Select/Special Topics in Classical Mechanics
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Module No. # 01
Lecture No. # 06

Equations of Motion (v)

Welcome back. This will be the concluding lecture on unit 1 which is on equations of motion.

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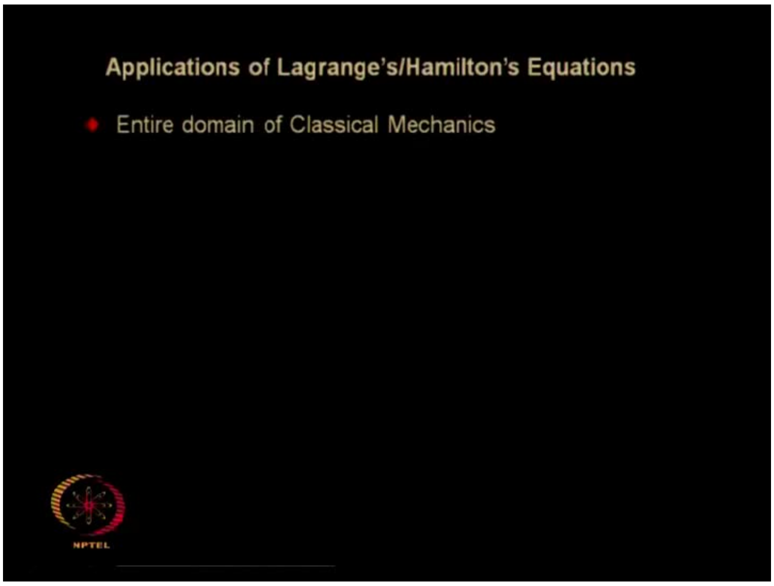

We will take up discussions on the applications of Lagrangian Hamiltonian formulations. It covers the entire domain of classical mechanics, but before I go further, let me answer a question that was asked, as to what is a saddle point? The context of this question is the extremum; can I use the board?

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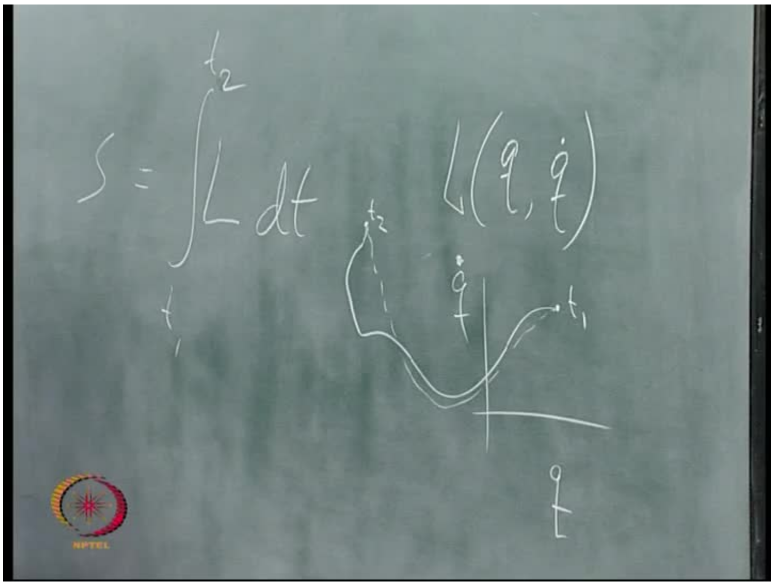

So, we have, this is our action and this action integral from t 1 to t 2 is an extremum so that any variation in this is 0. This is not 0; a variation in this would be 0. What kind of variation? a variation which comes about because of considerations of the Lagrangian being dependent on position and velocity. So, if you take a system, you draw a phase space and you have velocity on one axis and position on the other. So, these are two independent parameters; so, I plot them in orthogonal coordinates.

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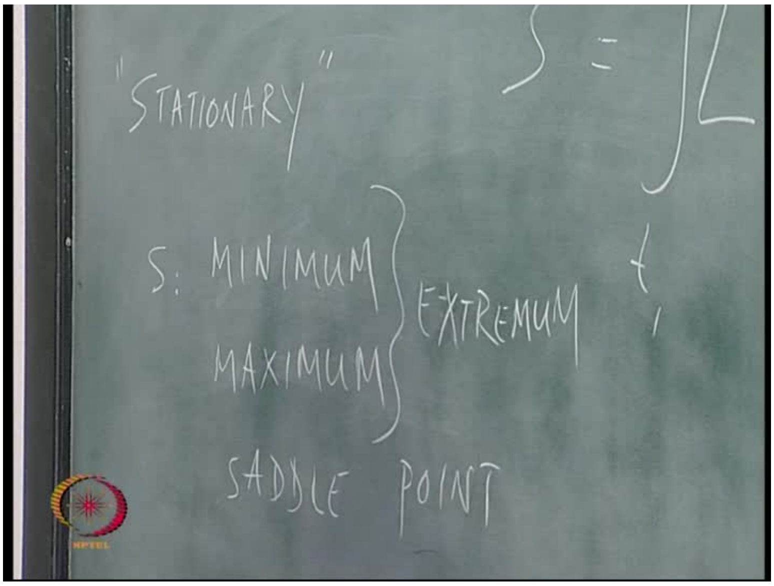

From the start, at time t 1, if the system is expressed by a point in the phase space and it goes to some other point at a time t 2, then if you consider any alternative paths, then with respect to variations in this path, the Lagrangian generates an action integral, and this integral is stationary. Stationary is a rather general word; it means that the action S could be a minimum; it could be a maximum. In general, these two are combined together. These are often referred to as extremum and it could also be what is called as a saddle point. Now, a saddle point comes into play when you have more than 1 degree of freedom.

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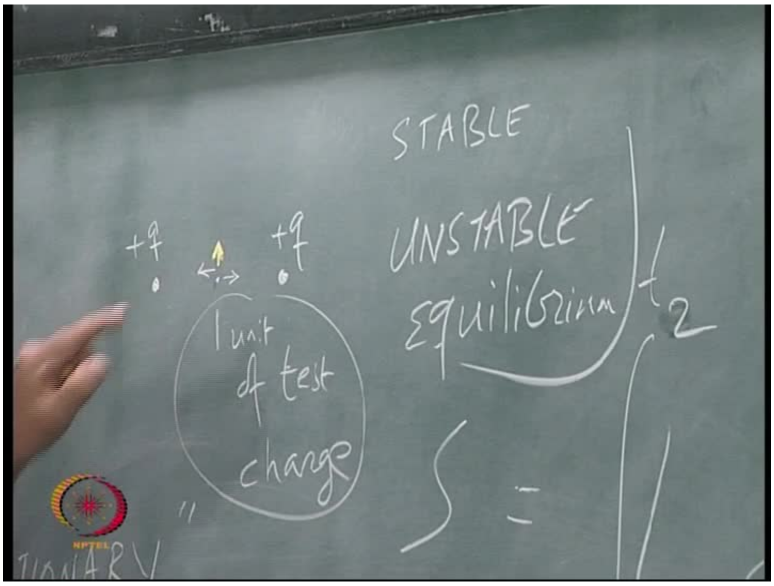

Consider 2 degrees of freedom. Let us say you have got 2 charges; both are unit positive charges and you keep one test charge exactly in the middle. Now, is this point a point of equilibrium? If it is, what kind of equilibrium is it, is a question that we will ask. So, both are exactly equal charges plus q and this is also plus q and this is one unit of charge; this is a test charge (Refer Slide Time: 03:58)

Now, what is going to happen is, if it is exactly at the center and you leave the system alone, what will happen to the test charge? It will remain where ever it is. It will be governed by the law of inertia; it is in a state of equilibrium. So, it will continue to be in its state since no other forces acting on it. If you displace this charge even slightly toward here - to the right, what will happen? and then what will happen? what will how will the What will be the dynamics? It will be sent back to the equilibrium point. If you move it to the left, what will happen? It will be sent back toward the equilibrium point. So, any displacement along this axis will create a situation in which the test charge will tend to come back to its original point.

So, the tendency to come back to the original point is what makes this point, a point of stable equilibrium. Because it is an expression of stability; if you displace it to the right, there is a tendency for it to come back; it is an expression of stability. Now, if you displace it not along the axis but along this point, along this direction, what will happen? It will be both are positive test charges; this is one unit of positive test charge. [Noise](Refer Slide Time: 06:34).

It will not come back. What will it do? It will be pushed further. The horizontal components of the forces will kill each other and the vertical forces will add up, and this test charge will be sent away further and further and further and further, till it is lost. So, any displacement along this axis tends to bring it back to the original point; whereas, any displacement along this point tends to push it away. So, that is an expression of instability or what is called as an unstable equilibrium.

So, now if you ask the question - is this point which is exactly in the middle of these two charges, is it a point of stable equilibrium or is it a point of unstable equilibrium? This question can be answered only in the context of where the displacement is. If it is along the axis, it is a stable equilibrium. If it is orthogonal to that, it is an unstable equilibrium. It is very similar and those of you who have done horse riding which is a great sport and an excellent sport and a wonderful sport; I strongly encourage it.

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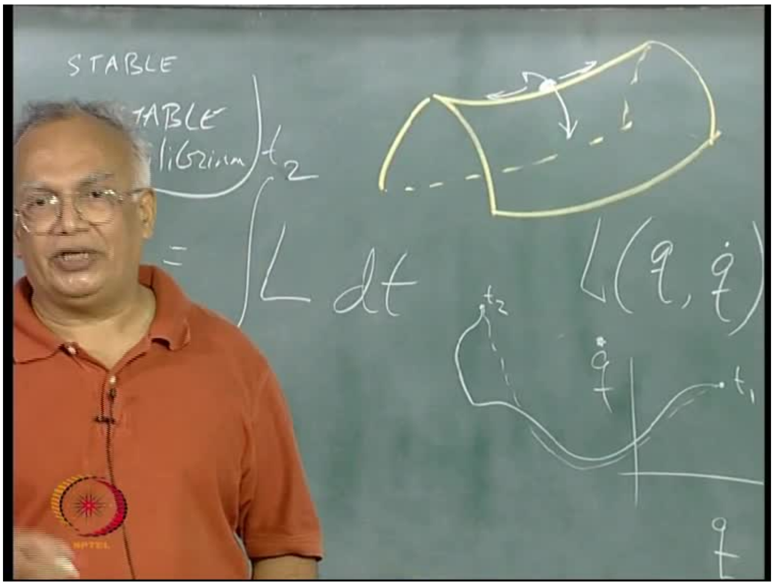

What do you do? You know, when you ride a horse, you ride on a saddle; you mount a saddle on the horse and you mount yourself on the saddle. Now, if you keep a marble at this point, if you displace it along the length of the horse, it will tend to come back. If you displace it along the width of the horse, it will tend to go away. That is a reason such a point is called as a saddle point because it comes very literally from what equilibrium at the center of a saddle is. So, that is a saddle point. A saddle point is an equilibrium point, but it is not unambiguously a point of stable equilibrium nor is it an unambiguous expression of an unstable equilibrium.

So, it depends on which degree of freedom is being changed. Is it along this axis, the orthogonal axis or any other? In a multi-dimensional space, when you have more degrees of freedom, of course a saddle point acquires a much more complex connotation, but the essential meaning is this.

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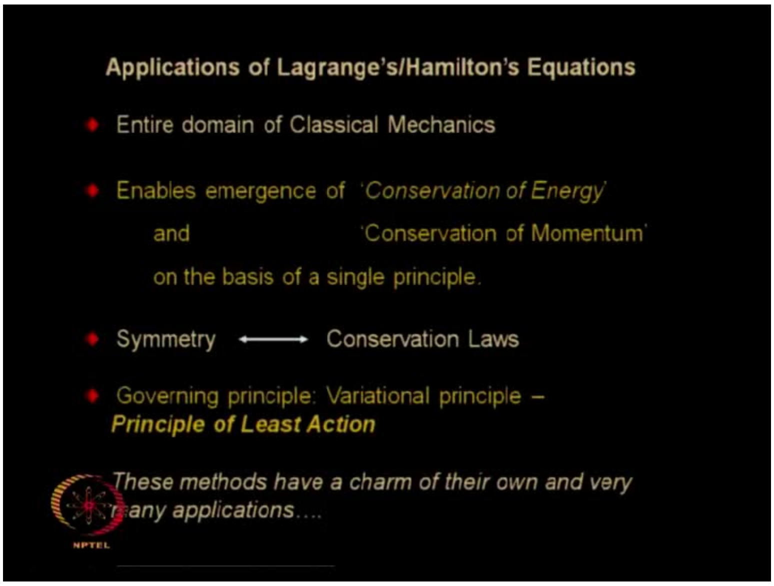

The Lagrangian principle of variation, the Hamilton's principle of variation really refers to an extremum condition. It is not getting into the details of whether that equilibrium is a point of stable equilibrium or is it an unstable equilibrium because at a stable equilibrium you have a minimum, and unstable equilibrium you have a maximum. So, we have not gotten into these issues about whether it is a minimum or a maximum, or is it a saddle point. These issues, we have not dealt with. These are more complex issues and a detailed discussion of this require the consideration of higher derivatives, not just the first order differentials but higher order differentials; It is well beyond the scope of this course; so, we will not get into those things. So, in general, it is a stationary point; that is the correct expression.

Popularly it is referred to as the principle of least action, but it is not automatic that it will always be least. It could also be a maximum. It is like Fermat's principle. We always think that light will take the least time, but it can also take the most time. If you have elliptic mirrors and if you have got a source of light at the center, this light could go along the minor axis of the ellipse which is the least distance, but it can also go along the major axis which is the most distance. So, it can be either least time or most time. So, Fermat's principle also accommodates both the possibilities.

So, anyhow, let me now proceed with some applications of Lagrange's and the Hamilton's equations. It covers the entire domain of classical mechanics. We have already discussed in the previous class that conversation of energy, conservation of momentum, everything comes out neatly out of a single principle, which is the principle of variation. We do not have to learn these as separate laws. Everything comes from the same principle and you interpret this principle in the context of different symmetries; you get different conservation laws, but everything is coming out of just one single principle which is the principle of variation. So, it connects symmetry and conservation laws very nicely.

At an introductory level, it gives us a good introduction to the Noether's theory; the governing principle is the variational principle. Principle of least action; again, I have used the term least over here. But for the sake of correctness, accuracy, and more generality, we should certainly refer to it as a principle where it is the stationary property which is referred to rather than a minimum or a maximum, or a saddle point. It is based on the principle of variation and these have applications in many other branches of physics.

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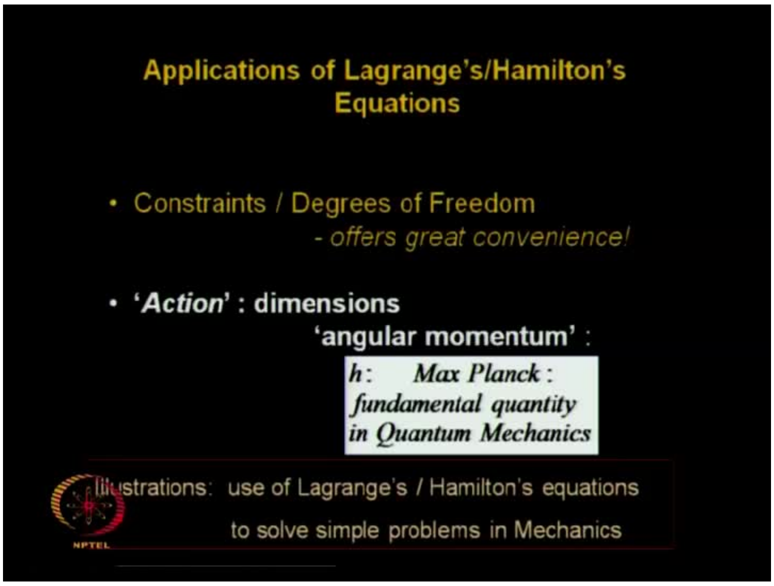

It also enables a treatment of constraints very nicely. I am not going to do it in this course, but constraints, I will mention what they are. Whenever you have a body like this and I can throw this body anywhere, hopefully the lid closed, and then no matter how it twists and turns, and flies, the distance between this point and this point will always remain the same (Refer Slide Time: 13:52).

It could be here, it could be here, it could fly somewhere else; which means that the trajectory of this point is not completely independent of the trajectory of this point. There is some correlation, some coupling, which always must be kept track off, which means that the degrees of freedom, if we look at two points, it is not really 6, but it is less; what make it less are these connections. These connections are called constraints. What one does in Lagrangian and Hamiltonian formulation is not deal with the actual coordinates; we deal with what are called as the generalized coordinates. In these generalized coordinates, you eliminate the constraints so that you deal with the minimum degrees of freedom which must be analyzed to describe motion. So, the Lagrangian formulation helps us do that.

One big advantage is that if you look at this action integral, it is what is this integral. It is integral of the Lagrangian over time. What are the physical dimensions of action? Lagrangian is t minus v. So, the dimensions of the Lagrangian are energy dimensions. So, the dimension of action is the dimension of energy into the dimension of time. So, the product of the dimension of energy and product with the dimension of time; if you do it in your notebook, in 2 seconds you will find that it has got the dimensions of angular momentum; energy times time, energy into time will have the dimensions of angular moment.

Angular momentum is a very fascinating creature. It has got very fascinating properties. It is this, which becomes a fundamental unit in Quantum theory. In fact, the Planck's constant which we write as h is sometimes called as a quantum of action. It has got the dimensions of angular momentum and this plays a very crucial role in quantum theory, as I am sure all of you would be familiar with, but those details are for a discussion in a course in Quantum theory, but that action is a fundamental quantity in Lagrangian and Hamiltonian formulation gives us a good starting point to adopt this to quantum formulation. Newtonian formulation really does not give us that strength.

Newton's laws can solve all the problems that Lagrange's problem can solve, but they cannot take you easily to quantum theory. It does not mean that you can derive quantum theory out of the Lagrangian formulation. No. you cannot because it is different scheme altogether and I will comment on it toward the end of today's class hopefully. It belongs to a different scheme, but this formulation is better adapted; not that it provides a platform to derive quantum theory out of it, but this formulation is better suited to build quantum theory on it.

In quantum theory, you will never deal with forces; you will deal with potentials. So, force will not come explicitly in quantum theory; it does not play a big role; potentials do so. This principle of variation in the Lagrangian and the Hamiltonian formulation provides a good starting point to build quantum mechanics.

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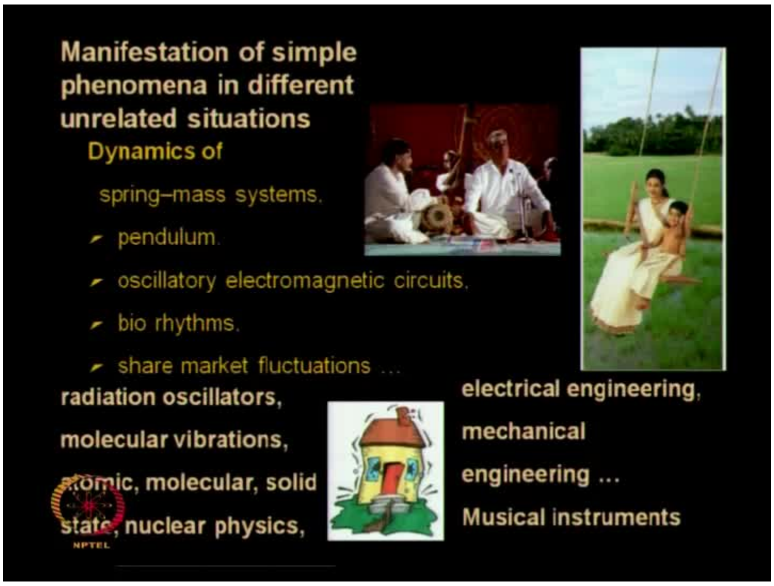

As I have been submitting all along that this course is aimed at under graduate students who are getting their first introduction to mechanics, it will not deal with any advanced applications of Lagrangian mechanics or Hamiltonian mechanics. The purpose of this discussion is only to reveal this alternative formulation of mechanics, which is in fact much better than the Newtonian formulation because of some of the reasons that I mentioned.

So, I will just show one or two simple illustrations of how it is actually used to solve mechanical problems and we will conclude this unit with these applications. What you see in this picture is very different kind of phenomenon; a mother and child enjoying the swing; you could be listening to music, and there are some connections in all of this. You know there is a common physics which runs in to all.

This is the physics of vibrations - the physics of oscillations. These oscillations could be: the oscillations of a mass spring oscillator; These could be the oscillations of a swing; These could be the oscillations in electromagnetic circuits; You can have an LC circuit - conductance and a capacitor. In biology, there could be other kind of oscillations; the market fluctuations where again some parameter is changing. Market goes up and it goes down, and I do not know anything beyond it, but do not tell my wife; she already knows it.

So, there are different kinds of oscillations which one can treat using certain common formalism and it is this commonality. I do not even know if there is a word like that. It is this commonality that we will discuss; what runs across these apparently different process? They are of importance in classical mechanics in quantum mechanics. You have radiation oscillators; you have molecular vibrations and atomic physics, nuclear physics, different branches of engineering, musical instruments, whatever.

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The study of oscillation began with whom else, The father of Experimental Physics, Galileo. What he did was to observe the oscillations of the chandeliers at the Pisa cathedral. You would have done in the experiment in your lab, perhaps with a stop watch or a clock and complained that it probably did not work. Galileo did this experiment by checking his pulse because that was the clock which he had; by checking his pulse, he could detect that there is a time period which seems to be constant. Of course, we know that there are friction losses etcetera; so, leave aside - the details.

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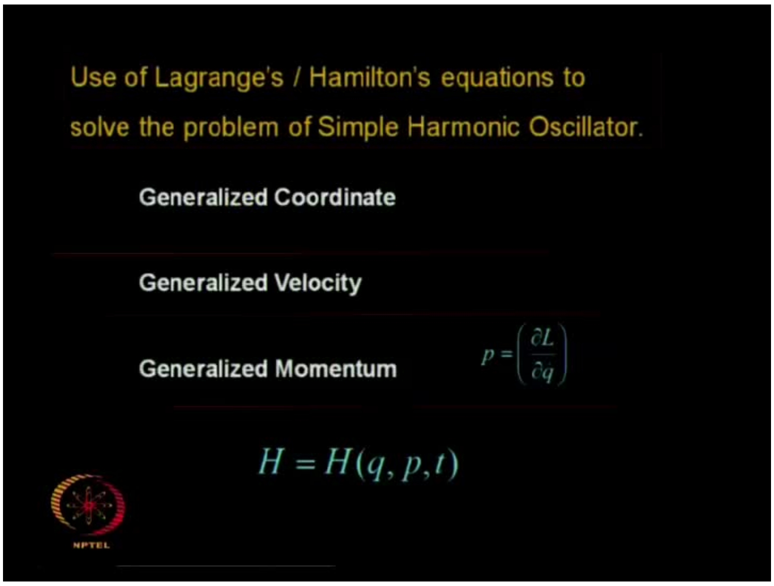

He recognized the repetitive nature of this oscillation. He was only 17 years old, which is perhaps as old as some of you are there about, and we will study these oscillations from the Lagrangian and Hamiltonian point of view. Just we illustrate how to use the formulations we have learned.

We will solve the Lagrange’s equation for a simple harmonic oscillator and also the Hamilton’s equations for the simple harmonic oscillator. We will take two kinds of simple harmonic oscillators. We will take the mass spring oscillators. We will also take the simple pendulum; both are simple harmonic oscillations at least to a good degree of approximation.

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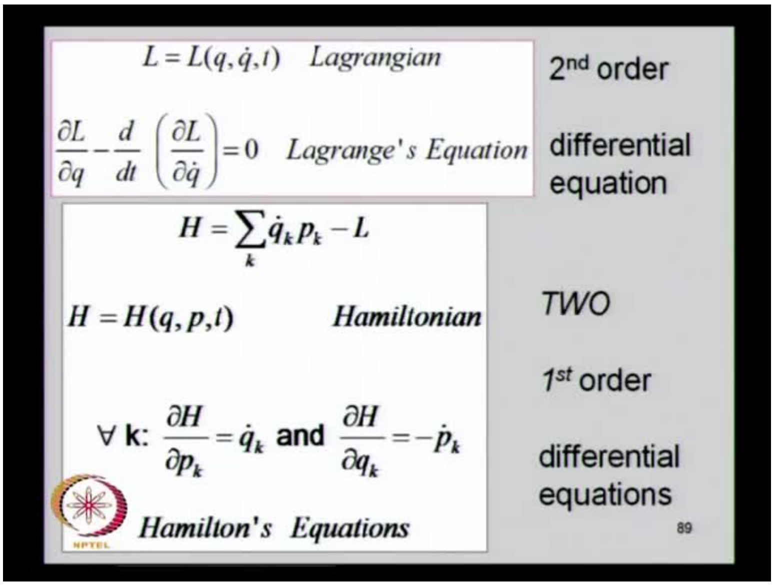

Now, if you look at the Lagrangian formulation, you find that you will have to set up a second order differential equation. If you work with the Hamiltonian formulation, you have first order differential equations. It is del H by del p equal to q dot and del H by del q equal to p dot, for every degree of freedom. So, these are first order differential equations, but there are two of them.

So, in any case, there is either a second order differential equation to solve or two first order differential equations to solve. On solving the differential equation, you will have to plug in the constant of integration. Once you integrate the equation, you will have to plug in the constant of integration. You will need two constants of integration to solve the second order differential equation, or one constant integration to solve a first order differential; but you have two of these. So, you always need as much information to solve these equations. Whether you use the Lagrangian formulation or the Hamiltonian formulation, you will always need two parameters. These are the initial position and the initial velocity, or initial position and the initial momentum, depending on whether you are using the Lagrangian or the Hamiltonian formulations.

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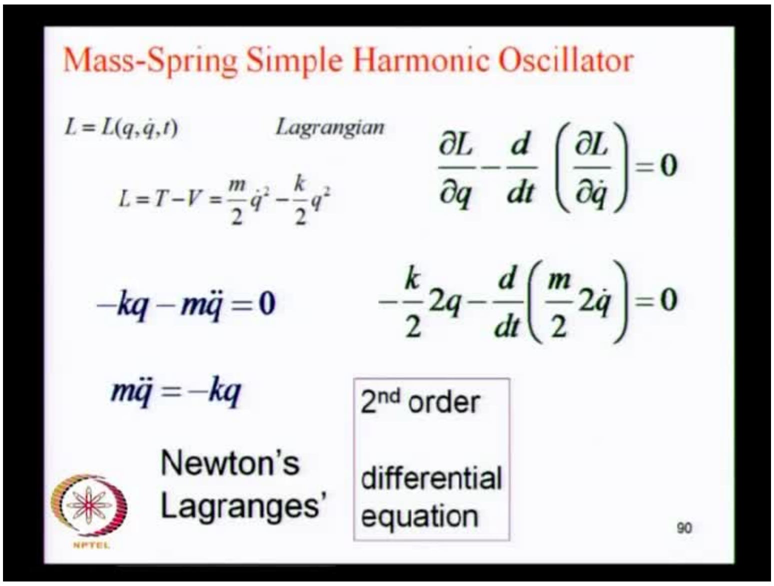

So, if you take a mass spring oscillator, the first thing to do is to set up the Lagrangian which is T minus V. So, the kinetic energy is half q dot square times the inertia. So, m by 2 q dot square minus the potential energy which is k by 2 q square; k is the famous spring constant. So, you have set up the Lagrangian.

Now, you can go ahead and get the Lagrange's equations. We already have it. So, all you have do is to take the partial derivative of L with respect to q, where u is the partial derivative with respect to q. this term is independent of q this term depends on q and its partial derivative is here. Here you need the partial derivative with respect to q dot, which is coming from the derivative kinetic energy with respect to the velocity; so it is m by 2 times twice q dot. The factor 2 cancels in both the terms and what you have is minus k q equal to m times q double dot.

What is that? Mass time's acceleration being the force, the force is always proportional to displacement q and always directed toward the equilibrium point, the proportionality is the spring constant and this is just the Newton's equation for a simple harmonic oscillation. You have already done.

We got this equation not by invoking the principle of causality of Newtonian mechanics, but we got it by solving the Lagrange's equation. So, the approach is different. Result will have to be the same; it better be the same; you do not want to run into any conflict in your results.

So, both the Newton’s equations as well as the Lagrange's equation are second order differential equations and the rest of the analysis will proceed exactly the same way as the Newtonian formulation does. You will have to plug in the constants of integration. You will have to provide the initial conditions because if the mass spring oscillator is oscillating like this, then depending on when you start the oscillations and at what velocity you launch these oscillations, the actual solution will be different. So, those initial conditions will have to be plugged in.

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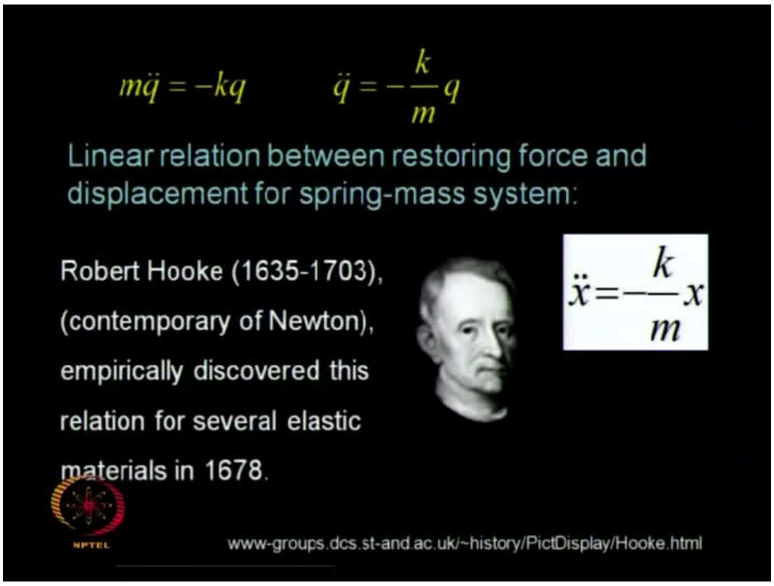

This was discovered also empirically by Robert Hooke who was a contemporary of Newton and he got it by observations on various elastic materials. So, this is famously known also as the Hooke’s law.

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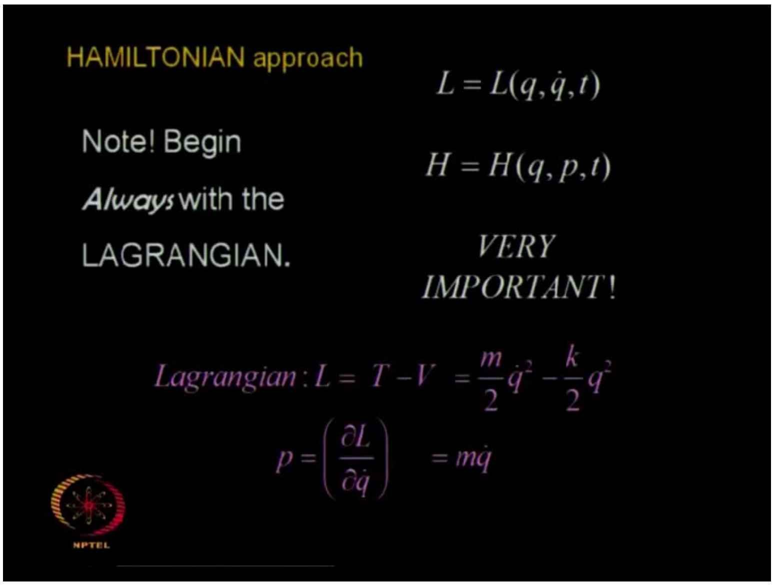

Now, let us solve it using the Hamiltonian approach. The important thing is - always begin with the Lagrangian. I warn you that I am going repeat this point that you must always begin with the Lagrangian. So, we set up the Lagrangian, but the other thing you have to keep track of is that the Hamiltonian formulation must be finally dealt with in terms of q and p, and not in terms of q and q dot. Momentum is defined as mass times velocity. It is directly proportional to the velocity in the Newtonian formulation in the variational calculus that we are using; that is not the definition of momentum. Momentum is del L by del q dot; whatever it turns out to be. So, we can set up the Lagrangian.

Since we have to set up the Lagrangian, we must obviously write an expression in terms of velocity because that is in terms of which the Lagrangian is defined. So, you cannot avoid velocity when writing the Lagrangian; but having done so, you must carry out subsequent transformations of that term and write it in terms of momentum, by getting the momentum as a partial derivative of Lagrangian with respect to q dot. So, those transformations will have to be carried out. So, you must remember that.

So, let us begin with the Lagrangian. It is T minus V. It is obviously written in terms of q dot square. So, the velocity appears explicitly in the Lagrangian, as it should. The momentum is now obtained as a partial derivative of the Lagrangian with respect to q dot, which is written as m times q dot. It appears in terms of the velocity q dot, but the

Hamiltonian, the Hamilton's principle function must be written not in terms of q dot, but in terms of p.

You cannot use the right hand side when you write p equal to m q dot. You obviously have two sides of an equation which are completely equal to each other, but one side of the equation which is in terms of velocity, has got no place in the Hamiltonian formulation. So, the equality has to be used within its own limitations.

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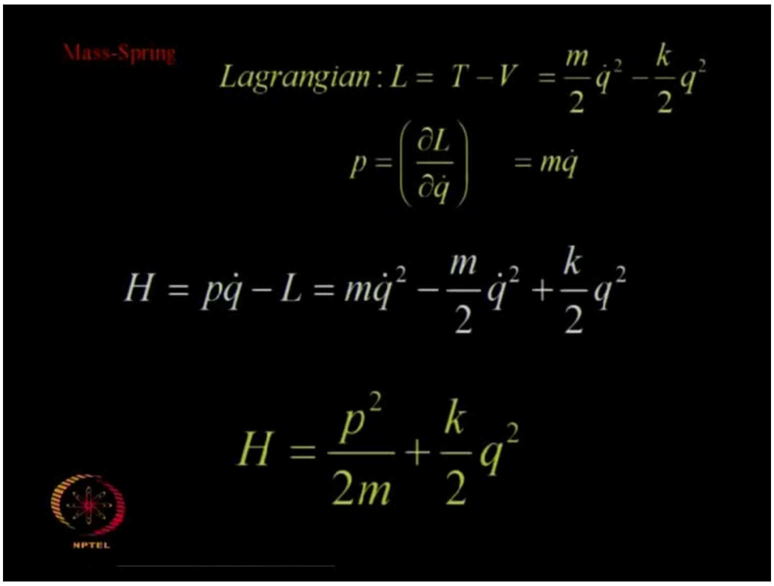

So, here you have, the momentum is m q dot. The Lagrangian must be written as this p q dot minus L and then transformed to get rid of the q dots, which must be written in terms of ps. So. Here, p q dot minus L gives you these expressions, but using this relationship, you must get rid of all the q dots and replace those by p by m. The Hamiltonian can be written only in terms of q and p. In the last step, you see that it is written as p square by 2 m plus k by 2, q square. The kinetic energy is written as p square over 2 m and not as half m v square. So, half m v square is kinetic energy. p square by 2 m is also the kinetic energy, but half m v square will be used in the Lagrangian formulation. p square by 2 m will be used in the Hamiltonian formulation; you cannot mix them.

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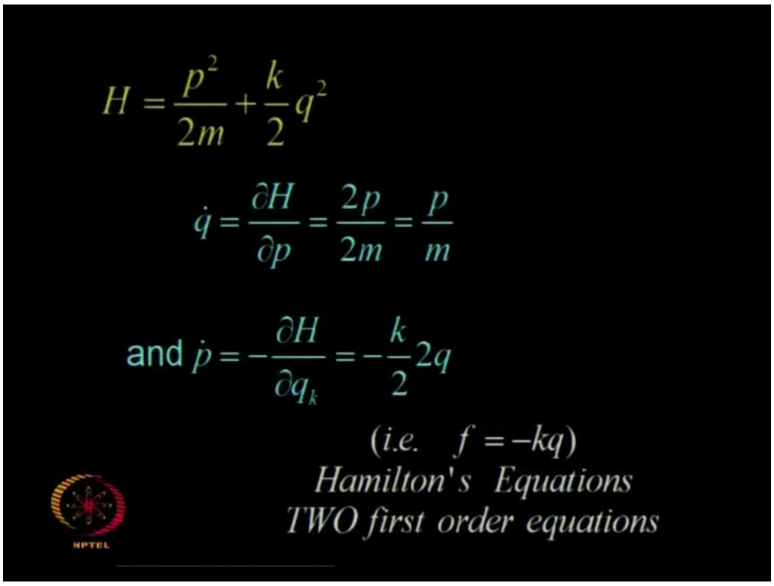

Now, you must always begin with the Lagrangian, get the generalized momentum, and then carry out the transformations to write the Hamiltonian in terms of q and p. Now, rest of it is simple because we have got the Hamiltonian in terms of p and q. We can take its partial derivatives; it this with respect to p and q. These are the Hamilton's equations that q dot is equal to del H by del p; p dot is equal to minus del H by del q. If you write these two partial derivatives using this formulation of the Hamilton's principle function, you get q dot equal to p by m, which is nothing but the Newtonian expression for momentum as mass times velocity.

This expression p dot is nothing but, the rate of change of momentum, which is nothing but the Newtonian force which is equal to a constant times the displacement. The proportionality is the spring constant and the displacement is always directed towards the equilibrium point. So, there is a minus sign. So, once again we have recovered Newton's equation. How did we get it? Not by using cause effect relationship; not by using Newton's laws, but using principle of variation and using Hamilton's equation; here they are. This is q dot equal to del H by del p p dot equal to minus del H by del q. This is what we have used and as a result of this which stems from the principles of variation, we get the same equation as we ought to. So, these are Hamilton's equations. These are two first order equations; again, you will need the two initial conditions.

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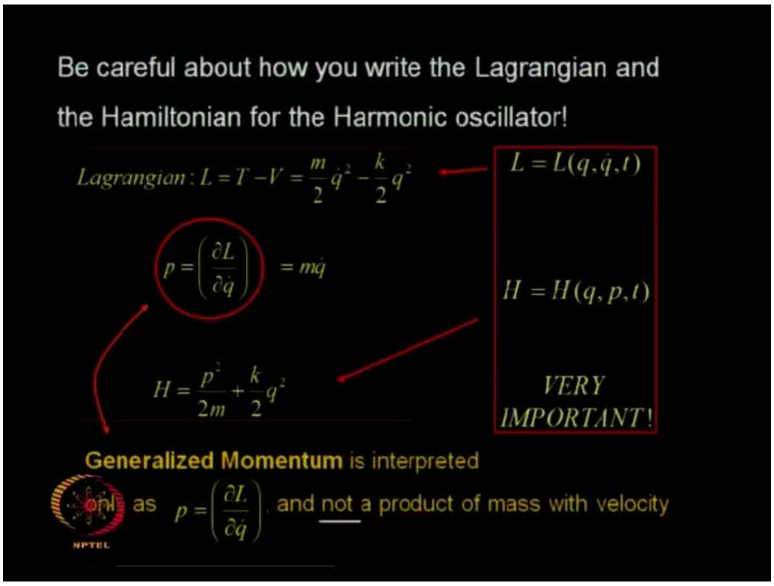

We have taken care that the Lagrangian is written in terms of q and q dot; the Hamiltonian is written in terms of q and p. So, the Lagrangian is written as T minus V and the Hamiltonian as sum of two quadratic term p square and q square, and not v square. So, we have taken care that the correct variable appears in each expression.

Momentum will however always be recognized as the partial derivative of the Lagrangian with respect to the generalized velocity; del L by del q dot is the generalized momentum; it will not be referred to as product of mass and velocity in the rest of the formulation in classical mechanics. In this variational formulation of classical mechanics, the mass times velocity has got no place.

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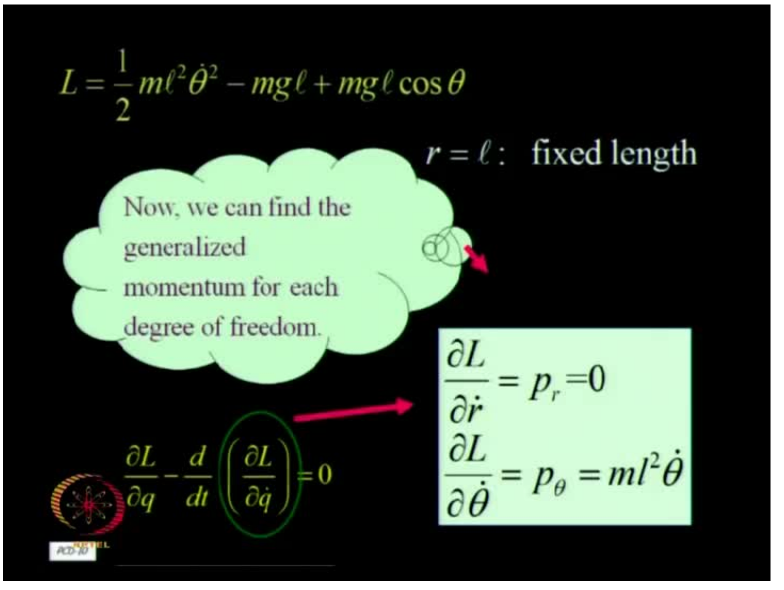

You can do the same for a simple pendulum. You have to be a little careful over here. Now, this diagram is almost self-explanatory. So, the potential energy will be given by this. If it is measured with respect to this reference point, then mg times this distance h will be the potential energy which is l times one minus cos theta; l being the length of the pendulum and we presume that this length is not changing; it is not elastic; it is rigid. So, you have got the potential.

We are going to set up the Lagrangian first, as we should always do. So, the Lagrangian is T minus V; the parameter r is time independent because the length between the point of suspension and the point at which the bob is. The mass is attached is not changing right; it is considered completely elastic. So, r dot will be 0. r will be constant and the value of the consent is equal to the length of the pendulum.

So, you have set up the Lagrangian; here T minus V. Now, with this Lagrangian, I have only separated these two terms over here; so, nothing big over here with this Lagrangian. We can get the partial derivatives of this Lagrangian with respect to the corresponding two velocities. It is independent on one of them. So, p r is 0 and p theta is m l square theta dot and you will immediately recognize that it is the angular momentum.

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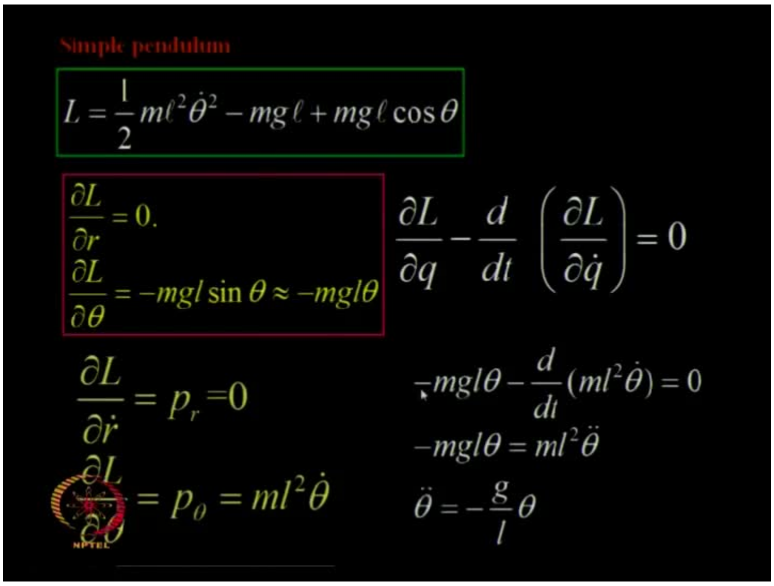

So, you have the Lagrange's equations, del L by del q equal minus d by d t of del L by del q dot equal to 0. You need the partial derivative with respect to theta which is coming from this term and this will be minus of sin theta, which for small angles is nearly equal to theta. So, here is an approximation. So, the simple pendulum is really not a simple harmonic oscillator except in this approximation because sin theta is not strictly equal to theta; it is nearly equal to theta; it is very nearly equal to theta, if theta is very small; it is very nearly equal to theta, if theta is very small.

It is not exactly equal to theta. In fact, it is an infinite series and you have got higher powers of theta. For small theta, these higher powers of theta become progressively small and you can throw them of. You can ignore them. You can say that they are so tiny that I need not worry about them; but the moment you say that it is too tiny to worry about, you are making an approximation; it is not exact. So, in this approximation, the sin theta is considered to be equal to theta.

You can develop the equation of motion. So, you have got minus m g l theta coming from this del L by del theta. The reason you have got theta here and not sin theta is because of the approximation I mentioned. This is del L by del q for q equal to theta when the degree of freedom is the angle theta and this is the time derivative of the corresponding canonically conjugate momentum. If you just cancel the common terms, the mass can be cancelled out, one of the powers of l is canceled out, and you find that theta double dot is equal to minus g over l theta. Now, this is an equation that you have met from Newton's formulation, but you have got it from Lagrange's equation; we have not got it from cause effect relationship.

Our answer to the description of how system evolves with time, makes no reference to a force. We say that a system evolves with time either; this evolution is governed by inertia if the body is in equilibrium. If the equilibrium is disturbed, then it is governed by the principle of extremum action that the integral L d t must be an extremum. It is this requirement which will determine how the system will evolve with time.

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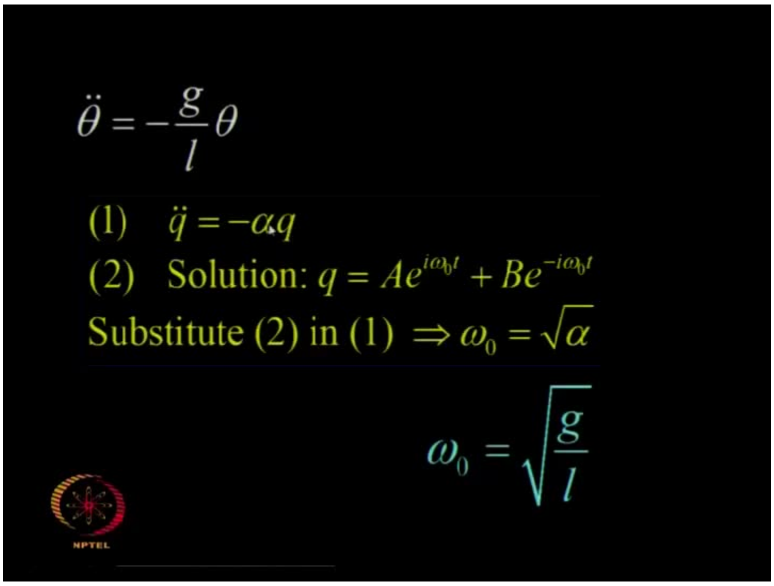

We have not referred to force; we have not talked about the cause effect relationship. So, we get this same equation and rest of the analysis is straightforward. Any time you have an equation, the second order derivative is equal to the coordinate itself, with a certain proportionality, no matter what this q is. It could be charge on a capacitor which is being discharged in an LC circuit, or it could the displacement of a mass spring oscillator, or it could be the angular displacement of a simple harmonic oscillator. So, it could apply to any domain: Physics, Electrical Engineering, Mechanical Engineering, Sound Vibrations, Elastic membrane vibrations; any oscillatory phenomenon. Whenever you have a different equation of this kind, it will always have the same solution which is the linear super proposition of these harmonic functions.

This solution will always have a natural frequency which will be given by this square root of the proportionality which comes here. Here, the proportionality is g over l. So, that is the natural frequency of the oscillator.

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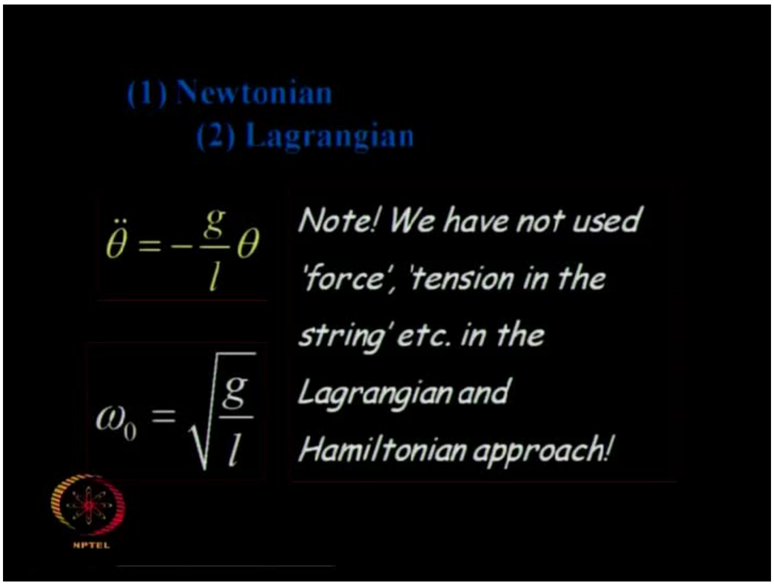

Both the Newtonian and the Lagrangian formulations give us essentially the same results as we already found for the case of the mass spring oscillator.

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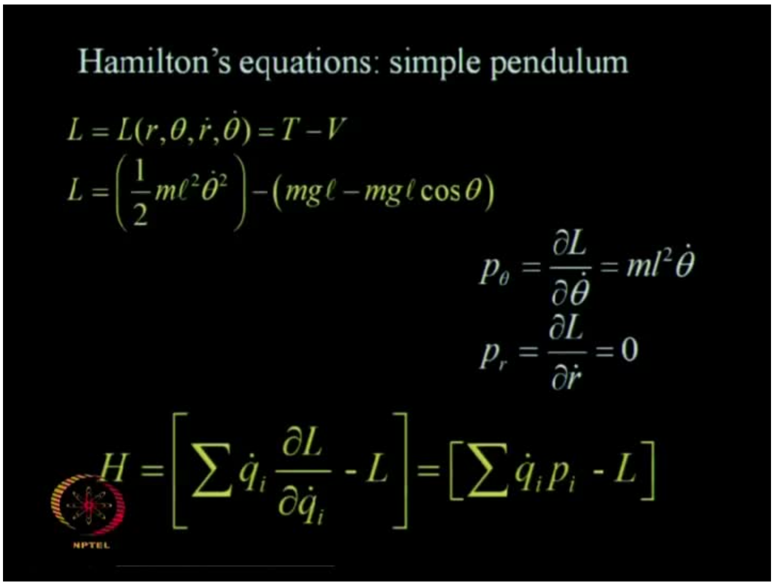

No surprise, there we can do the same thing using the Hamiltonian formulation which must begin by first framing the Lagrangian is T minus V. So, we do that. Then we find the generalized momenta from the Lagrangian by taking the derivatives of the Lagrangian with respect to the generalized velocities which are theta dot and r dot; r dot does not appear in the Lagrangian.

Now, you construct the Hamilton’s principle function. Now, this seems so simple for something like a mass spring oscillator or a simple pendulum; but I urge you to always begin with the beginnings; do not ever take shortcuts. You know that, for a given system you can write the Hamiltonian as total energy; you know it is a conservative system; you can go ahead and write the Hamiltonian as T plus V, but do not do it. Please first write the Lagrangian; then get generalized momenta; then write the Hamilton’s principle function which will be in terms of the velocity because it contains a Lagrangian as well; then transform it to generalize momenta.

There are these three, four steps which might seem so needless; if you know how to construct the Hamiltonian for a conservative system, ST plus V; do not do it; always write the Lagrangian and go through these 1, 2, 3, 4 steps. It is a good habit and you will find it extremely useful when you do quantum theory; because in quantum theory, when you set up the Schrodinger equation, you have to first compose the Hamiltonian and this is the way you compose the Hamiltonian.

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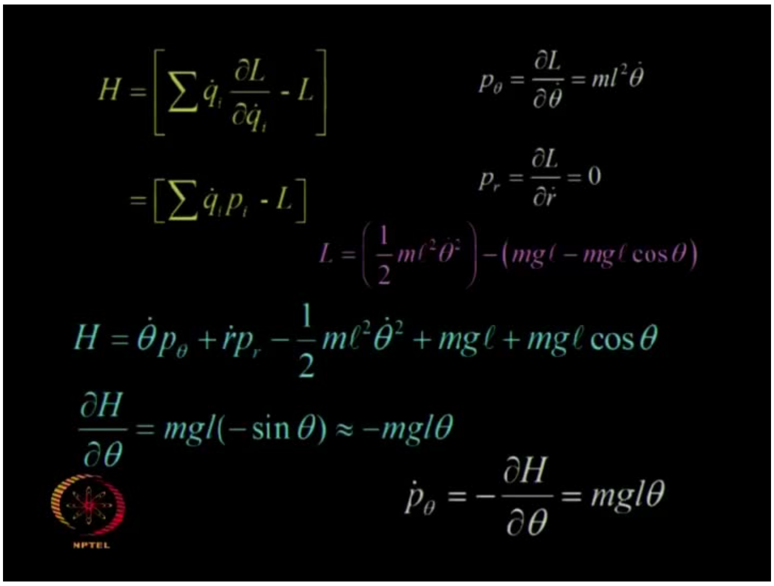

Essentially, we are going to illustrate here, how the Hamiltonian approach is applied to the case of the simple pendulum. As I mentioned repeatedly, the first thing to do always is to setup the Lagrangian and the Lagrangian will be in terms of the coordinates and the velocities. So, you see the theta dot coming over here. Then, from this, you determine the generalized momenta which are defined as the partial derivative of the Lagrangian with respect to velocity. So, there are two generalized momenta in this case; p r of course is 0 because we are considering a pendulum which is suspended by a string which is not elastic and its length will always be equal to L, and it is not going to change.

Once we have this, we set up the Hamiltonian in terms of what is generally called as the Hamilton's principle function, but then, we must transform it to momenta because the velocities should not appear in a Hamiltonian. So, here, we have the Hamiltonian in which we have expanded this summation. There are two degrees of freedom here; r and theta. So, we have theta dot p, theta plus r dot p, r terms over here. Then, from this, we subtract the Lagrangian. Now, we are ready to take the partial derivative of the Hamiltonian with respect to theta.

Now, this comes from this last term over here which is the only one which is theta dependent and its derivative gives us minus theta. Here is the position where we actually make an approximation of small oscillations in the diagrams. It is always shown with a large angle; just to make the point, but the whole approach of simple harmonic motion requires that this approximation is made. This is valid only at very small angles where you can approximate sine theta to be nearly equal to theta. This is the domain of application of this particular approximation; otherwise, of course motion is not simple harmonic.

So, here, you have the partial derivative of H with respect to theta in the small approximation limit. From Hamilton's equation which is p theta dot equal to minus del H by del theta, we get results. This minus sign is taken care of over here. So, minus del H by del theta will give us m g l theta.

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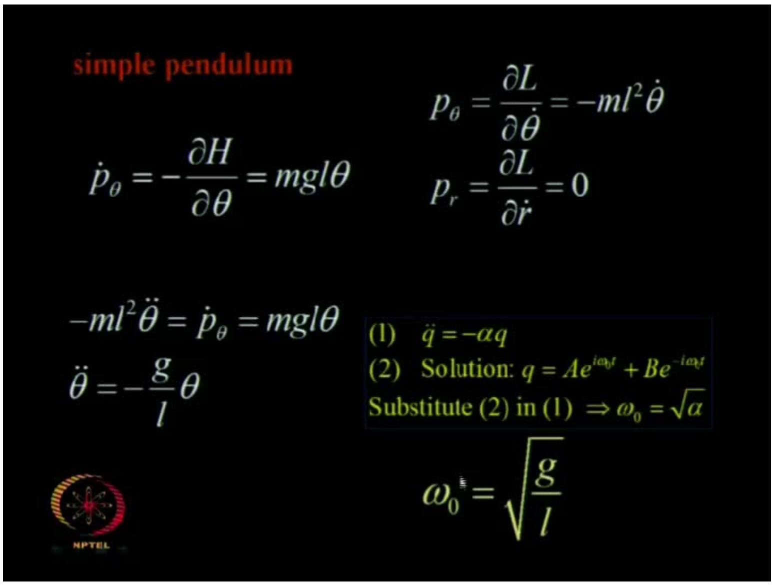

Subsequently, we already know that the generalized momentum is also given. This is how we obtained it in the first place which is the derivative of the Lagrangian with respect to the velocity. So, if we take the time derivative of this, we get another relation for p theta dot. So, let us do that and we will get p theta dot equal to m l square; of course it is constant. So, we get p theta dot equal to minus m l square theta double dot. From this equation, we know that p theta dot is equal to m g l theta. So, using these two relations, we get the relation; we get the result. Then the second time derivative of theta is proportional to theta itself. The proportionality is g over l and is always directed toward theta equal to 0, which is the equilibrium point.

This is the common equation for any simple harmonic motion, in which the acceleration is proportional to the displacement and always directed toward the equilibrium point. Any equation of this kind has got the general solution in which omega 0 is the natural frequency. We are quite familiar with the result from Newtonian mechanics as well and the solution is in terms of this circular frequency, which also called as natural frequency. It is the square root of this proportionality; in this case, it is g over l square root.

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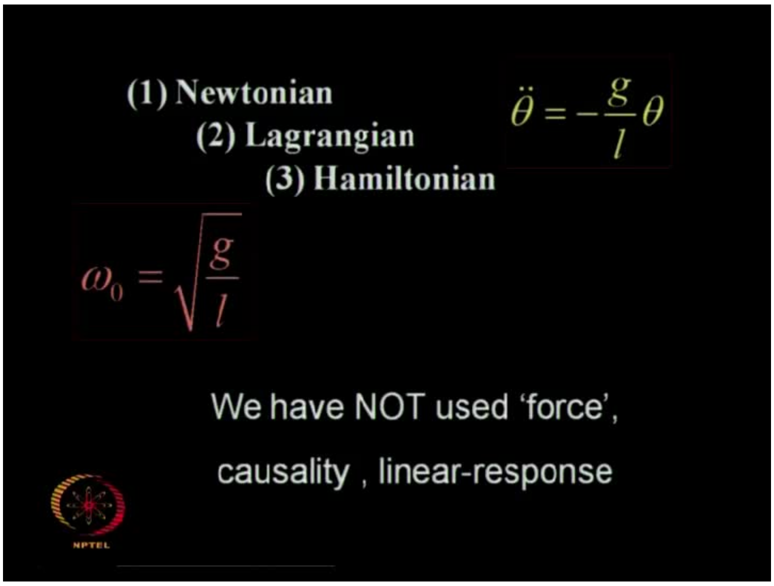

We find that all the 3 formulations of mechanics, Newtonian mechanics, Lagrangian, and Hamiltonian formulations give us essentially the same results. The point to be emphasized is that we have not used force, causality, and linear-response physics in this formulation; this is the most important thing.

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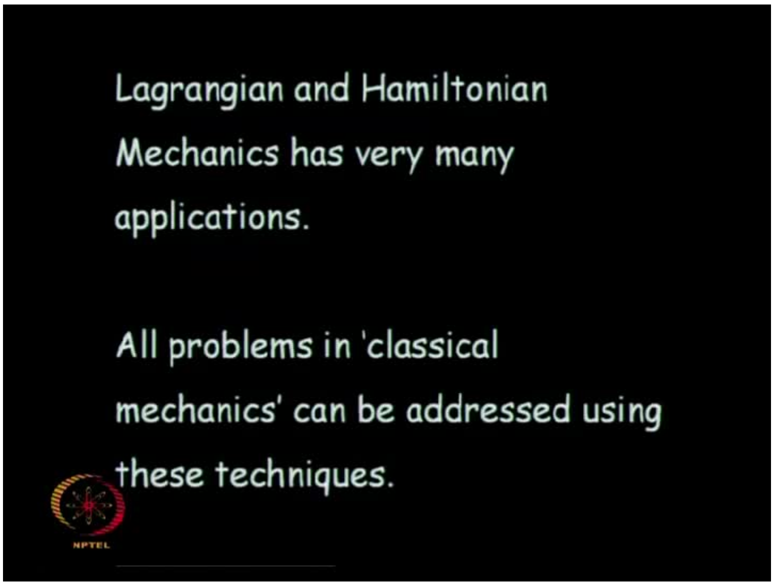

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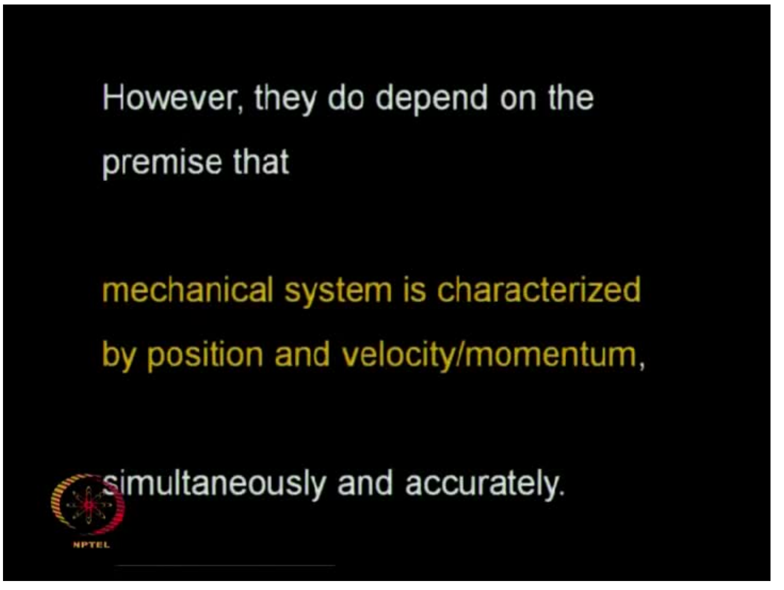

Now, the entire domain of classical mechanics comes under the umbrella of the principle of variation and the Lagrangian and the Hamiltonian formulation. They all have a common premise just like Newton's formulation. The original problem that we defined categorically that the problem in mechanics is to give an unambiguous interpretation to this mechanical state of a system, how do you characterize it, and how does this evolve with time? That is the problem

We learned how to formulate this problem and then how to solve it. We agreed that a system is characterized by position and velocity, or position and momentum equivalently. Their rate equations give us the equation of motion. The only difference was that in Lagrangian formulation. We have expressed it not in terms of position and velocity, but as a function of position velocity. The Lagrangian being the function of q and q dot in Hamiltonian formulation, we wrote it as a function of position and momentum which the Hamilton's principle function.

Now, naturally, all this assumes that the information about position and velocity can be obtained. If it cannot be obtained, where do you go and where do you get this information from? Who is going to give this information to you? Is it going to be given to you by the person who sets up a question paper in an exam that this is the position and this is the velocity? If nobody does, what are you going to do to get position and velocity or position and momentum? You need to characterize the mechanical state of a system.

So, you will have to conduct some experiment to get this information. So, you do some experiment; you carry out a measurement process; you measure the position, record it; say that this is my position. Then, you do another experiment; record your momentum and say that this is my momentum. Now, if you came back and repeat your measurement of position, what if the answer is changed? You cannot use the previous answer. Now, why should this answer be superior? If you think it is better, do it the third time. Now, what if you get a different answer, the third time? What will you do?

So, you have only made an assumption that repeated measurements of position and velocity will always give a consistent acceptable result which you will use to characterize the mechanical state of a system. That is an assumption; you never tested it and only an experiment will test it.

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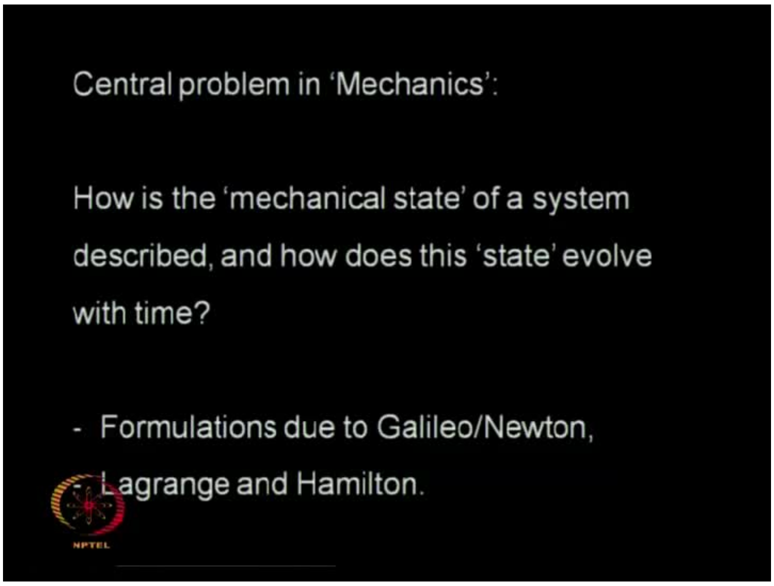

Now, this is the central problem in mechanics. You can do it in terms of position and velocity, or position and momentum. You have the same thing in Lagrangian formulation, or the Hamilton's formulation, or in the formulation of Galileo and Newton. But all of these, these 3 formulations are based on an assumption that you never test. You always assume if you repeat a measurement; you do not know what is going to be the outcome.

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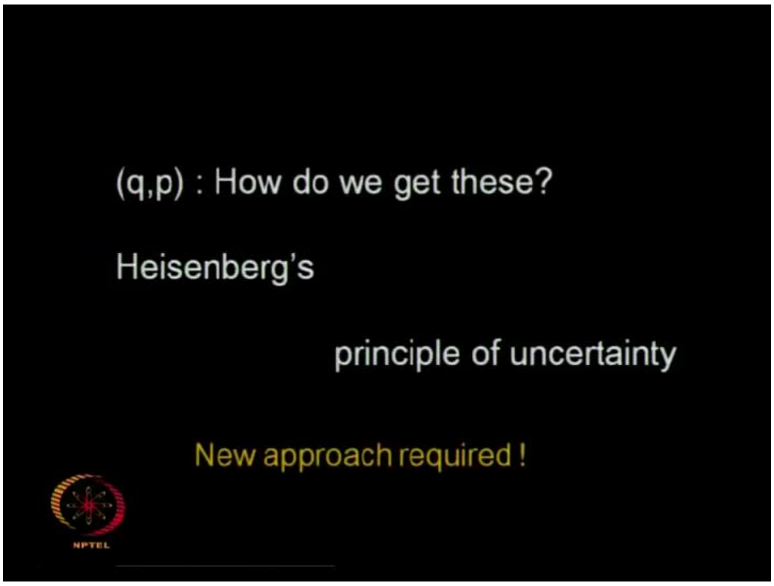

Now, what happens is that repeated measurements do not give the same results because when you make a measurement of position, you will get an answer. It is not that you will not get an answer; you do get an answer. Then, you measure the momentum. You will get an answer; it is not that you will not get an answer, but if you keep repeating it, if you keep getting different answers, then it means that these two parameters - position and momentum, cannot be used to characterize the mechanical state of a system. This is the essential content.

What is called as the Heisenberg's principle of uncertainty, that position and momentum cannot be determined simultaneously and accurately. More accurately, this principle is sometimes enunciated by saying that, a dog does not bark and bite at the same time. If he barks, he cannot bite; if he bites, he cannot bark. So, certain things are not simultaneously possible.

If you measure position, you will get some answer, but this measurement of position will end up disturbing the position, and disturbing the momentum in a certain way on which you have no control. This obviously means that, you cannot base your scheme of mechanics on the notion, but a point in phase space represents the state of a system. Then, talking about the evolution of this system as the trajectory the system will follow in the phase space such that it is governed by the principle of extremum action or by

Newton's cause effect relationship becomes irrelevant; you cannot use it; you need a completely new approach.

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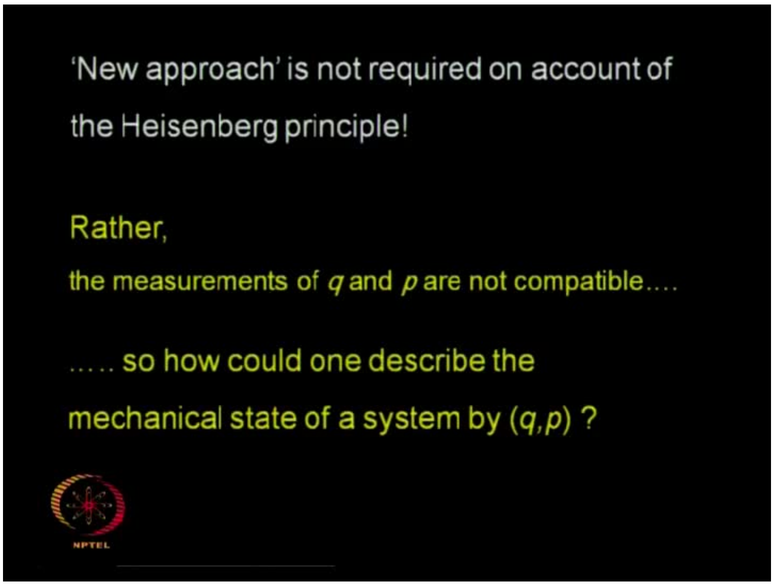

It is important to note that the new approaches are not required on account of the Heisenberg principle because how you put this idea across is important. It is rather that since such simultaneous measurements of q and p are not compatible, a mechanical system cannot be described by q and p.

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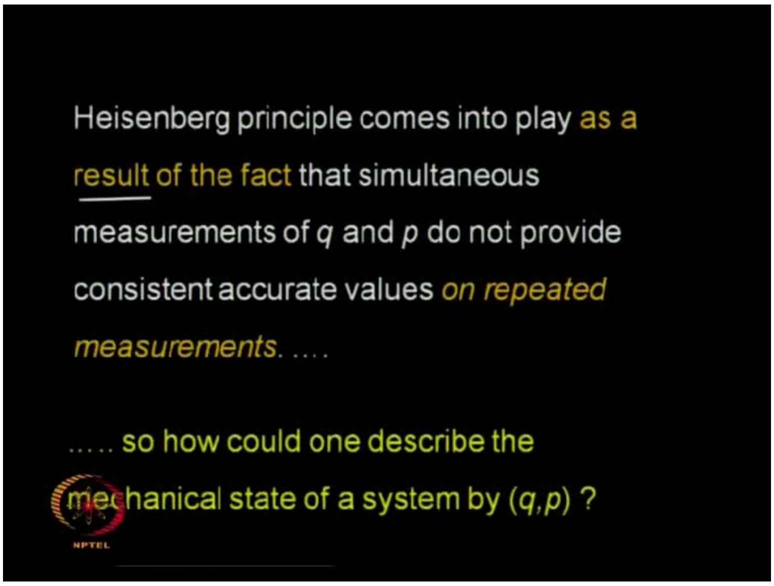

As a result of this, the Heisenberg's principle comes into play. So, the Heisenberg principle is not of course the cause; it is just a principle. It is an expression of the fact that simultaneous measurements of q and p do not provide consistent accurate values on repeated measurements. So, of course, where do we go? We started by describing the mechanical system by q and p. Now, we say it is not possible. So, we need a completely different approach.

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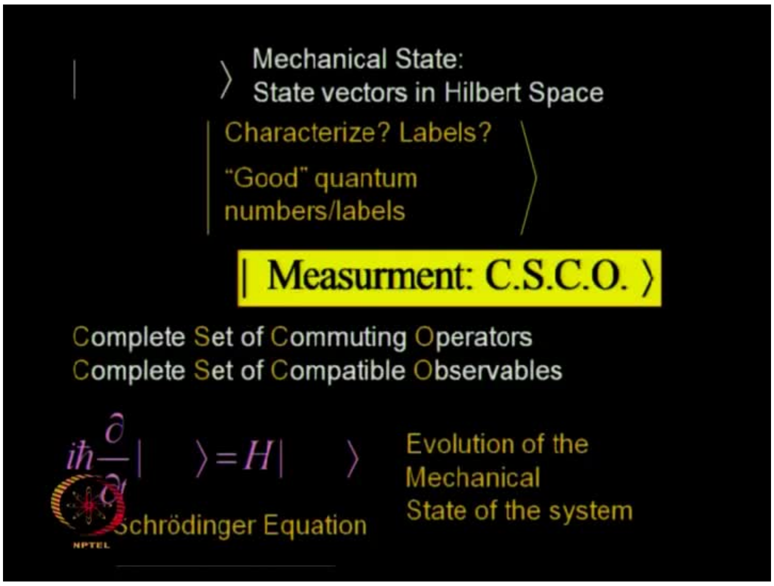

Then, this different approach comes in quantum theory, which is of course not the subject of this course, but what one does in quantum theory is that, you describe the system by what is called as a state vector in a Hilbert space.

Now, I have written a symbol for a vector in the Hilbert space. What goes into the simple blank space? How do you characterize this vector space? What labels will go into it and these labels will be called as good labels or good quantum numbers. What can give us good quantum numbers? measurements that you can carry out on the system; but only such measurements which are compatible with each other. So, the measurement of one does not disturb the measurement of the other. Because if it does, it is of no value.

If the measurements are not compatible, like position and momentum cannot be used as good quantum number or good quantum labels. So, you need what is called as a complete set of compatible observables. In quantum theory, these observables are quantized. So, they are expressed in terms of operators. This is the subject of quantum theory. Then, the equation of motion which governs the evolution of this state vector is the Schrodinger equation that the state vector evolves with time, and how do you describe it by giving its time derivative?

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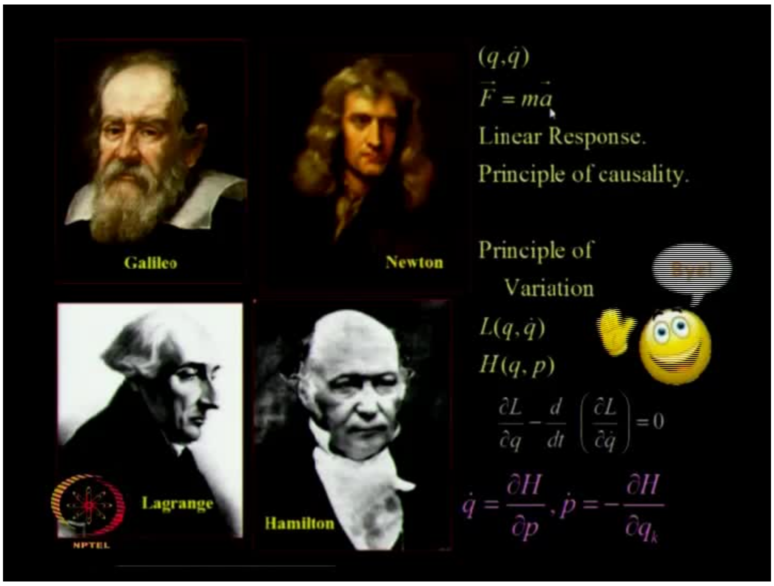

So, the time derivative of the state vector is given by this equation which is the Schrodinger equation. This describes the evolution of the mechanical system and this is obviously not the subject of this course, but quantum theory will begin from this point. We conclude this unit on the equations of motion by summarizing the main things that we have learnt and this is obviously the last slide for today’s class.

Our understanding of equilibrium comes from Galileo’s experiments in which he identified the complete equivalence between the inertia of rest and the inertia of motion. You remember the experiment I described about a person dropping balls from the top of the mast of a ship and it falls at the same place whether the ship is docked or it is in motion at a constant velocity?

So, the recognition of the equivalence between the state of rest or of uniform motion is the Galilean principle of inertia. It enables you identify inertial frame of reference as a frame of reference, in which motion is self-sustaining. Departure from equilibrium, we understood from Newton who explained that a departure from equilibrium express itself as a change in momentum at a rate which is dp by dt. I cannot help reminding you once again that dp by dt is so easy for us to talk about because we have learnt calculus and there was no calculus that Newton had learnt. He invented calculus; nobody ever taught him calculus; he invented it.

I was only 21, 22 years old at that time. Newton then explained the departure from equilibrium in terms of this linear response relationship; acceleration which is the response to this cause and this relationship is linear; the linear constant of proportionality is the inertia. Then, we found an alternative principle which has nothing to do with force, which has nothing to do with linear response, which is based on the principle of variation on the calculus of variation. The principle of extremum action provides us alternative formulations of mechanics which give us essentially the same results but in a much more compact way and much more beautiful way, which are nicely adapted to quantum theory. You can learn the connections between conservation and symmetry principles much more lucidly from this alternative formulation. This comes from Lagrangian Hamilton and then you get the equations of motions which are named after Lagrange and Hamilton.

So, with this, we conclude unit 1. If there are any questions, I will be happy to take; otherwise, good bye.

Sir, saddle point.

Yes.

Can we call center of mass as a saddle point?

No.

Center of mass of the system?

No, no.

[Noise]

Center of mass is a completely different property. Center of mass is a geometrical property; if you multiply the position vector by the mass of a many body system, add it up for all the particles divide it by the total mass, you get a geometrical property. It has nothing to do with the forces because the center of mass need not be in a state equilibrium at all.

There can be some net force active acting on it. A saddle point is a point of equilibrium; if the object is kept at a saddle point, it means that it can stay there forever and ever; you can change its position only if you displace it, and the displacement must move it to a point where the potential is different; otherwise, it could in a state of neutral equilibrium.

So, saddle point is a point of equilibrium. The only thing about a saddle point is that it is not unambiguously a point of stable equilibrium, no resist and unstable equilibrium. So, it is stable equilibrium with respect to one kind of displacement and unstable equilibrium with respect to another kind of displacement. So, these are two different ideas.

Any other question? Yes.

[Noise] How will you calculate because all of Lagrangian mechanics is based on energies: kinetic and potential energy. So, how will you calculate potential energy for non-conservative force friction and etcetera?

Actually, one can setup equations of motion in those cases - for non-conservative systems, and then one uses what is called as a generalized force. When you have a Lagrangian which is time dependent, which is what will happen when you have friction present. Basic fundamental interactions in nature, of course, are conservative. The fundamental interactions in nature, that you and I are concerned within day to day life, are only gravitational and electromagnetic. Gravitational interaction is a conservative interaction. The electromagnetic interaction which comes through the Lorentz force, for example, that is also a conservative interaction. Energy is not lost in that and no other interaction is involved in this, except quantum theory. Quantum theory is involved even in this, but I will not touch it at this point. But other than quantum theory, there is nothing else which is involved in in friction; yet, energy is lost.

The fundamental interactions are conservative. So, why is there a loss? It is because you have not taken other degrees of freedom into account. You are setting up the equation of motion for the bottle, and not for the bottle and the particle of the table. So, those systems in which you take into account all the degrees of freedom will be necessarily conservative which is what this formulation is meant for. And just the way in Newtonian mechanics, you can still write equation of motion, even if you do not take all the other degrees of freedom into account, by fudging them by what you call is the frictional force and it is by no means an exact force; it is by no means a fundamental force; any force in physics is made up of only the fundamental forces. Frictional force is not a fundamental force and the fundamental interactions are conservative. So, it is only because you have not taken into account all the degrees of freedom that friction comes into the picture. So, you can fudge for it in Newtonian mechanics by writing a frictional force proportional to the velocity; for example, as you often do in viscous medium.

So, one can do that in Lagrangian formulation also. Then, you can have terms which will contain such a force which is sometimes called as the generalized force, but I will not deal with those things. We have always confined our discussion to closed systems in which all the degrees of freedom are taken in to account.

(( )) one system then I would have taken into account all the [Noise]

You would but to take into account, friction. You will have to take into account not just the table and the bottle, but every particle of the table and every particle of the bottom of this bottle. Then write all the detailed pair interactions and before you solve that equation I would be long dead. Since you do not want to do that, you fudge it; you fudge it by saying that there is a frictional force. I will say it is some gamma times velocity; put it in the equation of motion; get some solution, and from some empirical information, say that this is my value for the friction constant; it has got no fundamental basis. So, you cannot do it.

You know from first principles; you can do it in some ad hoc manner. That is only because you are not taking into account all the degrees of freedom. Fundamental interactions are conservative; you and I do not create energy; we do not destroy it. The day you do, they will build a temple for you.

Thank you all very much.