Select/Special Topics in Classical Mechanics

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Module No. # 01

Lecture No. # 05

Equation of Motion (iv)

Greetings, we will resume our discussion on the Lagrangian formulation of equations of motion.

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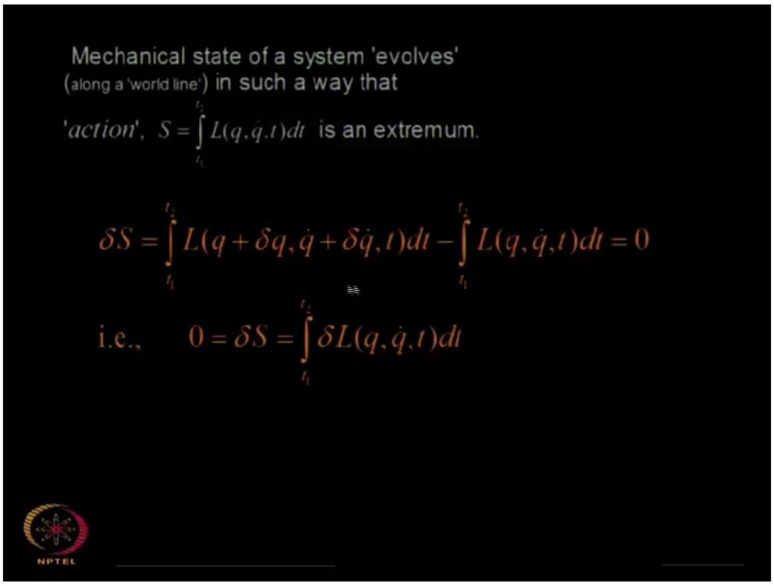

I will quickly recapitulate the essential consideration that we had undertaken. A mechanical system is considered to have evolved in such a way that action would be an extremum. That is the principle of extremum action as it is called. This is also referred to as the Hamilton's principle and this is a big departure from the Newtonian formulation. In Newtonian formulation, we agreed that a mechanical system evolves in such a way that it responds to the stimulus, which is imparted to the system by a force. So, there is a cause-effect relationship, which is the principle of causality. It is the important consideration and fundamental consideration in Newtonian mechanics.

The entire evolution of the mechanical system is described in terms of the Galilean principle of relativity. Secondly, any departure from equilibrium is explained in terms of the cause effect relationship, which is contained in Newton's second law; the principle of causality. So that is the governing principle of Newtonian mechanics. The governing principle over here has nothing to do with causality. It has nothing to do with force. It deals with the principle of variation. It stipulates you to set up what is called as the Lagrangian for the system. We have not yet mentioned as how to setup the Lagrangian. We have not gone there. We only know that it is some function of position and velocity. In general, it is a function of q, q dot and t.

What kind of a function it is? We have not described and never mind, we will work with it. At some point, we will have to answer this question, what is the recipe to set up the Lagrangian? So, we will take it up at that point. In the meantime, we carry forward the same starting point as in Newtonian mechanics. The mechanical system is described by a point in phase, space by position and velocity, rather than by position velocity. Directly, it is represented by a function of position velocity, which is the Lagrangian. So, the Lagrangian is the function of q and q dot. If you define an integral, it is known as the action. Action is the integral L dt from t 1 to t 2. Let this action be an extremum. This is the governing principle, on which the evolution of a mechanical system is described.

Now, what this led us to? These are some of the major steps that we discussed in the last class. So, I will not go through this in any detail, but quickly recapitulate that. If this is an extremum, then any variation in this would vanish and variation with respect to different paths in the phase, space in the position velocity phase space that a system can take. So, these are like different world lines or different paths that the system can take. Variations with respect to these parts will lead to an action integral, which will remain stationary and it will not change. If you go from one path to a neighboring path, which is somewhat close to the previous reference path, along which, the system will evolve. This differential increment in action would vanish, which we have restated in the equation at the middle of the screen here.

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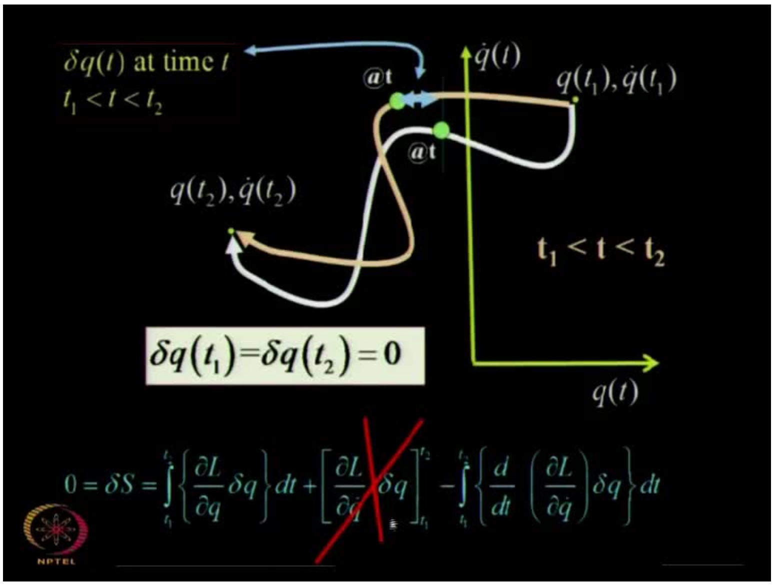

Now, these are the alternative paths that one can think of. So, this is a position, velocity phase space. We consider the evolution of the system from time t 1, where the point with coordinates q and q dot describes the state of the system. At a later time t 2, this system gets to this point along a certain path. This is some wiggling arbitrary path. It could even cross itself, whatever it is. It is some path that the system would take at later a time, t 2 and it gets to this point.

Now, I have show an alternative path over here in different color, which is slightly different color and it is beige. I think I am not good at identifying colors. So, I suspect this as beige. You can think of a slightly different path at some intermediate time between t 1 and t 2. The system would be at different points at some specific points on each of these two paths. So, let us say that if it were to evolve along the white path and if the system is at this point at a certain time t, then at the same instant of time, if the system has to evolve along the beige color path, the system would be here.

The horizontal distance between these two is the delta q at the time t and this delta q at the time t. As you can see from this figure, it is 0 at the start and at the end, it cannot be anything else. If it is non-zero, it can only be in between t 1 and t 2, but not at the start and not at the finish. Those points are fixed and there is no variation in delta q.

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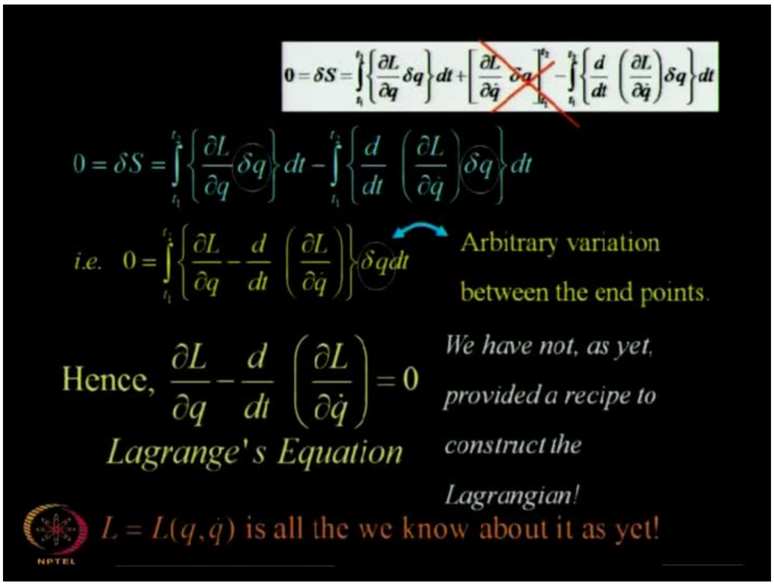

Here, you have the term, in which delta q vanishes and the remaining terms can be combined together. So, this term vanished and the remaining terms have been put together in this expression here. Now, you are left with this term, which is the first term amongst these three. The last term and these two terms are put together; this is an integral del L over del q delta q dt. So, this del L over del q delta q dt this is coming from the first term and from the second term, you have this expression here.

Now, in this, delta q is appearing in our analysis because we are choosing different paths. There is no reason to suspect the beige colored path that we had in our previous figure. It is the only alternative to the white colored path that we have considered. We could think of n number of alternatives paths or any different variation or anything, which is other than the path that the system would actually take. So, any of those paths could be taken. Therefore, this delta q is completely arbitrary. There is nothing special about that particular delta q or delta q dot that we had in the previous figure. That was just one of the many infinite variations that one can think of. So, this variation is being arbitrary.

Now, you are confronted with a very simple situation that you have a definite integral from t 1 to t 2. It is an integral over time, so that is indicated by this dt at the end. What is being integrated out? It is the product of two factors. One is a factor in this beautiful bracket and the other factor is delta q. The integrand is expressed as a product of two factors of which, one is completely arbitrary. No matter, what this arbitrary delta q is, the total integral must goes to 0. For arbitrary delta q, this can happen if and only if, the factors inside this beautiful bracket is identically 0. So that comes as a necessary condition that the principle of variation holds. This is the condition or the factor, which is inside this beautiful bracket. It must go to 0 and this is what we call as a Lagrange's equation.

How has it emerged? It comes as a necessary condition that action is an extremum. So, we begin with the requirement. We demand that action must be an extremum and then ask, if action is to be an extremum. What is the condition that must be satisfied? A certain equation must hold the term, which is inside this beautiful bracket. So, the equality of this term inside the beautiful bracket is the necessary equation, which must be satisfied and this equation is called as the Lagrange's equation.

Now, we have the Lagrange's equation between do not have the Lagrangian itself we have not discussed even as yet what is the recipe to construct the Lagrangian the only thing we know about it is that it is a function of q and q dot so we will have to of course address this because otherwise if we do not know what the Lagrangian is we cannot get it is partial derivative with respect to q and we cannot proceed nor can we get its partial derivative with respect to q dot because we have to know what is the mathematical dependence on q of L and how does L depend on q dot so unless we know this we cannot really do anything with this equation. So, let us see how we can get it.

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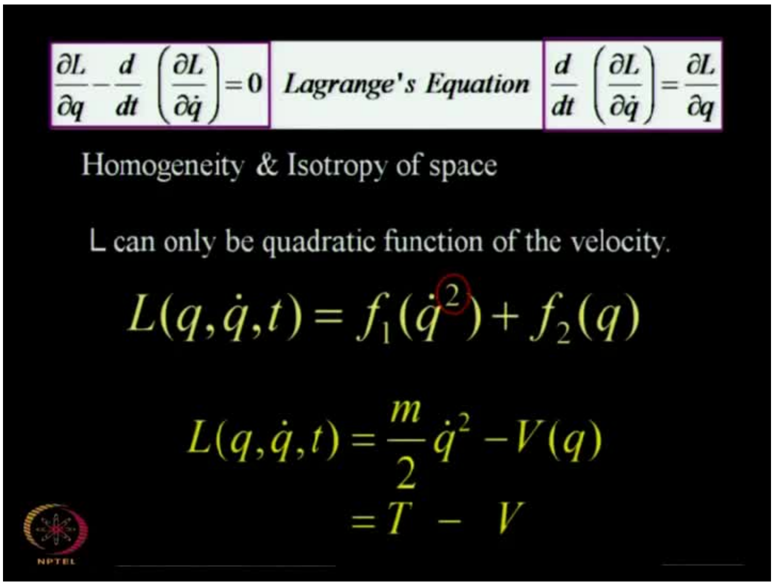

This is the Lagrange's equation. What we do know is that space is homogeneous and isotropic. The space that we are dealing with is isotropic. Its properties are same in all directions. Therefore, the Lagrangian cannot depend on any specific direction because properties of the space are the same. That is what is meant by isotropic. When Properties of the space is isotropic, the Lagrangian cannot depend on the velocity. So, we are looking for a Lagrangian, which depends on velocity, but not on the direction of velocity. So, what is the quantity that you can construct from the velocity, which does not depend on the direction of the velocity? It can be V dot; V being the velocity and the dot product of V with itself is V dot V. This does not depend on the direction of V and it is a scalar.

You can construct a quantity, if it depends on velocity. The only function of velocity, which is independent of direction is the quadratic function of velocity. The speed comes from the square root of V dot V. So, it must depend quadratically on the velocity. Therefore, we pick a function f 1, which is a quadratic function of q dot and q dot is our velocity. So, we expect the Lagrangian to be a quadratic function of velocity.

We take the simplest functions. As they say, do not trouble trouble, unless trouble troubles you. Why do you look for more complicated function? Look for the simplest function. What is the simplest function that you can think of? Position, some function of q known as 2 q. It is the simplest function of velocity, which does not depend on velocity; it will be a function of q dot square. That makes the Lagrangian a sum of f 1 and f 2. Here, f 1 is a function of q dot square and f 2 is a function of q.

Now, we have made some progress. We still have not identified, what is the function f 2 of q. Is f 2 equal to alpha q? Is it alpha q plus beta q square? Any polynomial function of q would meet the requirement that it is a function of q. So, we have still not pinned down the exact functions and that is what we will do. Now, we take the simplest function of q, which depends on the position. We know that the potential energy depends on position. So that is a mechanical property that we are familiar with. We expect it to be of great value in any discussion on mechanics. So, we take the potential and we propose that the function f 2 q to be chosen. So that is the negative potential energy. We make this proposal and we will still have to justify this proposal. We have made this proposal with a certain hope that it will turn out to be a good proposal. We have not provided reasons for it because we could think of other functions as well

At this point, we propose that the function f 2 q is chosen, so that it is negative of the potential energy function minus V of q. The simplest function of quadratic function of velocity is the kinetic energy, half mv square. So, we propose n 2 q dot square, in which we have identified the inertia. What we have done is we are choosing f 1 to be the simplest function of q dot square and the simplest function of q dot square is just some constant times q dot square. The constant that we choose is half the inertia half the mass. Why do we make that choice? We still have to rationalize and so we make this proposal that f 2 q be chosen. So that it is minus V of q. We propose that f 1 q dot square is proposed as m by 2 times q dot square. This will identify the Lagrangian's T minus V and we hope to justify this choice.

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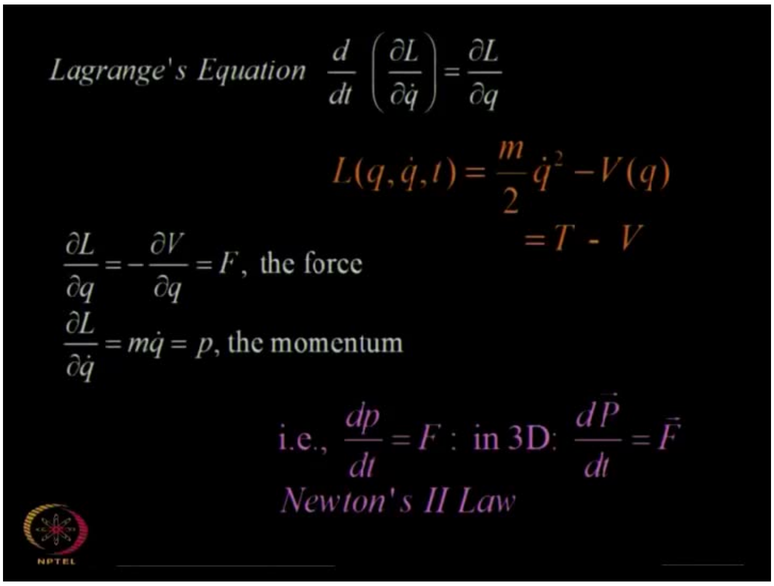

Now, what justifies this choice is a question. We must now answer. Now, just ask yourself, what will this combination be? Here, m by 2 q dot square will be the kinetic energy. This is now giving us the difference between the kinetic energy and the potential energy for the Lagrangian function. Now, you can take the partial derivatives because you have written L as a function of q. So, the partial derivative of L with respect to q will be simply minus del V by del q. You can also take the partial derivative of the Lagrangian with respect to q dot because that will simply be m by 2 times twice q dot. So, you can determine these derivatives. You can put these partial derivatives in the Lagrange's equation and ask yourself, what do you get? The partial derivatives of the Lagrangian with respect to q, which is the negative of the partial derivatives of the potential energy function with q. The negative gradient of the potential in Newtonian mechanics is the force. So, we get the force from this term and from the other term, we get the momentum. In the Lagrange's equation, we have the time derivative of the momentum. So, you will get dp by dt, which is the Newtonian force and essentially, what you get is Newton's second law. This is what justifies our choice of f 1 and f 2.

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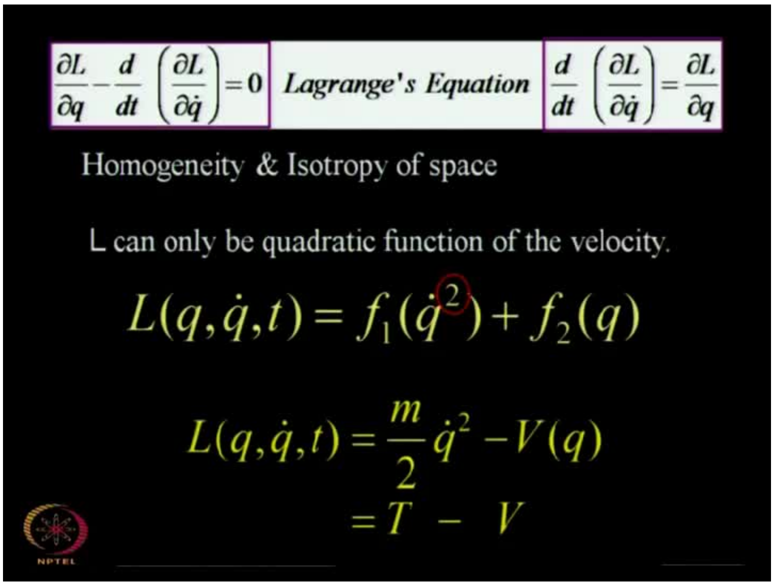

Let me go back to the previous slide one more time that the proposal here was the Lagrangian. It can be expressed as a quadratic function of q dot and a sum of another function of q.

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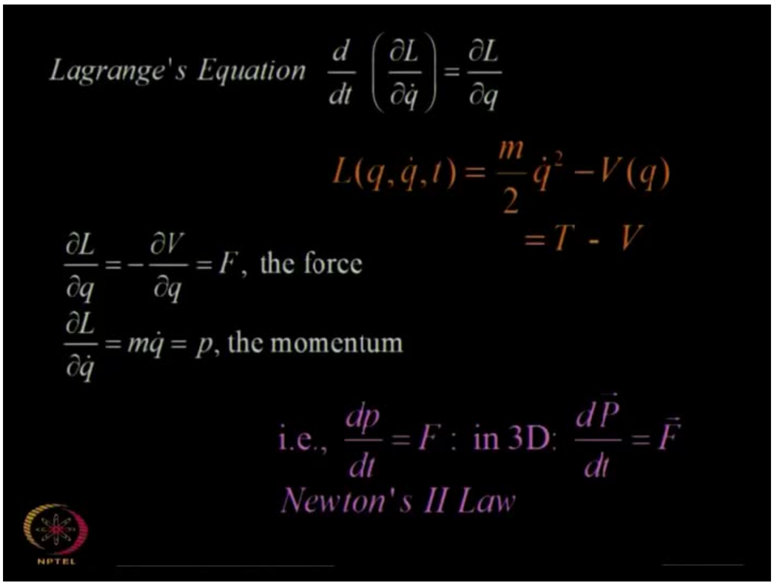

We choose these functions to be the kinetic energy and the potential energy with the minus sign over here. This particular choice turns out to be extremely profitable because it gives us results, which are completely consistent with Newtonian formulation of mechanics. Now, Lagrangian formulation or a Hamilton’s formulation is an alternative formulation. It is not going to give us anything different. If it contradicts with Newtonian mechanics, there will be trouble. So, it is important that the results which come out of this are completely consistent with Newtonian mechanics. This consistency is now explicitly manifest. What makes it possible for us to attain this consistency? It is the proposal that the Lagrangian has written. It is m by 2 q dot square and we did not provide any rationalization for the factor m by 2. We said that we would take the simplest function of q dot square, which is just a linear function of q dot square like alpha times k q dot square or k times q dot square. I suggested to choose the proportionality to be half the inertia, m by 2.

Now, we see that the particular choice m by 2 q dot square gives this del L by del q dot to be inertia times a velocity, which is the momentum. That choice turns out to be a productive and a useful one. This is what enables us identify the Lagrangian as T minus V. I will not say this as a very standard recipe to construct the Lagrangian. When you go to more complicated situations, one has to keep track of many other fairly subtle factors. You have to ask- is this Lagrangian invariant with respect to Galilean transformations? If you are doing relativistic mechanics, you will have to ask. Is Lagrangian, which is set up will be consistent with the Lorentz transformation? So, there are many additional questions that you need to answer.

In this introductory discussion, there is no room for such involved issues. We do have a basic foundation and this works for a good part of mechanics. You can define the Lagrangian as T minus V. This also suggest us a recipe to do other problems because here after, we will not recognize momentum as mass times velocity. It is a Newtonian definition of momentum. We will rather use the Lagrangian definition of momentum and it is the partial derivative of the Lagrangian with respect to the velocity. So, del L by del q dot is our definition of momentum. Whenever we talk about momentum, we will talk about del L by del q dot and not mass times velocity. This is what is called as the generalized momentum.

The definition of generalized momentum is that it is the partial derivative of the Lagrangian with respect to the velocity. We can see a suggestion towards this and from this correspondence, which we see over here.

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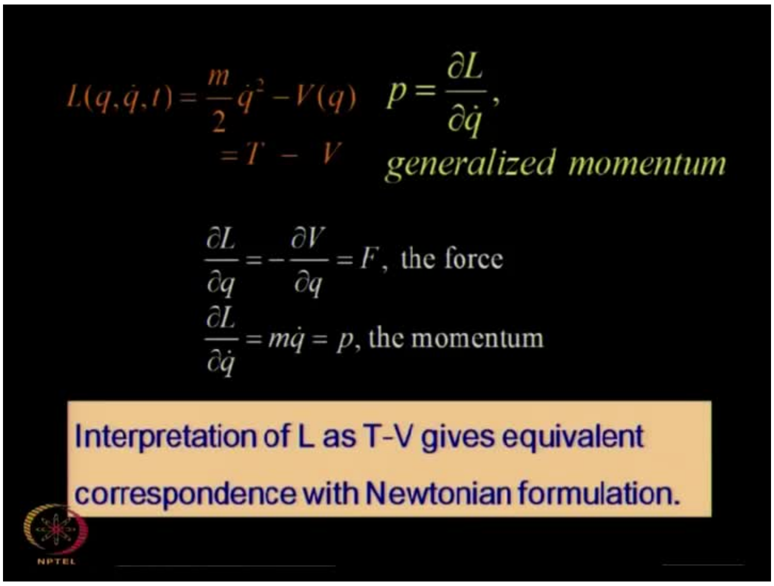

This is our generalized momentum, del L by del q dot and that is the term we shall use in the future. What we have achieved by this particular recognition of Lagrangian as T minus V? It is a complete equivalence with Newtonian formulation. The foundations are completely different. The foundation in Newtonian mechanics is the principle of causality, the cause effect relationship, the linear response formalism.

The foundation of Lagrangian mechanics is the principle of variation. They have nothing to do with each other. These are completely independent formulation and you cannot derive one from the other. These are independent formulations, but they must converge and they must provide you with the same results as long as you work within the domain of classical mechanics. Things change, when you go over to quantum theory and that is a different story. In classical mechanics, Newtonian formulations and the Lagrangian or Hamiltonian formulation must give you the same results.

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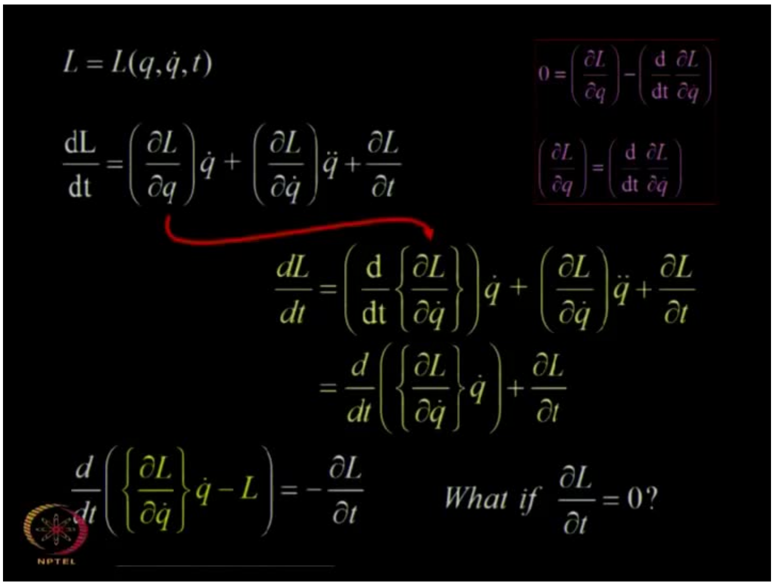

Now, let us write a more general Lagrangian, which is a function of q q dot and t. You will very soon see, when t must be included and when it need not be included. If you have an isolated system, there are no missing degrees of freedom. I discussed this in a different context earlier, but I think that sample is a very useful one. If you are moving an object of a surface; you know that there is friction at the surface and if you set up the equations of motions only for this bottle, but do not include the particles of the table, which are also interacting with this, then you do not have an isolated system. Then this system is interacting with something, which is not taken into account in your equations of motion and it will show up as missing degrees of freedom.

When you do not have any missing degrees of freedom, you will not have the Lagrangian to have any explicit time dependence. The Lagrangian will always have implicit time dependence and the implicit time dependence means that the Lagrangian depends on time by the virtue of its dependence on position and also by the virtue of its dependence on velocity. It is because of that reason, the Lagrangian depends on q and q depends on time. The Lagrangian becomes a function of time. So, such a dependence is what is called as implicit dependence. If there are missing degrees of freedom, there may be an explicit time dependence. So, this is the general expression for the time derivative.

The Lagrangian will come from its implicit dependence on time, through its dependence on q, which in turn depends on time. So, there is dq by dt over here. Dependence of the Lagrangian on the velocity depends on time. So, there will be a d by dt of q dot, which is q double dot, which is the middle term that you see. There will be a possible explicit time dependence, which is shown in the last term. So, this is the complete expression for the total time derivative of a Lagrangian.

This is the Lagrange's equation. These two brackets must be equal to each other. Their difference goes to 0. So, del L by del q is d by dt of del L by del q dot and this del l by del q is replaced by d by dt of del L by del q dot in this step over here. You have this term, which is del L by del q dot time the second derivative of q. You have the explicit time dependence of the Lagrangian coming in the third term. If you look at these two terms, they are algebraically completely equivalent to the derivative of a product of two functions. The derivative of a product of two functions is one function multiplied by the derivative of the second plus the second function times the derivative of the first. It is q double dot and so you have only rewritten this algebraically as d by dt of a product of these two functions. The last term shows up over here.

Essentially, if you take this term, it is a total derivative of the Lagrangian. This is a total derivative of a product of these two functions. So, I move this term to the left and it comes to the other side or rather I move this del L by del t to the other side and move this to the right. So, just a rearrangement of these terms gives us this relationship over here. The total time derivative is the difference between the first term and the Lagrangian. The first term is a product of two functions.

We are looking at the total time derivative of the Lagrangian. It comes from implicit dependence on time through the explicit dependence on position and velocity, which are explicit functions of time. A possible explicit dependence of the Lagrangian on time directly comes from the missing degrees of freedom. So, we make use of the Lagrange's equation. From the Lagrange’s equation, we know that these two terms are equal. The difference is 0. We express del L by del q as the time derivative of this quantity over here. We already know the generalized momentum and that is our definition of the generalized momentum. We combine the first two terms and write this as a time derivative of a product of two functions.

Now, if we rearrange the terms in this last equation, we get the time derivative of this difference equal to minus del L by del t. So, you move this del L by del t to the other side and move this d L by d t to this side. So, you have only minus del L by del t on one side. This tells us something very interesting. In case, the Lagrangian does not have any explicit time dependence, then the partial derivatives of Lagrangian with respect to time will vanish. So, we recognize the condition under which, the total time derivative of a quantity vanishes. If del L by del t is equal to 0, which is the meaning of the Lagrangian not having any explicit time dependence. If the Lagrangian does not have any explicit time dependence, del L by del t will go to 0 and d by dt of this bracket must vanish. If the derivative of a certain quantity goes to 0, the quantity must be a constant.

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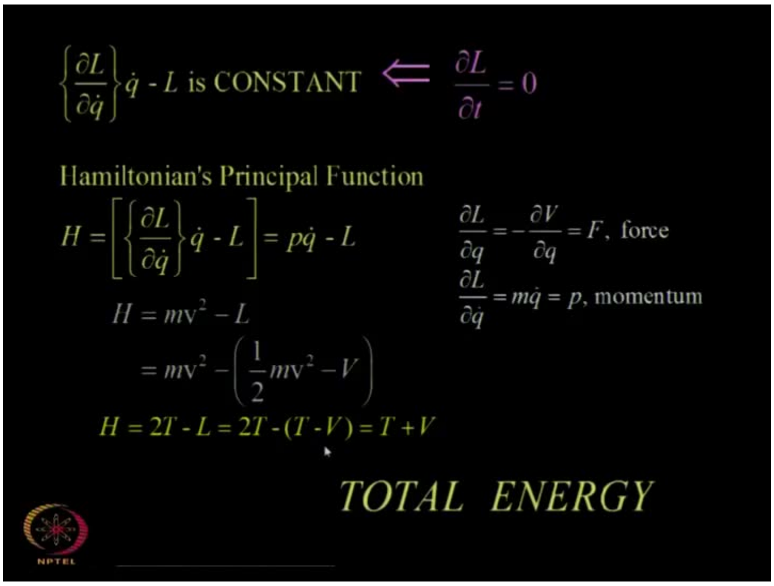

Here, del L by del t is 0. Now, it guaranties that this particular function is the constant and this function is called as the Hamilton’s principle function. This is the definition of Hamilton’s principle function. It is del L by del q dot times q dot minus the Lagrangian and del L by del q dot It is our definition of the generalized momentum, so the definition of our Hamilton's principle function is that it is p q dot minus the Lagrangian. This is the definition and meaning of Hamilton's principle function.

We know that it is a constant, if and only if the Lagrangian does not have any explicit time dependence. We have also seen the correspondent with Newtonian mechanics through these partial derivatives, which we have shown earlier. We can ask ourselves- what is it that we can learn further from this correspondence? If you see this n p q dot minus L from the Newtonian perspective by exploiting this correspondence, you see that p q dot is mass times velocity. It is mv square and you have mv square minus Lagrangian.

The Lagrangian is kinetic energy minus the potential energy and therefore, the Hamilton's principle function is 2T minus the Lagrangian. It is 2T minus the difference of kinetic energy and the potential energy, which essentially gives a meaning to the Hamilton's principle function. What comes out of this is T plus V, which is obviously the total energy of the system. The Hamiltonian, which we have defined is the Hamilton's principle function. It can be immediately identified with the total energy. It is a conserved quantity, a constant quantity and the constant c is not a matter of faith.

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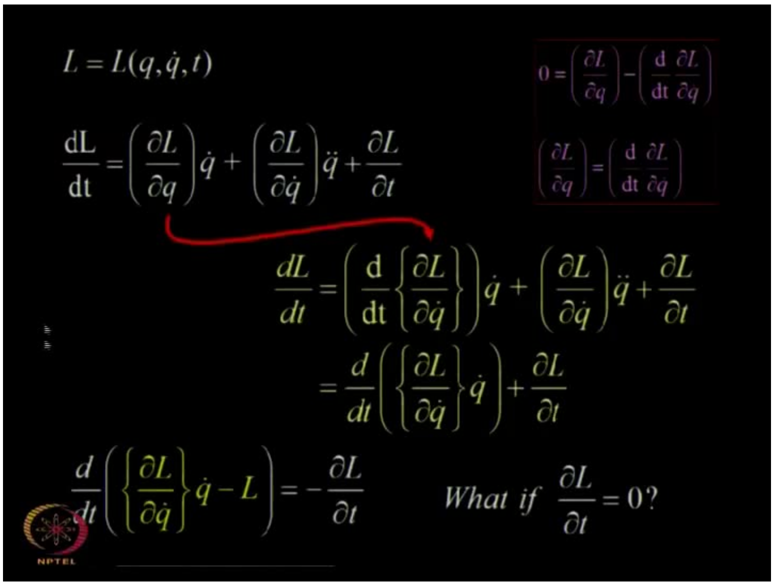

It is expressed by the previous result that it has come from this d by dt of this Hamilton's principle function. It goes to 0, whenever del L by del t is 0.

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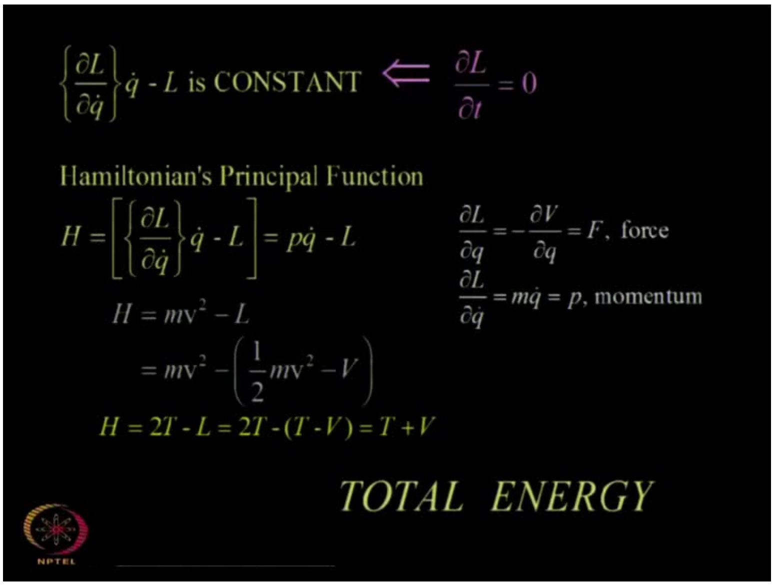

It is not just a matter of faith, it has come as a result of this particular condition. Now, what is the condition that the Lagrangian must be? Whatever time dependency it has, it must be only through q, q dot and not directly. So, there should be no explicit time dependence that the Lagrangian can have.

If the Lagrangian does not have any explicit time dependence, will the Hamilton's principle function correspond to the total energy? It is a conserved quantity. Do you see the Noether's theorem talking to you in this result? The Lagrangian is independent of time is a symmetry that the Lagrangian will remain the same, whether you formulate it yesterday or today or tomorrow or even day after tomorrow. It is the symmetry with respect to translation along the time axis, any time in the past or present or future. The Lagrangian remains the same because no information is missing; no degrees of freedom are lost. You cannot do that if you were to include friction because every time you do this, there will be differences in the interaction. They are not taken explicitly in your analysis, so they are left out.

When the Lagrangian is independent of time, you have symmetry with respect to change in time. Time is the parameter with respect to which the Lagrangian is invariant. This invariance is the symmetry principle. Associated with this symmetry, there is a conserved quantity, which is the energy. The energy is a constant and you can see a manifestation of the Noether's theorem associated with every symmetry principle. There is a conservation law and vice versa. Now, this has come so neatly from the Lagrangian formulation.

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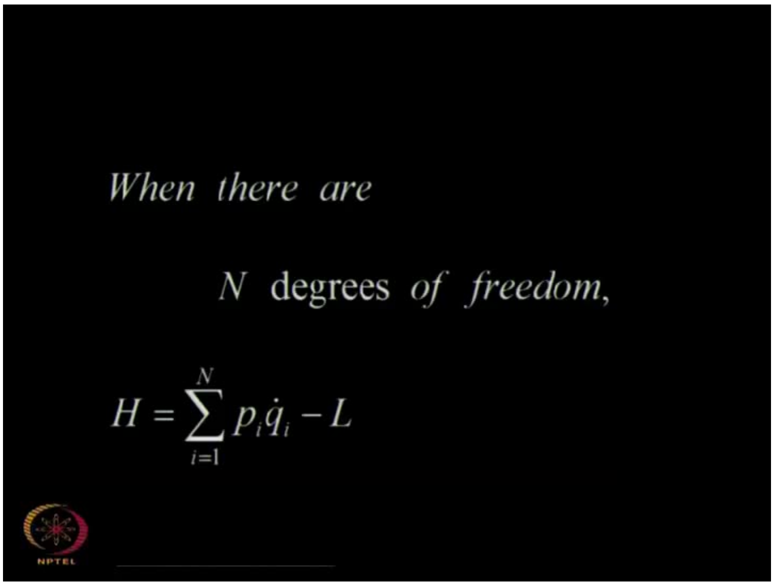

You can generalize this. If there are N degrees of freedom, you can put a subscript p and q dot and write the Hamilton's principle function as the summation over i going from 1 through N. So, this is a straightforward extension. Let me remind you that the object of this course is not to get into the details of Lagrangian formulation or into the details of Hamiltonian formulation. This is an introductory course in classical mechanics. This is what students would take as one of the first courses in classical mechanics after high school or at some undergraduate level. At that level, one would not take up any problems involved in classical mechanics in the Lagrangian formulation or Hamiltonian formulation.

We will not take up systems with many degrees of freedom and so on. We will only illustrate some examples. The focus of this course is to reveal an alternative formulation of classical mechanics as much as Newtonian mechanics. It is inspired by the principle of inertia. Galileo's interpretation of equilibrium goes into the first law. Newton's interpretation of departures from first law or from equilibrium generated by interactions revealed by the cause effect relationships contained in the equation f equal to m a, which is the second law. An alternative formulation of mechanics exists and it has a charm of its own. It has a beauty of its own, it is based on a completely different principle. It is this alternative, we are introducing to the students in this course.

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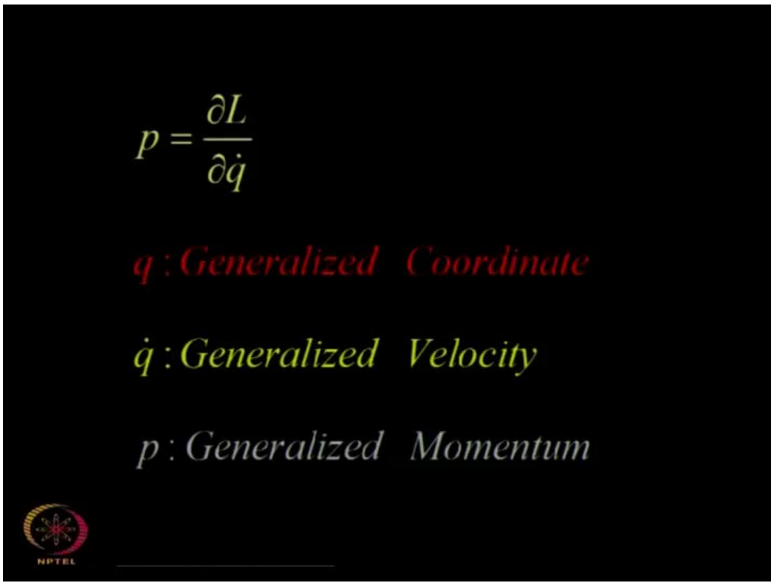

The quantities that we are going deal in this formulation will be the generalized coordinate, the generalized velocity and the generalized momentum. So, we shall no longer talk about coordinate, if it is just a position coordinate in a Cartesian geometry. The definition of the generalized momentum requires the Lagrangian to be set up because unless you setup the Lagrangian, you cannot determine its partial derivative with respect to q dot. So, you set up the Lagrangian and take its partial derivative with respect to q dot. Now, you know what the generalized momentum, until then it is unknown to us.

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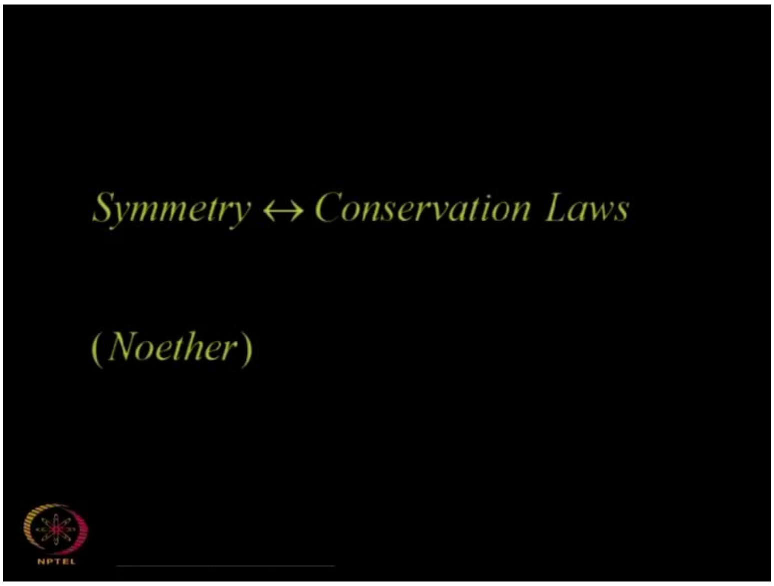

We have seen that it reveals a connection between symmetry and conservation law. We have already seen such an illustration. Because of the fact that this formulation is so general, this can be easily extended to other expressions of the Noether's theorem and I will illustrate another one.

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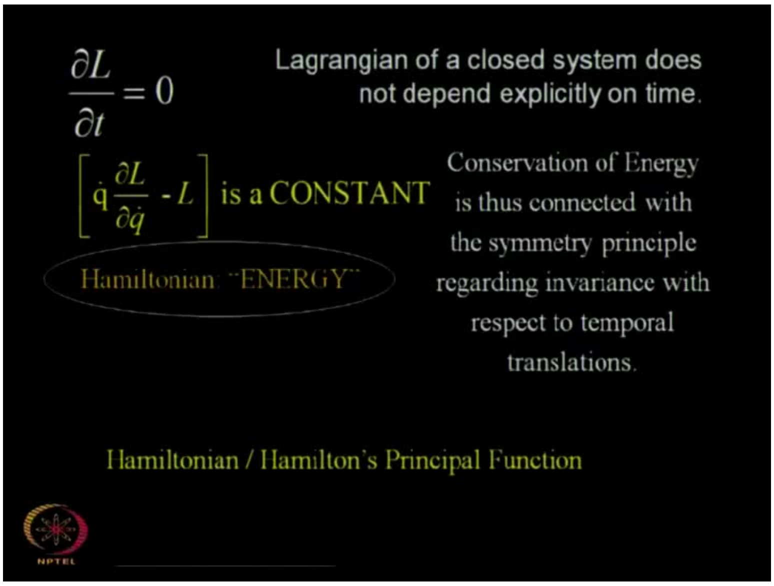

This is the one that we have already seen that the Lagrangian of a closed system does not depend explicitly on time. Therefore, del L by del t is 0 and the symmetry principle associated with this is the conservation principle that the Hamilton's principle function is a constant and you call this as the total energy. So, there is a connection between symmetry and conservation law that we have seen over here. It is in this context that you call the Hamilton's principle function as the energy. Otherwise, you simply call it as Hamilton's principle function, but when it is a constant, you call it as the energy.

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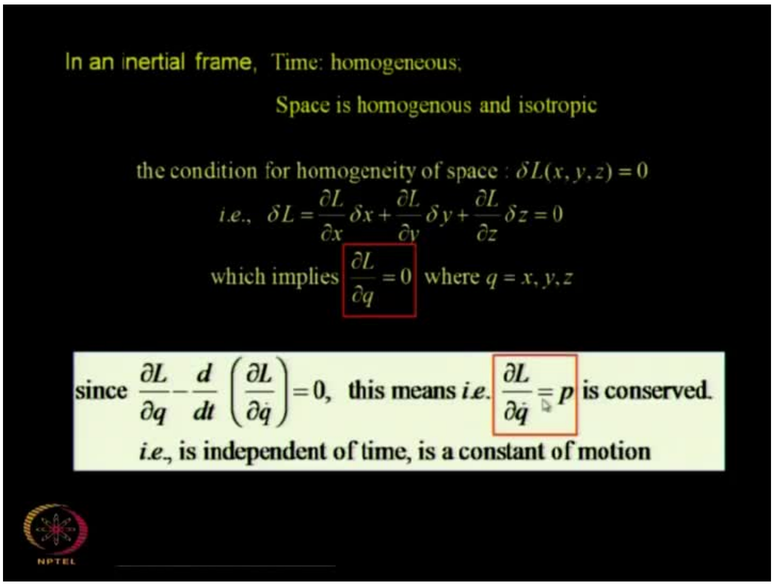

Now, let us consider another symmetry. We will now consider symmetry with respect to space. We considered symmetry with respect to time, del L del t being 0. It is symmetry with respect to time that is it is the same Lagrangian any point of time. So, its partial derivatives vanishes. Now, we consider changes in the Lagrangian because of translational displacements of the system. Is there any invariance coming because of this? This invariance would obviously come in homogeneous space. If the space is homogeneous, you cannot have a different Lagrangian. So, delta L changes in the Lagrangian because of changes in the coordinates.

Just to illustrate this, I will make use of the Cartesian coordinates, which is the simplest or most familiar coordinate system that people use. So, this delta L will be determined by partial derivative of L with respect to x times the displacement, delta x plus similar term from the dependence of the Lagrangian, if any on y. Likewise, this term involves the z coordinate. If this delta L is equal to 0, no matter what delta x is or no matter what delta y is or no matter what delta z is. The sum of these three terms is not going to 0 because delta x, delta y, delta z takes some special values, which adjusts with each other. So that the sum of the three terms can go to 0.

You can always have 2 plus 1 minus 3 equal to 0 sum of three terms going to 0. Then, none of the three terms is individually 0. This cannot be the case because delta x, delta y, delta z are arbitrary displacements. So, no. matter what delta x is, no matter what y is and no matter what delta z is, if delta L is 0. This is the statement of symmetry and it is coming from the homogeneity of space and del L by del q must be 0, no matter what delta x, delta y, delta z is. So, this result del L by del q equal to 0 comes from the consideration of space being homogenous. Now, we makes use of the Lagrange's equation because del L by del q minus time derivative of this term is equal to 0 and del L by del q is 0. It means that this time derivative of this factor is 0 and this is of course the momentum. The time derivative of the momentum is 0 and if the derivative of a function goes to 0, the function must be a constant. So, we have exactly got a conservation principle that is absolutely right.

You see that the conservation of momentum keeps coming every time, when there is symmetry. Earlier, we met the symmetry with respect to time. Now, we meet symmetry with respect to translational displacements in homogeneous space. We have discussed this earlier in the context of Newtonian mechanics. Here, you see how it comes very naturally and easily out of Lagrange's equations. You know that whenever the Lagrangian is independent of this coordinate q, is… What would happen, if del L by del q goes to 0 every time? The Lagrangian is independent of a degree of freedom q and the corresponding momentum will be constant.

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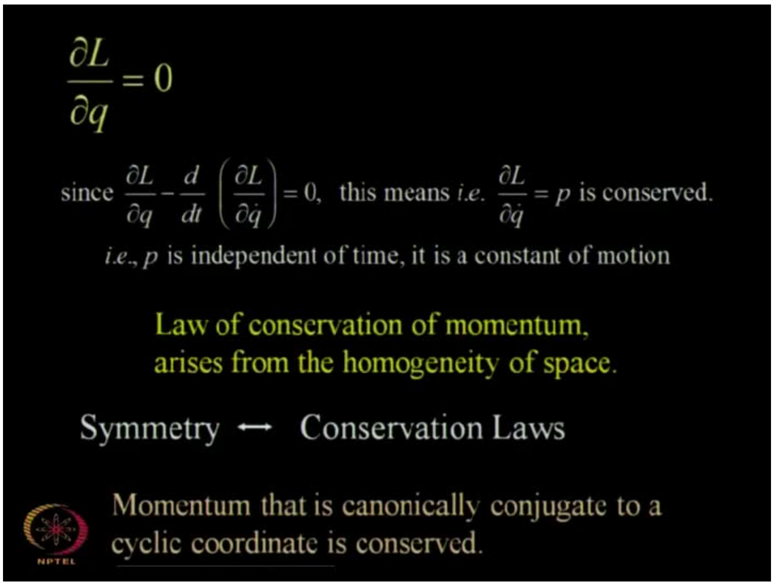

Now, this is a connection between symmetry and conservation law. So, del L by del q equal to 0 guarantees that del L by del q dot equal to p is conserved. The law of conservation of momentum arises from the homogeneity of space. This is very nicely seated by this very well known law that the momentum, which is canonically conjugate to a cyclic coordinate is conserved. Whenever, the Lagrangian is independent of a coordinate, this coordinate is said to be a cyclic coordinate and that is the meaning of a cyclic coordinate. Cyclic coordinate is a one, which does not appear in the Lagrangian.

If the Lagrangian is independent of the coordinate, its partial derivative with respect to q vanishes. Therefore, the partial derivative of the Lagrangian with respect to q dot is the corresponding velocity, which gives you the corresponding momentum. This correspondence is implied by the term of canonical conjugation. This canonical conjugation refers to this particular explicit correspondence between momentum and position. So, the momentum p is said to be canonically conjugate to a coordinate. It is given by the partial derivative of the Lagrangian with respect to the corresponding velocity del L by del q dot. So that is the momentum, which is canonically conjugate to a cyclic coordinate is conserved.

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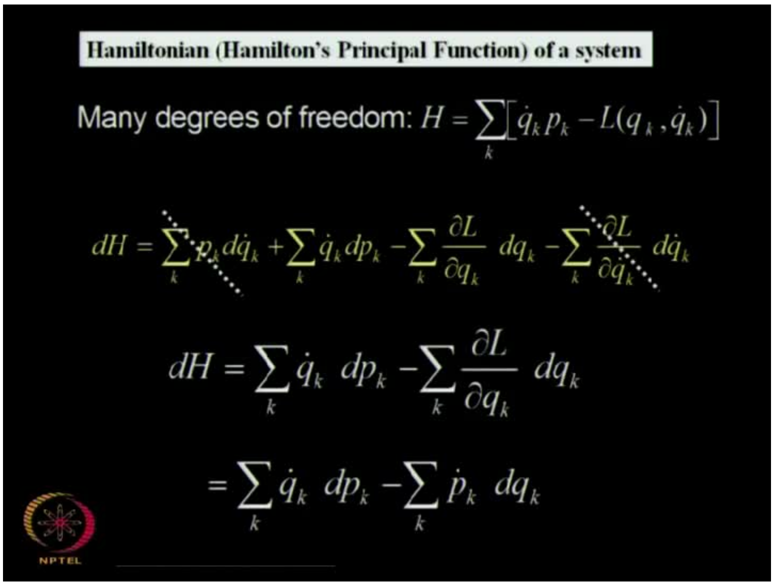

Like I said, extend this for many degrees of freedom. You will have to sum over all the degrees of freedom. So, it is essentially the same original expression for the Hamiltonian, which you wrote for all degrees of freedom. Now, you make demands of the properties of the Hamiltonian. What would be a differential increment in the Hamilton's principle function? So, this dH will get it and it is the increment from changes in q k dots, which are changes in the velocities, changes in the momenta as the product is coming in the Hamiltonian, because of changes in the Lagrangian.

So, the increment in dH of the Hamilton's principle function comes from the change in this product. This product can change because of either of these two terms. So, it will be the first function times increment in the second and the second function times the increment in the first, so they both are contributors. So, the change in the product, q k dot p k will be a sum of first two terms. You must subtract from this because of this minus sign. The increment or the change in the Lagrangian is due to the dependence of the Lagrangian on q, a change in q and also the dependence of the Lagrangian on the velocity and the change in velocity. Of course, you must sum over all the degrees of freedom.

Each term has been summed over k, but there is something we ought to notice. Some of you would have already noticed that if you look in the first and the last term, both involve increments in the velocity dq dot. For each degree of freedom, a degree of freedom is necessarily independent. The corresponding coefficients must be equal because this one comes with the plus sign and this one comes with the minus sign. So, these two terms must cancel each other. They must cancel each other because del L by del q dot is nothing but the momentum itself.

What is p? It is the partial derivative of the Lagrangian with respect to the velocity. So, those two terms cancel and you are left with the middle two terms. In the middle two terms, the coefficient of dq in the second term is del L by del q. It is equal to the time derivative of the momentum. As we know from Lagrange's equation that del L by del q must equal to d by dt of the corresponding moment. So, del L by del q is now identified as p dot. The subscript k must be kept off the track and you must sum over all the terms.

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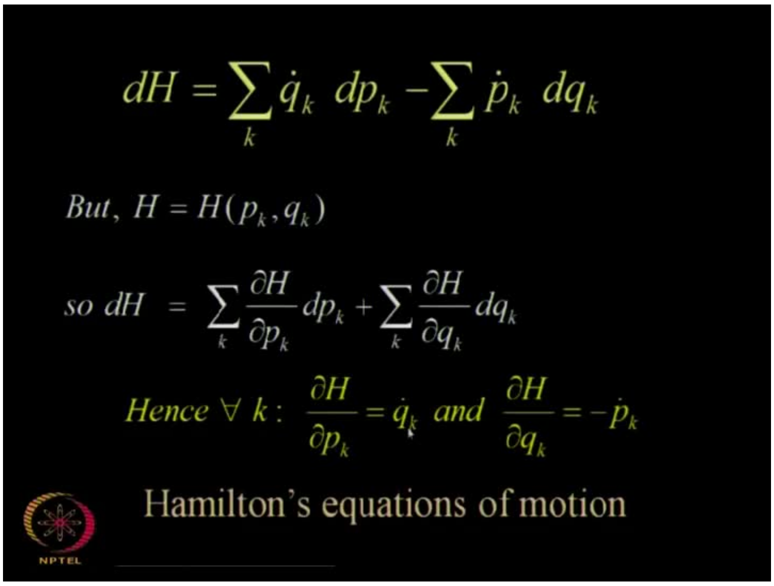

Now, this is our expression for the increment in the Hamilton's principle function. Hamilton's principle function's in a way as Hamiltonian mechanics is done. It must be written as a function of position and momentum and not as a function of position or velocity. Now, these are two different formulations - the Lagrangian formulation and the Hamiltonian formulation. I am going to rub this point and I am going to hyphen this any number of times, even at the cost of repetition.

The Lagrangian formulation must be used in terms of position and velocity. The Hamilton's formulation must be used in terms of position and momentum. So, we express the Hamiltonian as a function of position q and momentum p. It is either position or momentum or both momentum and position. It is the same thing and does not matter which one you write before. These are classical dynamical variables and the order in which you write them does not matter.

This dependence on position and momentum gives the expression for the differential increment in the Hamilton's principle function as dH equal to the partial derivatives of the Hamiltonian with respect to p times the increment in p. Where can this increment come from? It can come only from increments in p and q because that is what the Hamiltonian depends on. The corresponding coefficients will obviously be the partial derivatives of the Hamiltonian with respect to the corresponding Hamiltonian parameters. So, del H by del p times dp plus del H by del q times dq. I think it is always good to see this physics behind the partial derivative, so that you do not look at it as just mechanical algebraic mathematics or calculus through each term. It is physics, which is talking to us that there is an increment in Hamiltonian. This increment can only come from the increments in momenta and from the increments in the coordinates because those are the quantities on which, the Hamiltonian depends on. So, dH becomes this and this must hold good for each degree of freedom, independently.

This is how we have two expressions for the differential increment in the Hamiltonian. One is this and the other is this. Both involve arbitrary increments in the momenta and arbitrary increments in the coordinates. They all add up to the same increment in the Hamilton's principle function. So, the corresponding coefficients must be equal and they better be right for each degree of freedom.

The coefficient of dp in this is q k dot must be equal to the coefficient of dp in this term. It is del H by del p so del H by del p. It becomes equal to q dot and the coefficient of this dq, which is p dot must be equal to this coefficient of dq over here. There is a minus sign over here, so this is carried over here. So, these equations that you see at the bottom, del H by del p equal to q dot and del H by del q equal to minus p dot are known as Hamilton's equations of motion.

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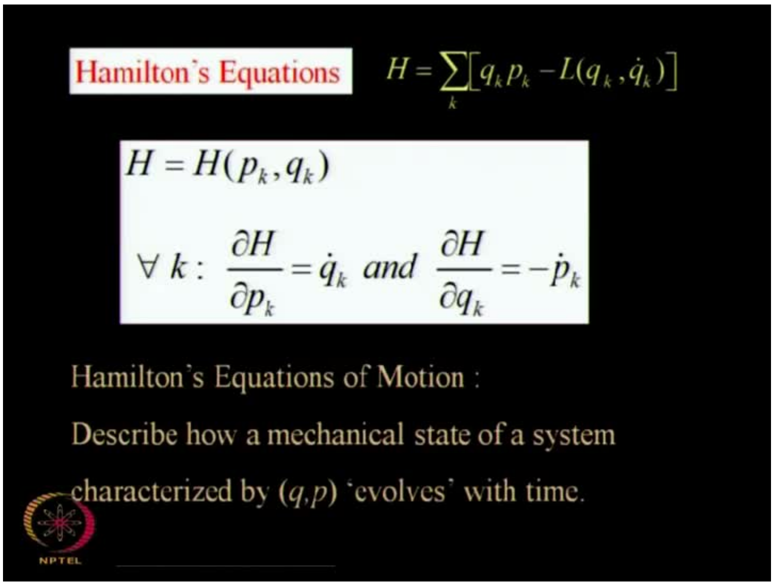

What do they tell us? They tell us how q and p evolve with time. That describes how a mechanical system evolves with time. That is the fundamental problem in mechanics. How do you describe? How do you characterize the state of a mechanical system? How does this evolve with time? It is time dependence and it is time derivative. What mathematical equation provides this time evolution? It is an equation of motion; it connects the position, velocities and accelerations. It provides an equation of motion and these are known as Hamilton's equations of motion.

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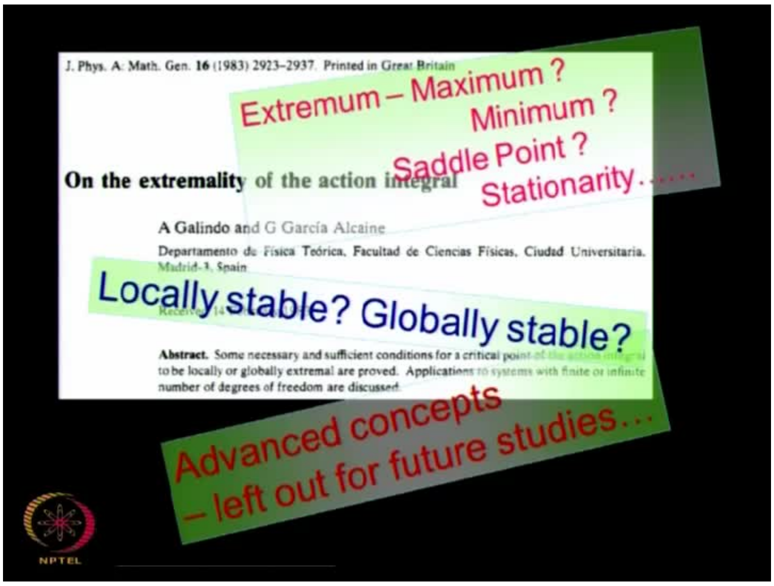

There are some issues, which I have not gone into a great depth. I mentioned them for those who are interested, intelligent and who would like to take proactive steps to read further, but I am not going to discuss that in this course. I have emphasized that this alternative formulation is based on the principle of variation. It requires the action integral to be an extremum. I pointed out this extremum could be a minimum, it could be a maximum, it could be a saddle point. In more general terms, it is called as the stationary point.

There are complex questions that one can ask. Is the stationary, local or global? These are fairly complex issues, fairly advanced concept. They go well beyond the scope of this course. I will not get into those details. Usually, we know that even from elementary calculus, we distinguish between a minimum and maximum, only by looking at the second derivatives. So, there are additional differentials, which come into play. I will not discuss that and with this, we are ready to conclude this particular class.

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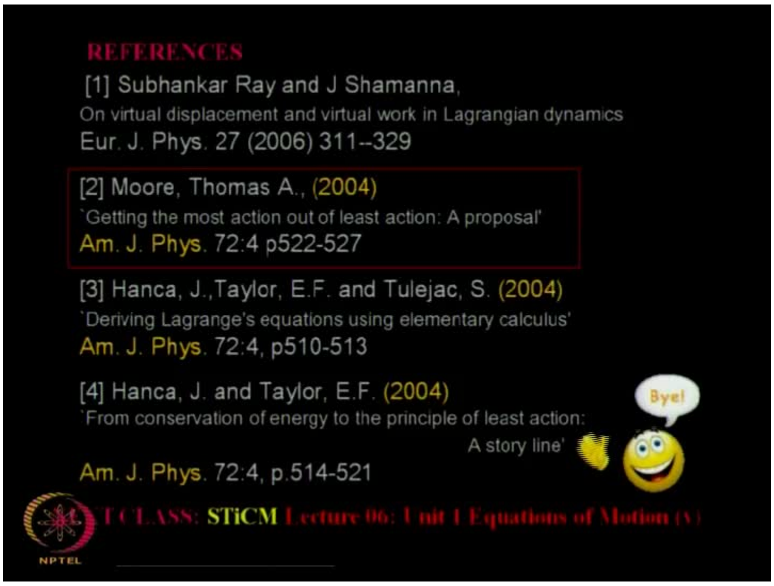

I will mention a few references. There are some excellent papers in the American journal of physics, which you will find very instructive. In particular, I like this title very much - Getting the most action out of least action. This is really a very catchy title for a paper because the least action that he is talking is obviously the principle of least action and not the least action. We always like to do, which is do nothing. So, the least action is what Thomas Moore refers to. He gets a lot of it and the title is very nice - Getting the most action out of least action. There are other papers by Taylor and his collaborators, Hanca and so on. You might find these papers quite instructive and with that we will conclude this class.