Statistical Mechanics
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Lecture – 20
N Spins in a Uniform Magnetic Field

So, good morning students. Today we will talk about Magnetic system of N Spins in a constant magnetic fields.

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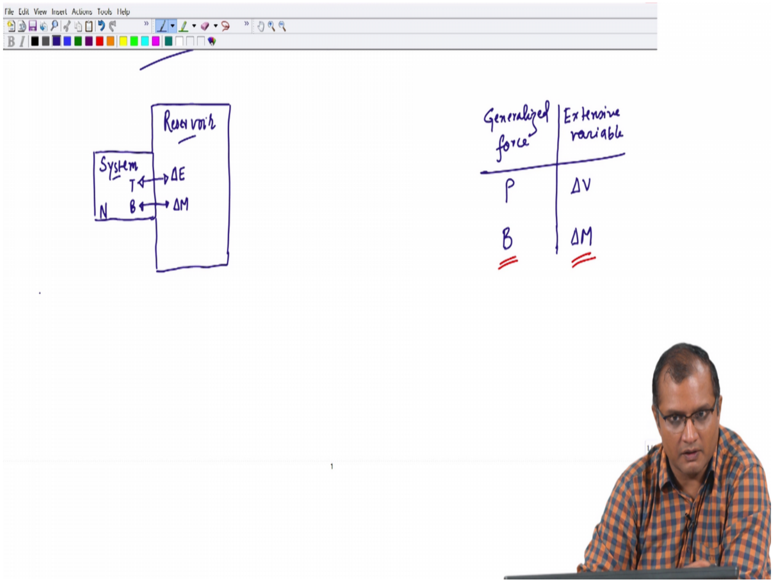

I am going to take a constant and uniform magnetic field. So, this is another system in the Gibbs canonical ensemble. The microstate in this system is given by a combination of three thermodynamic variables which are N, the magnetic field B and temperature.

So, the pressure in the previous ensemble of the ideal gas has been replaced with a new variable which is magnetic field. Now because this is a system of spins we are going to maintain the system of N spins in the constant magnetic field. So, it is like if you consider pressure as a generalized force and you want to keep pressure constant then you have to allow for exchange in volume because, PDV is the work done on the system against an external pressure. So, this is the energy scale ok.

So, I will write this as the if you want to write it the systematically, it would be better to sort of understand what we mean by a generalized force and the corresponding conjugate extensive variable. So, in the previous canonical Gibbs canonical ensemble where we maintained the systematic at constant pressure. So, the generalized force that we subjected our system to was the pressure and we could do it by allowing the system to exchange volume. So, the extensive variable that was allowed to fluctuate was volume that is the conjugate variable, in the sense that PDV is an energy scale neither is P force nor is delta V displacement, but the product of the generalized force into generalized in into you know generalize displacement is always energy.

In this system I am going to maintain constant magnetic field and the extensive variable conjugate to this magnetic field is the magnetization. So, the system will be having a fluctuating magnetization. So, some of the fins can be instantly flip down or flip up in response to a constant external magnetic field ok. So, as again B is magnetic field which is not force, dimensionally it is not equal to force and neither is magnetization dimensional equal into displacement. But the product of B into delta M has a dimensions of energy like we have been saying some time now that journalize force are to be realized not as force themselves they are thermodynamic variables that we are going to keep constant.

But conjugate to each of these generalize forces there exist an extensive variable such that the product of force and the variable extensive variable is always energy. So, our system is now a magnetic system which is under constant magnetic field. So, they are going to be dealing with constant B and hence the system will be allowed to exchange allowed to have fluctuating magnetization. So, let us think of draw small schematic that I have a system that is in contact with reservoir as before and this reservoir has a two fold task ok. So, the reservoir and this is my system the number of spins in my system is constant that is easy because I am not allowing any particles to leave the system.

So, the box are has a volume has a has valves that does not allow exchange of particles that is easy. Then I will be allowing the system to interact with the reservoir in the sense that the system is allowed to exchange energy with the reservoir to keep the temperature constant ok. So, this will keep the temperature constant. So, there is a give and take of energy between the system and reservoir. In addition to this the system is allowed to exchange or have its magnetization changed which will be a consequence of a constant magnetic field exerted on the system ok.

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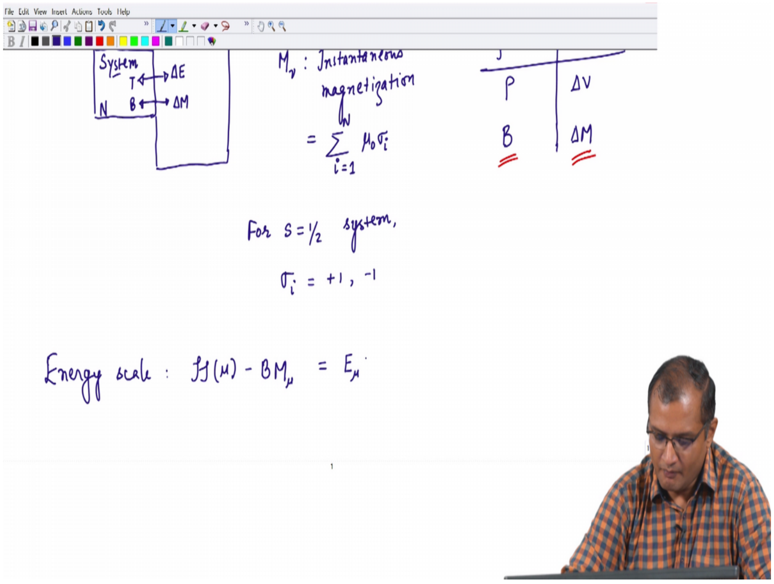

So, now if I write down the. So, this is a system under Gibbs canonical ensemble which means the magnetization here that fluctuates between the microstates can be represented as V mu and I am going to call it as a instantaneous magnetization this instantaneous magnetization is nothing, but that magnetization of that particular state that you are observing at any instant and this is given as summation over all the spins that I have taken. So, i will run from 1 to N into the Bohr magneton or magnetic moment per unit spin into the excitation of a particular spin.

So, this is a spin half system, which means the spins can be either along the magnetic field or anti parallel to magnetic field. So, here for I have taken for very you know for the sake of convenience I have taken spin half system, I can have my excitation to be only plus 1 or minus 1 means parallel to magnetic field or anti parallel to magnetic field. Naturally in the presence of an external magnetic field you would have more and more spins aligned in the direction of magnetic field because, that is the that lower the energy of the total system.

So, alignment is preferred over non alignment and because of that I am going to write down the energy scale of the problem or the total energy of the system as some internal Hamiltonian and minus BM ok. So, this is the magnetization instantaneous magnetization ok. So, as you can see I have to take the magnetic moment of the mu th microstate. So, I am going to call as the energy total energy of the mu th microstate. So, that is my energy scale.

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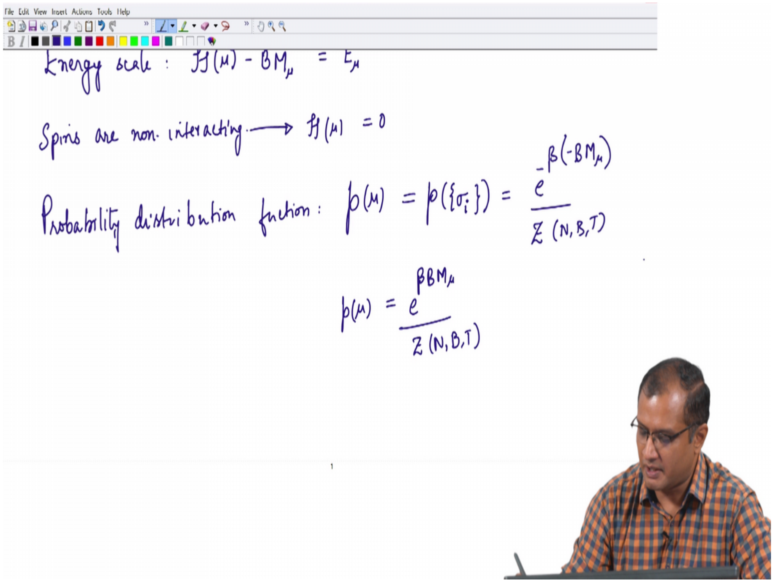

Now our spins are non interacting to take a very simple example which means there is no interaction between spins. So, there is no internal energy in the absence of a magnetic field. So, that basically makes my Hamiltonian in to be 0. So, there is no internal Hamiltonian the corresponding equivalent would be in ideal gas you had taken the Hamiltonian as sum of pi square over to M; now there is no such translational momentum here for the spins they are sitting on you know their latest positions.

So, the only internal energy this spins can have is if they are talking to each other the interaction energy and we are taking non interacting spins. So, there is no internal Hamiltonian for the system the only energy the system will pick is by coupling to an external magnetic field. So, the only energy scale is just minus BM and M is that magnetization of the instantaneous microstate. So, then you can write down the probability distribution function of the micro states as simply.

Now microstate here is nothing, but collection of all these excitations I am going to write it as collection of all these sigma is because I have used the label sigma is ok. So, the particular values of sigma is that you take constitutes a microstate ok. So, here i that we have taken is basically 1 of the N particles in the system ok. So, this is the meaning of the excitation fine now this can be written as you can write down this as e to the power minus beta into the energy scale which is just minus B times M mu divided by the partition function ok. So, this simply becomes e raise to beta BM mu upon the canonical partition function fine. So, now we can compute what is the canonical partition function because that is required to build the connection with thermodynamics.

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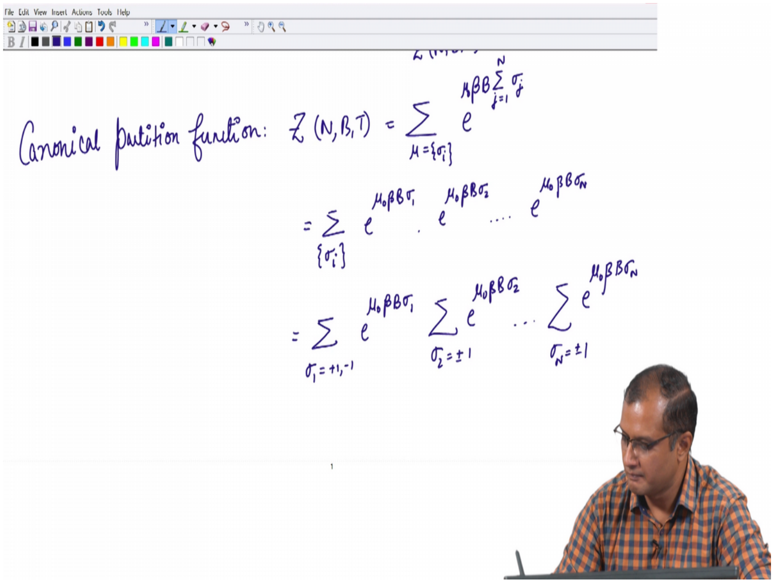

So, I can write down the canonical partition function or the Gibbs canonical partition function as simply summation overall microstates which is nothing, but the collection of all these sigma is e to the power beta B and the magnetization is basically this expression ok. So, I have taken the magnetization as microstate is nothing, but the symbol of a microstate is mu here.

So, we can just simply take this expression put it here. So, it will be nothing, but summation over all the spins I am going to take mu naught outside and simply write down sigma j ok. Now this sigma js are the values of sigma j that you have taken in the microstate fine. So, I can write this as summation overall the micros summation overall the microstate which is basically combinations of the sigma is e raise to mu 0 beta B into sigma 1 into mu 0 beta B sigma 2 all the way to e 0 beta B sigma N, because the exponent has sum of terms if you spit up into products.

Now you write down since there all independent of each other always sums are independent of each other you can write this as nothing, but a summation sigma 1 being plus 1 or minus 1 e to the power mu 0 beta B sigma 1 and then you can write down summation sigma 2 as plus 1 minus 1 e raise to mu 0 beta B sigma 2 all the way to the last particle.

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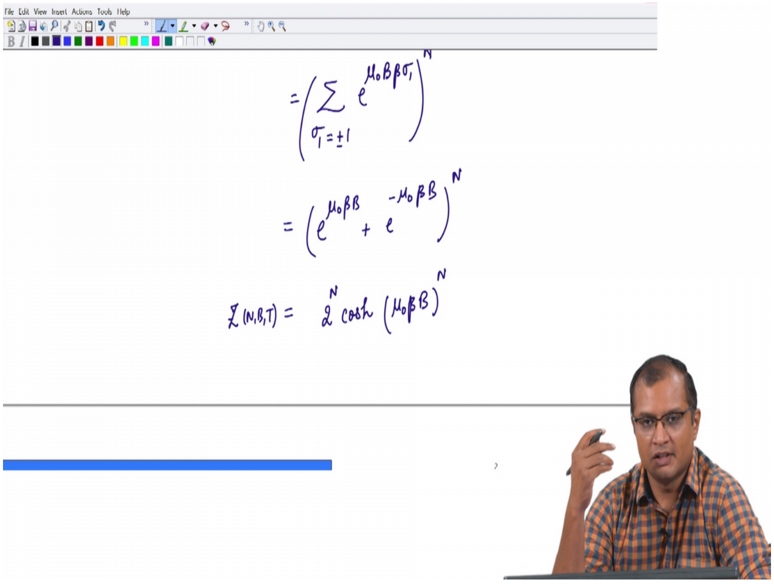

And if you see that these are all independent of each other it is just amounts to write down just 1 of them and raising e to the power M. So, the product of sums becomes sum of products now you can right it as simply e to the power mu 0 beta B plus e to the power minus mu 0 beta B the whole to the power N. And this is nothing, but the expansion of twice for cosine hyperbolic mu 0 beta B to the power N. So, I am going to write it as 2 to the power N and cos hyperbolic to the power N fine.

So, this is the partition function for the NBT system that we have chasseing and this opens up the route to thermodynamics in the sense that I can now compute what is the average magnetization of my system. Because, a system as a fluctuating magnetization the spins are constantly in a between different microstates the spins are differently align. So, each microstate has a different magnetic moment. So, you would be interested in the average magnetization of the system because, magnetization of a micro microstate is a random variable and when you are dealing with random variables you are interested its averages or moments.

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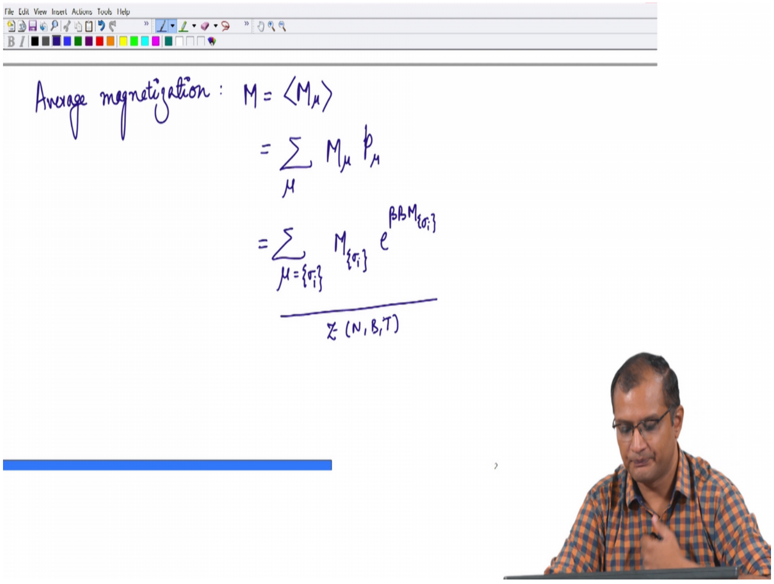

So, average magnetization is then very easily computed. So, if you want to compute the average magnetization this is nothing, but the average of the you know magnetic moment of each realization or each microstate. So, you can compute this from the from statistical mechanics by simply summing overall microstates M mu P mu. So, here you can write this mu as we already know that this is nothing, but the collection of all the excitations.

So, if you specify the values of these excitations it constitutes a single microstate, if you change this sigma is it becomes a new microstate. So, you sum over all such combinations and sample this magnetic moment in this PDF and which is given as e to the power I have already written the expression of the PDF is beta B M and this has to be divided by the partition function.

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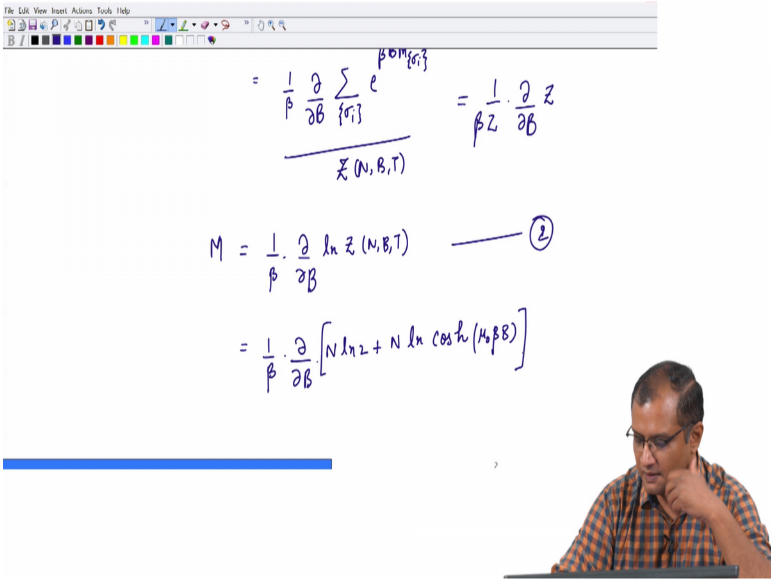

And well you know the trick here this is nothing, but you have to pull out an M so, which means you have to take the derivative with respect to beta B. So, this is d by dB of here exactly the partition function. So, this is my partition function that I am going to write down. So, this is nothing, but 1 upon beta d by this is not complete derivative. And you got a beta Z in the denominator ok. So, this is nothing, but 1 upon beta d by d beta d by dB of lon Z ok. So, that is the value of the average magnetization. So, I am going to call this as let us equation 2 and I am going to call the partition function has equation 1.

And so, I can just go ahead and take the logarithm of this partition function compute M M. So, this is nothing, but I can write down as a 1 upon beta d by dB of lon Z would be lon Z would be lon 2 raise to N which is N lon 2 plus I have yeah N lon of cosine hyperbolic mu 0 beta B fine. So, you can easily see that there is only the second term that is going to contribute.

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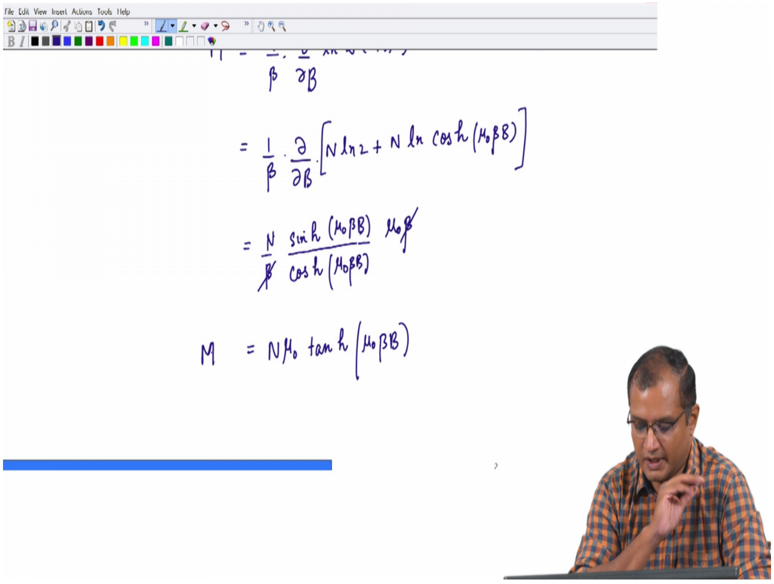

So, you can write down this as 1 upon beta N comes out that makes the magnetization and extensive quantity. And the derivative of logarithm of cos hyperbolic is nothing, but 1 upon cosine hyperbolic and the derivative of cos hyperbolic is sin hyperbolic and I will have a mu 0 beta to complete the derivative ok. Now you can write this as you can knock off a few terms and write it as N times mu 0 into tangent hyperbolic of mu 0 beta B. So, this is my magnetization for this N N N spins in the magnetic field B ok.

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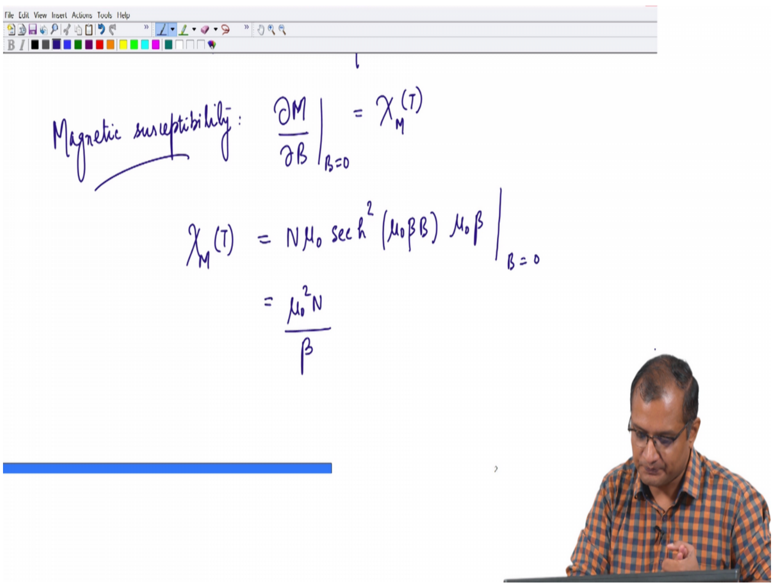

So, now with the magnetization in the thermodynamic limit we are in a position to compute the magnetic susceptibility which is nothing, but derivative of magnetization with respect to field at 0 field. So, which means that this is magnetic susceptibility at some temperature T, it tells you basically how M can be expanded in powers of B for small value is a field. So, if M has a Taylor series expansion around origin then this. Susceptibility is nothing, but the first sub leading term in the in the in the Taylor expansion.

So, here we can compute the chi of M at some temperature t as simply the derivative of the magnetization that we have constructed. So, you can complete the derivative is as a constant free factor tangent hyperbolic becomes second hyperbolic square of mu 0 beta B and to complete the derivative I have a mu 0 into beta outside ok. So, this is nothing, but mu 0 square N upon beta and the entire derivative is taken at B equals to 0. So, this just becomes mu 0 N mu 0 square N upon beta because second hyperbolic 0 is 1 is 1 upon cosine hyperbolic 1 0 which is already 1 this is basically the magnetic susceptibility.

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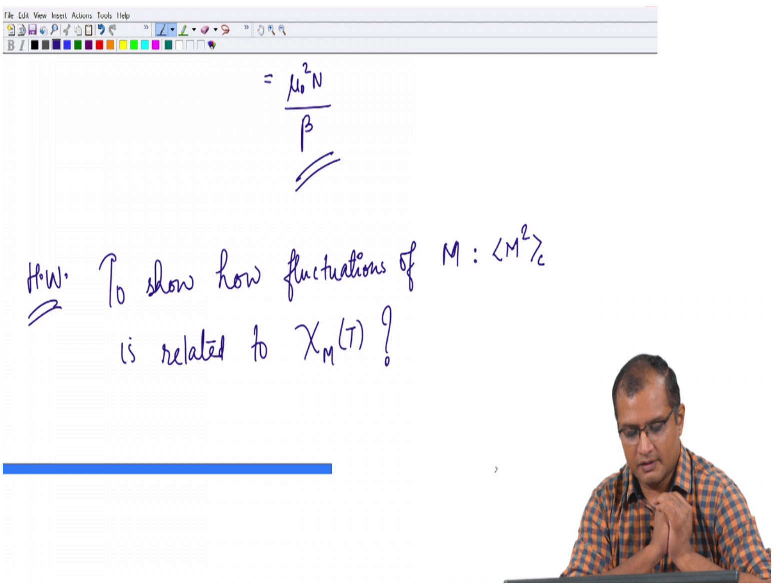

And there is a homework task that I would give you in this lecture is to show how fluctuations of magnetization. Fluctuations of magnetization which means fluctuation is represented as the second cumulant is related to susceptibility magnetic susceptibility ok.

So, that is your home work task you can do this exercise and learn something about fluctuations in the magnetic system ok. So, you will be you will be surprised to learn or you would be happy to see that these fluctuations again have a very important meaning in canonical ensemble when quantities are not really consort. So, let us meet in the next class and I will show you the result and then we will also startup with the final ensemble of this course which is called as the grand canonical ensemble.

So, we will break here and we meet in the next class.