everything is connected. The area is of course finite you actually saw it in a finite part of the complex plane, but the perimeter the length of the border is infinite. This would remind you of the Koch curve and if you take any part of the border zoom in you will find more of the border, you will see more structure. And the border will have so much information in it in fact has huge amount of information in it that you 0 in and zoom into any small part of the border, you will again see huge amount of information so it really has infinite detail.

(Refer Slide Time: 32:37)



Now, here are some other objects, which will remind you of self-similarity it is not just the cauliflower, but the internal structure of the lungs, graphs of stock market data, many of these things have got something in common. So, if you study the Mondelbrot set and examine the behavior then from the mathematical forms, you can study many other branches of physics, chemistry, various sciences, biology, finance, economics, what not. And if this is what science is about that you find mechanisms to study these then studying the Mandelbrot set. Obviously, has a large number of applications, which is why there is so much interest in this field.

(Refer Slide Time: 33:40)

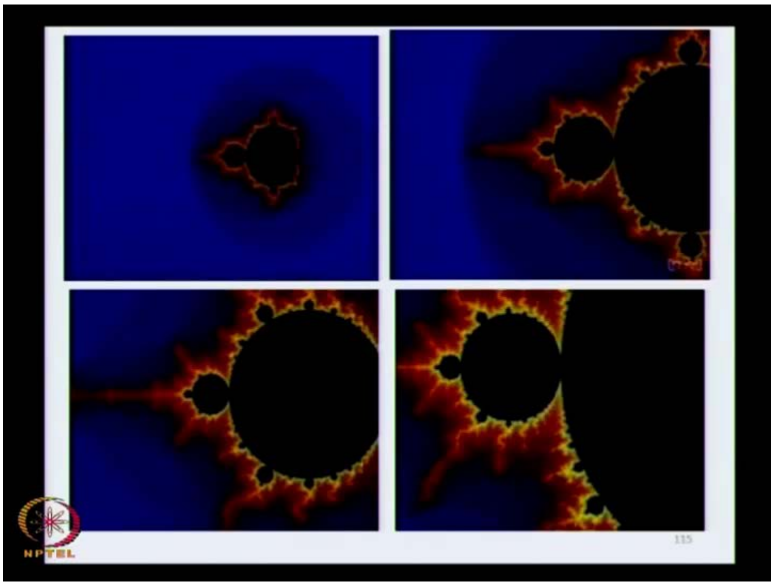


I will like to show some pictures, there is a very lovely video which is available you can get it even on a DVD, which is John Hubbard's video, but I will show you some still pictures first just to get some general idea. So, here you begin with the Mandelbrot set can you see that. Now, what you have done in the second slide is to zoom in the central part of the previous slide. So, every time I show you the next picture, it will be a magnification of this central portion of the previous one, and you already see that this is the magnification of the central portion of the previous picture. Now, let us do it again and this is what I had mentioned earlier that as you zoom in you see more and more detail to such an extent that you have infinite detail over there.

Look at this now, we have enlarged this further this is a magnification of the central part. Lets go further, now, what is in the central portion has been selected for magnification. So, this part now begins to appear like this and notice that these edges at the boundary what looks like fibers or the serpentine kind of shapes. Now, watch the structure of these serpentine shapes. what is going to come out of it is something which even after seeing it means, I have obviously seen all these things it is hard to believe that this is what is going to come out of this, but it actually happens. So, let us zoom in further you get some further detail over here and you begin to see these serpentine shapes.

Now, you have zoomed in on this part. So, these serpentine extensions, which come out of these bulbs these decorations and now, you, see that this decoration is a very appropriate name, because it really look so beautiful. And now, you are beginning to see some further structure and does this look anything, like what you have ever seen earlier, I mean including in the few minutes when we started showing the Mandelbrot sets not yet at least; right it is also different. So, now, we have zoomed in again on this middle picture, middle part, we have zoomed in again on this middle part. So, here is the middle part and that is now beginning to look larger. Now, zoom in on this again, it becomes like this enlarge this further zoom in, see how beautiful it is see how much structure is there, you did not expect it when you just saw the first one, did you? But, now that we are here, you see how much structure there is, but you will also see self-similarity, have you seen the self-similarity? Yet not quite, but you will let us look at the next one here it is this is the next one, what you have done is to zoomed in on this central part, which is this. This is the middle part this is the one on which you have zoomed in it generates this pattern and may be you notice that there is some black spot over here and watch it grow, watch it grow, because that is the one on which you are going to zoom in now. So, it is already beginning to look like the Mandelbrot set that you began with and now you recognize the self-similarity.

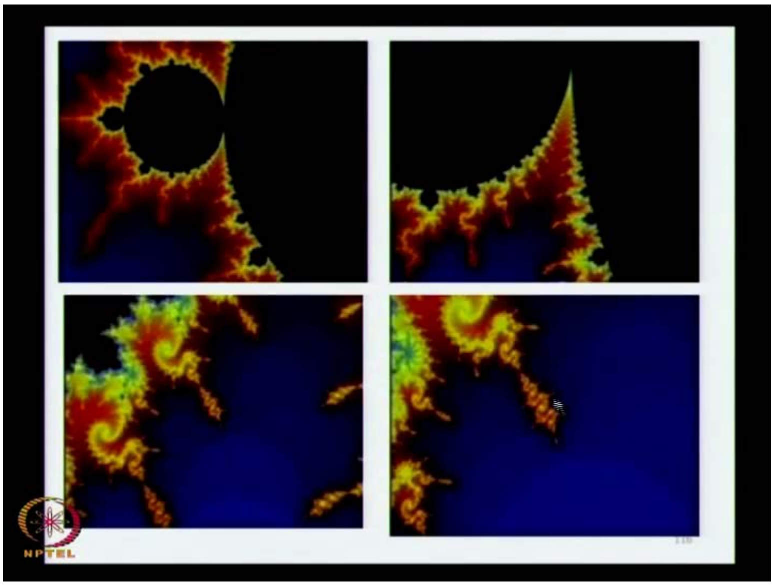
(Refer Slide Time: 38:38)



Zoom in further, there you are, go further, its more familiar, now right go further, and this is the kind of thing that happens that you zoom in and this is the self-similarity. And self-similarity reappears at a certain scaling factor and we have seen how this is connected with the logistic map and here is another set of pictures of the same thing.

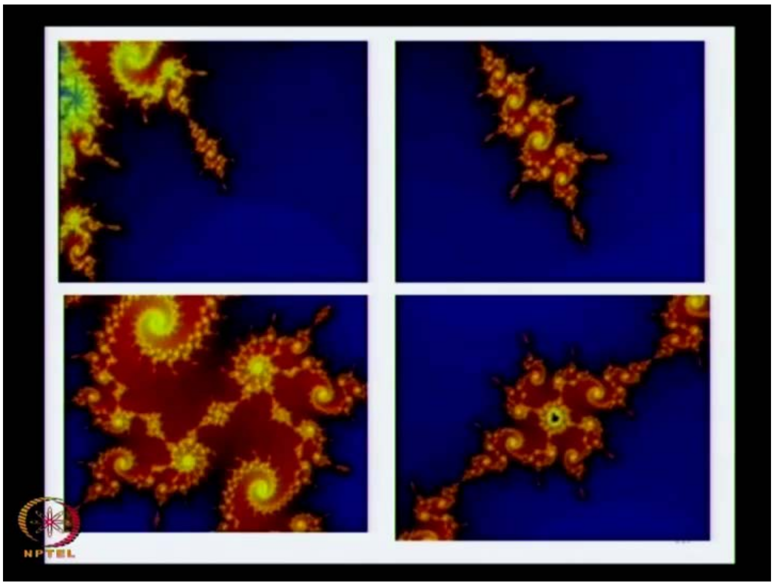
So, once again we are zooming in the central portion but these are slightly bigger slides. So, I think you can see it more closely. Now, we have zoomed in on this central part.

(Refer Slide Time: 39:18)



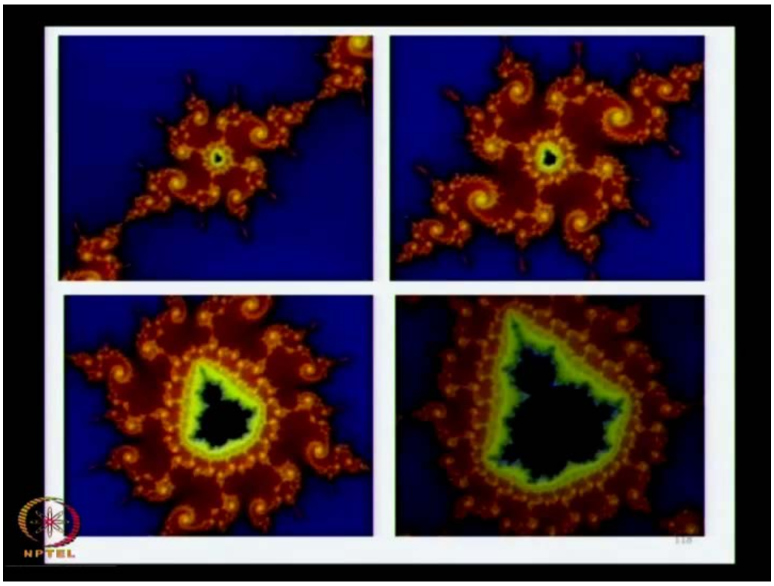
So, this is the part we are zooming in, so we see more detail over here now we are going to zoom in over here. So, you begin to see this. Now, you get one of these the serpentine pictures.

(Refer Slide Time: 39:40)



And then, if you watch this grow further, this is the one that you had, you watch it grow further, it becomes you zoom in, you look at more detail and then there is beautiful picture but then there is something at the centre over here.

(Refer Slide Time: 40:08)



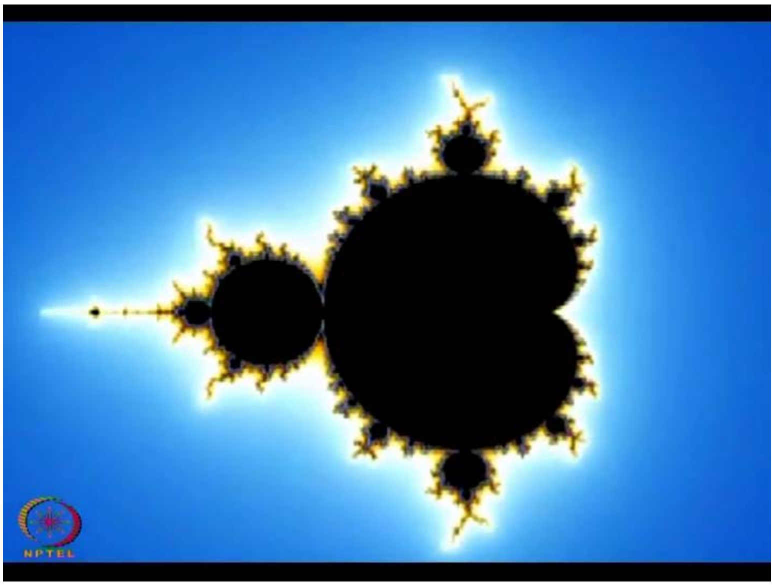
And it is going to grow on you as you watch it, there it is and you see that the Mandelbrot set that you began with just the same kind of picture starts coming out of it the only thing is that there is a different scale factor which is involved.

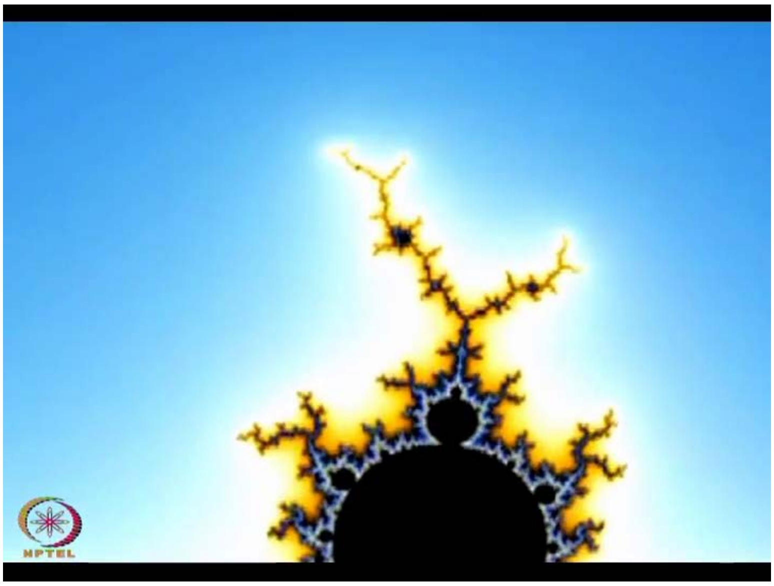
(Refer Slide Time: 40:20)

Mandelbrot Zoom

M. Eric Carr / Northlight Computing

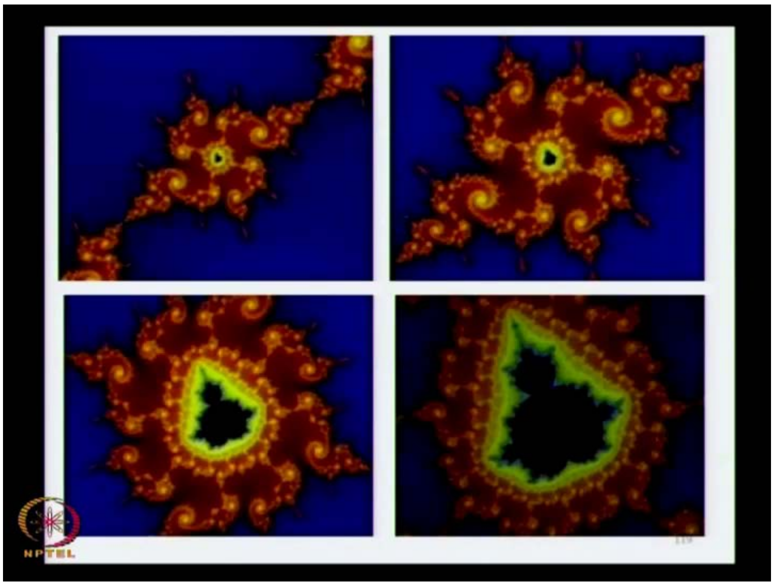
Music: "Mandelbrot Set" by Jonathan Coulton
(used under the Creative Commons license)





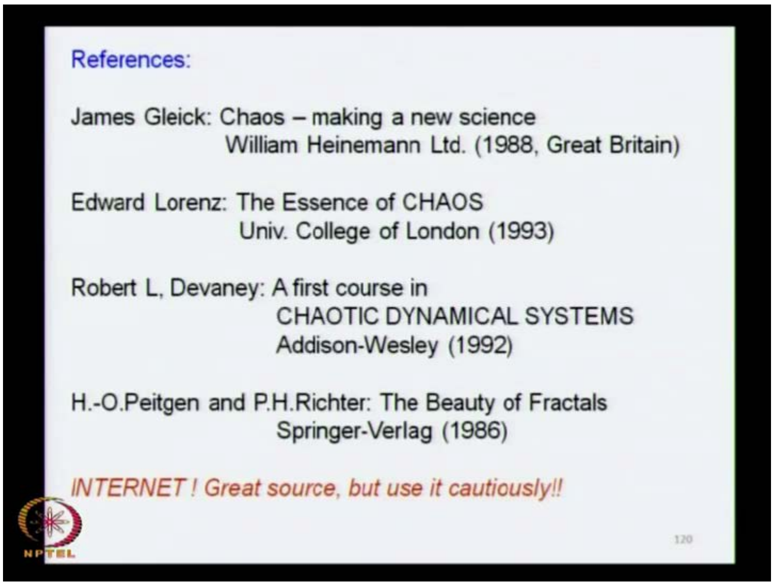
So, this is really amazing these are very fascinating objects and I will like to draw your attention to some links on the you tube, you can get there are some very nice, you know movies and there is one that we downloaded for demonstration over here ((music is been played over here 40:37 to 40:57)) what you really see in this video is infinite detail in the decorations and as you zoom in essentially, what you are doing is you are iterating and you are carrying out, further and further iterations and there is very challenging programming, which is involved in this because you must associate some colours, with these pixels. So, that it really highlights these features and make them look so beautiful for us these are extremely beautiful pictures that you can generate using Mandelbrot sets ((music is been played over here 41:34 to 41:45))

(Refer Slide Time: 41:45)



And what you see is that as you zoom in, what you have as decorations, which are on the boundary or the border of the first Mandelbrot set that you started out with. It has got so much information inside it that when you zoom in, you begin to see these extensions. These fibers and these fibers have got some central part and you zoom in on that and what comes out of it where if you zoom in further a shape is a form which looks exactly like what you really began with. Now, this treatment really needs to be qualified it does look like the Mandelbrot said that you began with yet it is different.

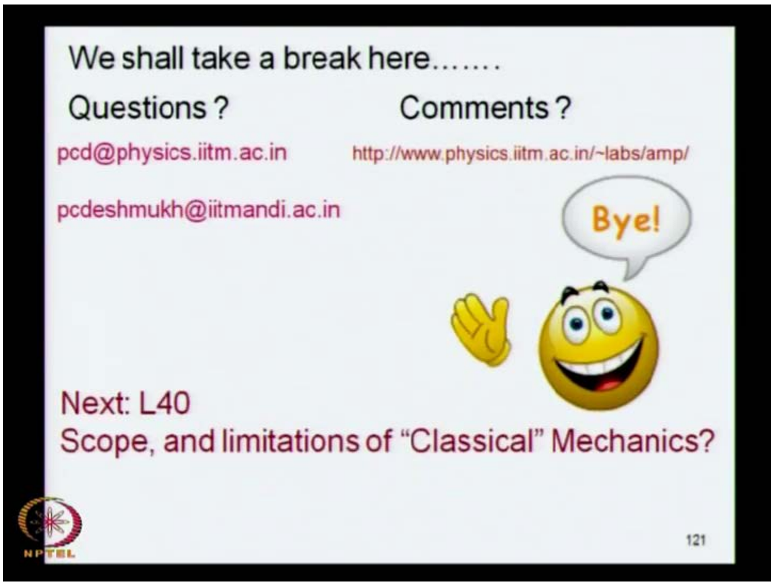
(Refer Slide Time: 42:50)



This is not infi0nitely self-similar. So, it has got a very complex behavior and I will conclude this class by the a few references, which you might want to read from James Gleick of course has an excellent book and now later editions have also come out. So, the first edition came in the 80s and then Lorenz has got a very nice book on the essence of chaos, and Devaney's book is very nice this is a first course on Chaotic Dynamical Systems which you might like to read. There is again some very beautiful pictures and this Peitgen and Richter's book the beauty of fractals. And then of course as I mentioned during our discussion that the internet itself is a great source, you always have to use internet with a lot of caution. So, never trust it fully.

But then it is a great source, it is a great source and I certainly recommend. So, I am not one of those who will say that do not ever learn from the internet yes you may but use caution. So, with that we conclude this discussion on chaotic dynamical systems. This pretty much brings us to the conclusion of this series of topics that we did as special topics or select topics in classical mechanics and in the next class I will generally provide a general overview of what exactly is the scope of classical mechanics and also what the limitations are. So, that will pretty much conclude, this short course which I have personally enjoyed very much I have learnt a lot while preparing for these classes myself and I hope that at least some of you have been able to learn at least, some during this course.

(Refer Slide Time: 45:05)



So, thank you all very much. Is there any questions? I will be very happy to take it, otherwise, its goodbye for now.

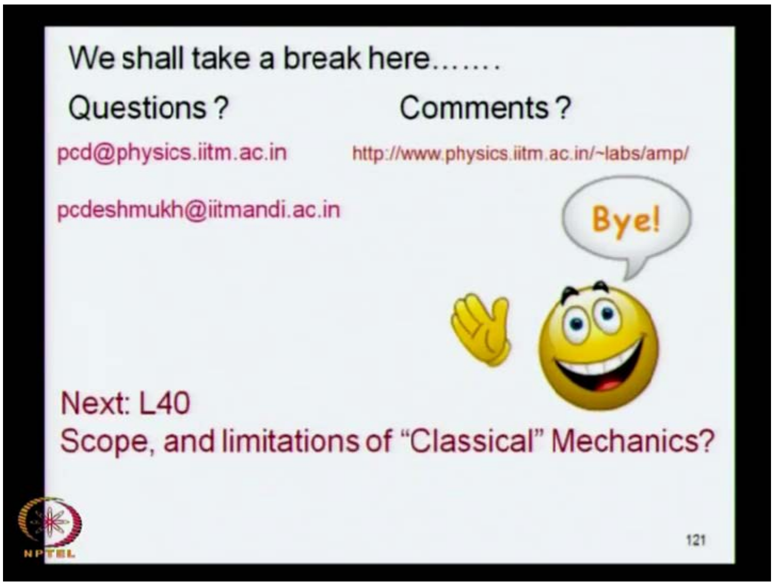
In the logistic map also the self-similarity it is repeated?

Yes

As we go along the r,

It is that is the common feature between the Mandelbrot set and the logistic map that is a reason you see that the there is a scaling factor. The Feigenbaum constant appears it appears in the Mandelbrot set through the points of the intersection of the Mandelbrot set with the real axis, so that is the connection.

(Refer Slide Time: 46:10)



Any other question, if not it is goodbye for now and then we will take a short break and then I will summarize the scope and limitations of classical mechanics. Thank you.