Theory of Yarn Structure
Prof. Dipayan Das
Department of Textile Technology
Indian Institute of Technology – Delhi

Lecture – 23
Tensile Mechanics of Yarns (contd.,)

Welcome to you all to this MOOCs online video course, theory of yarn structure. We have started discussing module 8, tensile mechanics of yarns in the last 3 classes. We talked about mechanics of parallel fiber bundles, then we started discussing about stress-strain relation. So in the last class we have derived so far this relation. So our aim is to find out a relation characterizing fiber strain and yarn strain.

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$$(1+\varepsilon_l)^2 = \frac{(1+\varepsilon_a)^2 + (1-\eta\varepsilon_a)^2\tan^2\beta}{1+\tan^2\beta}$$

$$= \frac{(1+\varepsilon_a)^2 + (1-\eta\varepsilon_a)^2\tan^2\beta}{\sec^2\beta}$$

$$= (1+\varepsilon_a)^2\cos^2\beta + (1-\eta\varepsilon_a)^2\sin^2\beta$$

$$= (1+2\varepsilon_a+\varepsilon_a^2)\cos^2\beta + (1-2\eta\varepsilon_a+\eta^2\varepsilon_a^2)\sin^2\beta$$

$$1+2\varepsilon_l+\varepsilon_l^2 = \underbrace{(\cos^2\beta+\sin^2\beta)}_{=1} + 2\varepsilon_a(\cos^2\beta-\eta\sin^2\beta) + \varepsilon_a^2(\cos^2\beta+\eta^2\sin^2\beta)$$

So this is your fiber strain, this is your yarn strain. This is contraction ratio and beta is the angle of fiber and we obtain this relation. Now you will work from here. So 1 + tan square beta is X square beta. We do not change the numerator 1/sec square beta is cos square beta, so tan square beta * cos square beta is sin square beta right. Now we expand it 1 + 2 epsilon a + epsilon a square cos square beta + 1 - 2 eta epsilon a + eta square epsilon a square sin square beta.

Then we see that cos square beta + sin square beta + 2 epsilon a, no it is 2, yeah, 2 * epsilon a cos square beta – eta sin square beta + epsilon a squared cos square beta + eta squared sin square beta cos square beta + sin square beta all right. If we expand at the left hand side also then what we obtain is square and this is = 1. Now we assume that the strains are small. Fiber strain as well as yarn strain all are small in magnitude.

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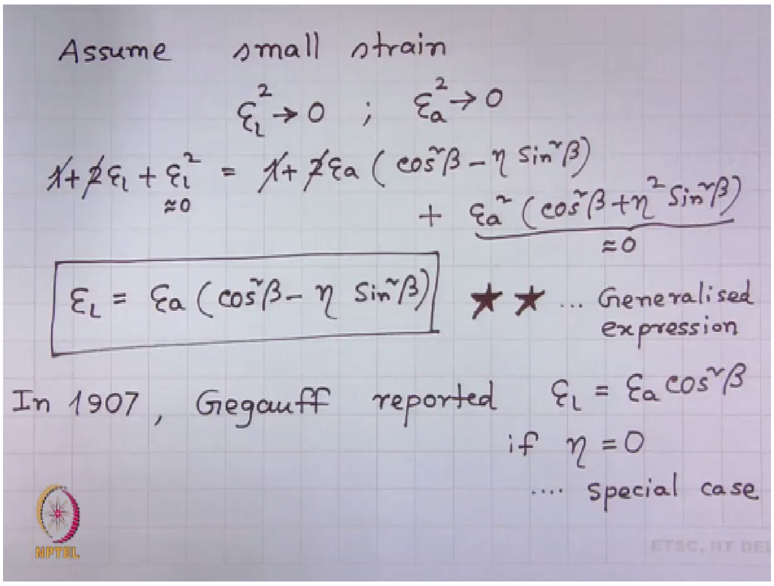

When fiber strain is small so square of a small quantity will be further small tends to 0. Similarly, when axial strain is small square of a small quantity will be further small, this tends to 0. So if we rewrite our earlier expression 1 + 2 epsilon a cos square beta - ETA sin square beta + square cos square beta * square sin square beta. So when this tends to 0 these tends to 0, this 1 and 1 cancel out.

So we obtain these 2 too cancel out, this is approximately = 0, this also 0. So what we obtain is * sin square beta. This relation is a very important relation because it characterizes what will be fiber strain if we apply a given amount of strain to yarn. So fiber strain depends on yarn strain, twist angle beta and contraction ratio eta. These 3 quantities determine fiber strain.

Long ago in the era of 1907 a very famous scientist Gegauff reported this relation. This relation can be obtained from this expression if we obtain this = 0. So this is a very special case. When there is no contraction, no radial contraction in yarn fiber strain = yarn strain * cos square of twist angle; however, this expression is a more generalized one. If we know contraction ratio, twist angle, yarn axial strain will be able to predict fiber strain using this relation.

So we derived a relation between fiber strain and yarn strain. Now we would like to derive a relation between fiber stress and yarn stress. So what is the relation between fiber stress and yarn stress, this we are going to derive now.

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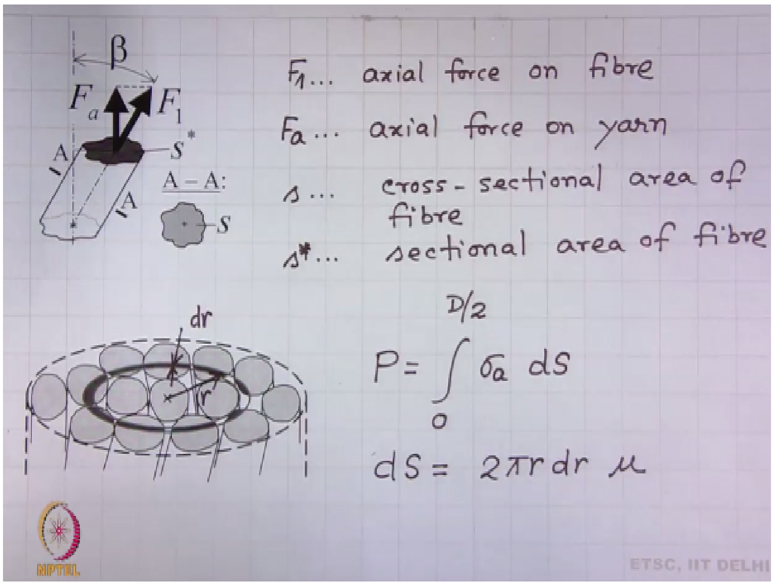

For this derivation we take help from this figure. What you see is an oblique fiber, if an axial force F subscript 1 is applied to this fiber and F subscript a is the axial force to the yarn, so F subscript 1 stands for the axial force to the fiber where as subscript a represents axial force in yarn. Beta is the twist angle. The cross sectional area of the fiber is small s. As a result, the sectional area is S star.

So let us write what is what here. F1 axial force on fibre, F subscript a axial force on yarn, cross-sectional area of fiber and is the sectional area of fiber right.

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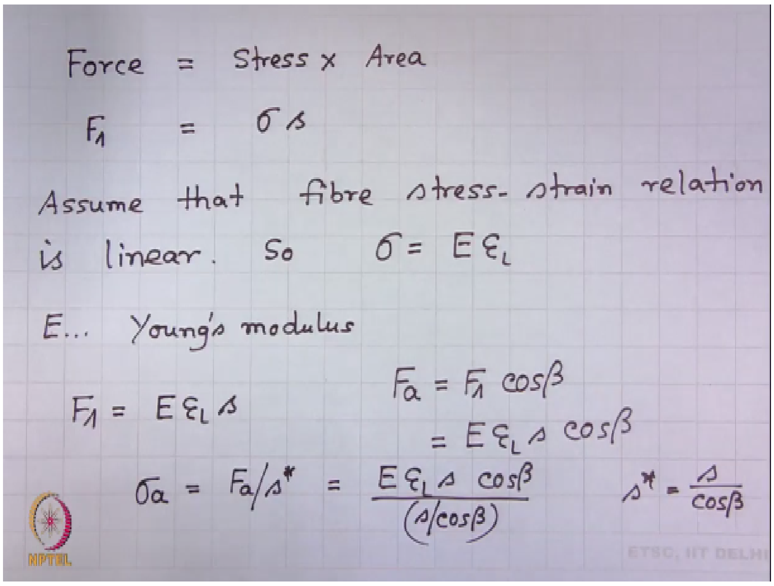

We know that force is area, so F1 is the force let us assume stress is sigma and what is the area? small s. We assume that fiber stress-strain relation is linear. So sigma is = E times epsilon L, Hook's law. Sigma is fiber stress, E Young's modulus and epsilon L we know fiber strain. So if we substitute this relation here what we obtain is * s. So axial force on fiber is = initial modulus of fiber, axial strain on fiber and cross sectional area of fiber.

Now we go back to this drawing again. We know the quantity of F1, we are interested to find out F subscript a. Beta is the twist angle. So F subscript a = F1 times cos of beta. So let us write that. F subscript a = F1 times cos of beta. F subscript a = F substitute 1 times cos of beta okay. So what is F subscript a, F subscript a is from here. Young's modulus fiber strain, fiber cross section area * cos beta.

Then what is this trace developed along the axial direction of yarn? What is this trace developed along the axial direction of yarn? Stress = force per unit area. So force is F subscript a. What is the area? This area is S star. So sigma subscript a = F subscript a/S star. Stress = S star. What is F subscript a, E epsilon L s cos beta/S star. S star we know from module 2 is s/cos beta. So S star is s/cos beta.

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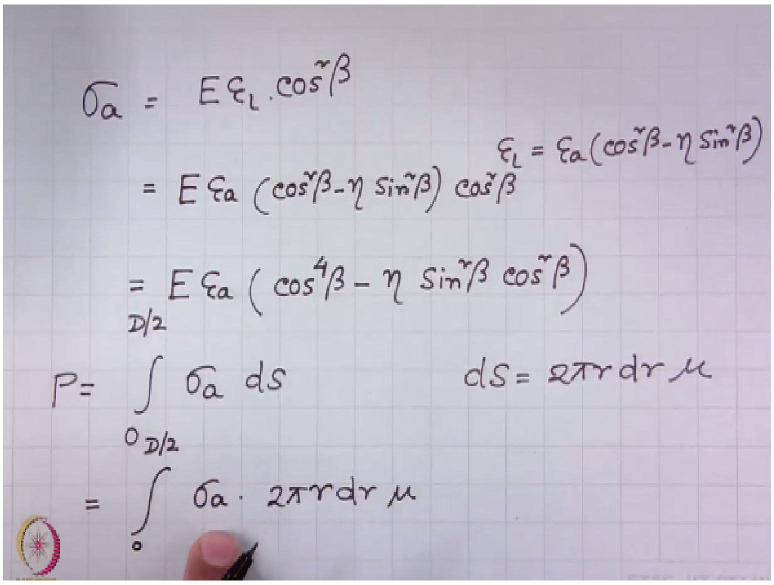

So sigma a becomes E epsilon L cos square beta because you see here this s and this s cancel out. So sigma subscript a = E Young's modulus epsilon L fiber strain * cos square beta, beta is twist angle. Right now we know E subscript L = this form. We have derived this expression a few minutes before. If we now substitute this here * cos square beta. So E epsilon a cos to the power 4 – beta.

What is this expression? This expression is the axial stress of one fiber along the yarn axial direction. So now we have to find out what is the total axial stress or total axial force developed in yarn. For that purpose, we come back to this drawing. What you see here a cylindrical yarn circular cross section is shown here, lot of fibers are present here. We consider an annular ring which is situated at a radius r and whose thickness is dr.

So if we find out what is the force developed, force on fibers in this annular ring then if we integrate that expression from 0 to d/2 we will obtain the total force developed in the yarn right. So the total force developed in the yarn is the integration of stress * area. So what is stress? Stress is sigma a, stressed on one fiber and so the total area available is d subscript s, d subscript s is the area of fibers available in this annular ring.

If we integrate that from 0 to d/2, we obtain the total force. So what is ds? d subscript s stands for the elementary sectional area of fibers. Elementary substance cross section area of fibers. These we define in module 2. So we know mu is = fiber area / yarn area. So fiber area = yarn area * mu. What is the yarn area here? 2 pi r dr * mu. So this comes from the definition of packing density.

Packing density is defined by fiber area ds/yarn area 2 pi r dr, so ds becomes = 2 pi r dr * mu. So now if we write this expression P sigma a ds 0 to rho/2 then ds = 2 pi r dr * mu. So if we substitute sigma a * 2 pi r dr *mu. If we substitute sigma a here, then we will get a relatively long expression.

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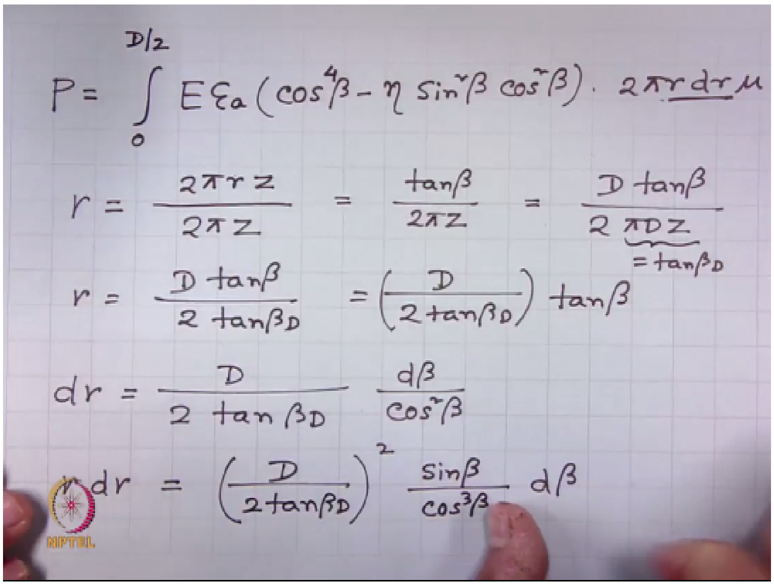

So p = 0 to D/2 E * epsilon a cos to the power 4 beta – eta sin square beta cos square beta * 2 pi r dr times mu right. This form of integration we see 2 variables, one is beta, second is r and we know that this r and beta are related because tangent beta = 2 pi r z. So we have to either convert this beta * r or this r * beta, so that we will be able to do the integration. So let us convert these r * to beta, so let us do that.

How we do that r. What is r? r can be written by 2 pi r z/2 pi z. What is 2 pi r z? 2 pi r z is tangent of beta, 2 pi z. In the denominator we multiply capital D. Similarly, we have to do the same for the numerator, so let us do that, D what is this pi DZ? tangent beta D. So r becomes D tan beta/2 beta D. Now this D/2 times tangent beta D is a constant, tangent beta right. So dr is D/2 tangent beta sec square beta. So that is = cos square beta.

Now what is r dr, here it is r times dr. So D/2 square * tan beta/cos square beta. So sin beta/cos cube beta, d beta. We will substitute this into this expression. So this expression will become further long. Let us do that.

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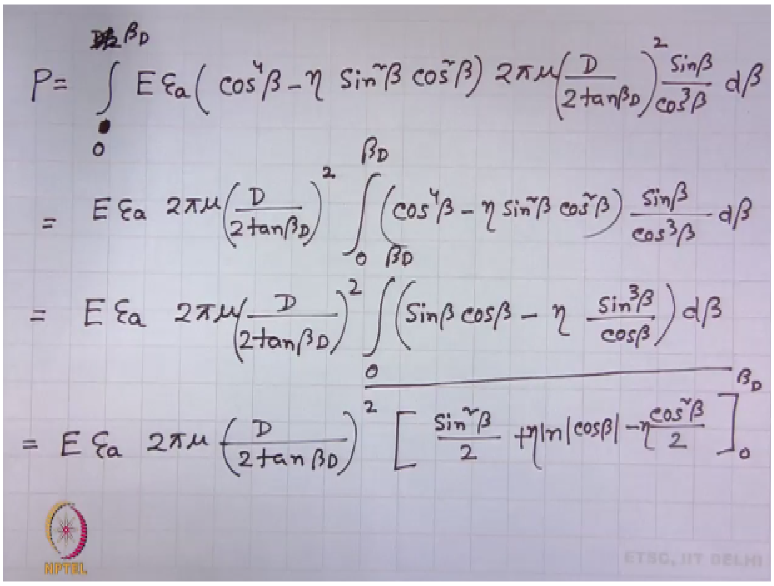

So p = 0 to D/2 * epsilon a cos to the power 4 beta 2 pi mu * r dr, when r = 0, this will be 0 and when r = D/2 the angle will be beta D. So the limit will change from 0 to beta D. So what we obtain is E epsilon a 2 pi mu * D/2 tangent beta squared integration cos to the power 4 beta * d beta. So epsilon a 2 pi mu squared integration sin beta cos beta – eta sin cube beta/cos beta * D beta right. This limit is 0 to beta D, so we have to solve this integral. So there are 2 indefinite integrals. Let us solve them.

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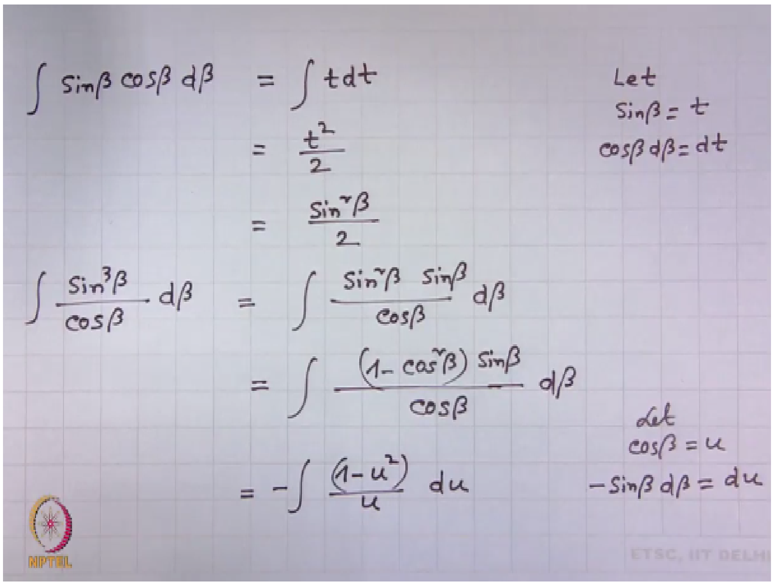

First is sin beta, cos beta, D beta, let sin beta = t, so cos beta d beta = dt. So this integration will become tdt, integration of tdt, t square/2, what is t square? sin square beta/2. So you come back and write the solution one by one. Sin square beta by 2 - what is the next integral? Next integral is sin cube beta/cos beta * d beta. This we can write as * d beta. Let us assume cos beta = u, -sin beta d beta = du. So 1 - u square/u du.

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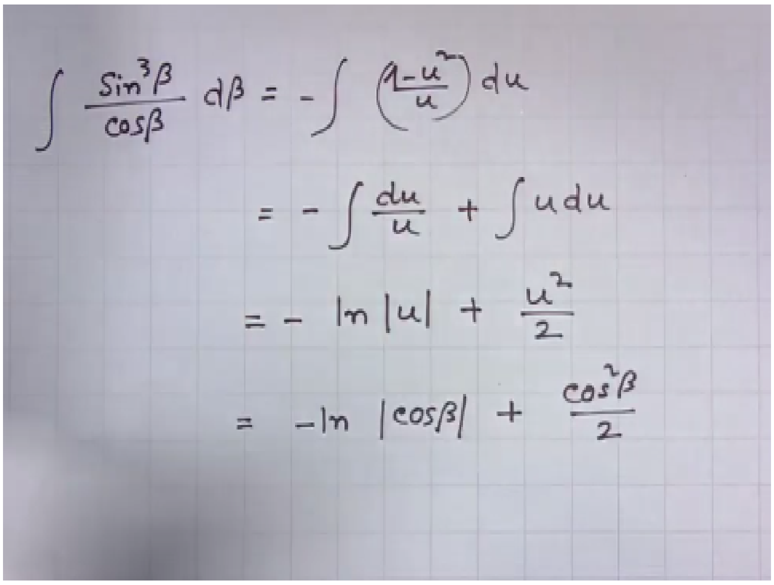

So we continue integration sin cube beta/cos beta d beta - 1 - u square/u du. So –du/u + udu, -du/u ln u + u square/2. What was u? u was cos beta, so –ln cos beta + cos square beta/2. So this is the solution of this integral, we come back and we write this -, so - * - will be + ln cos beta - cos square beta/2 and this will be multiplied here, this will be multiplied here. So we will now write this.

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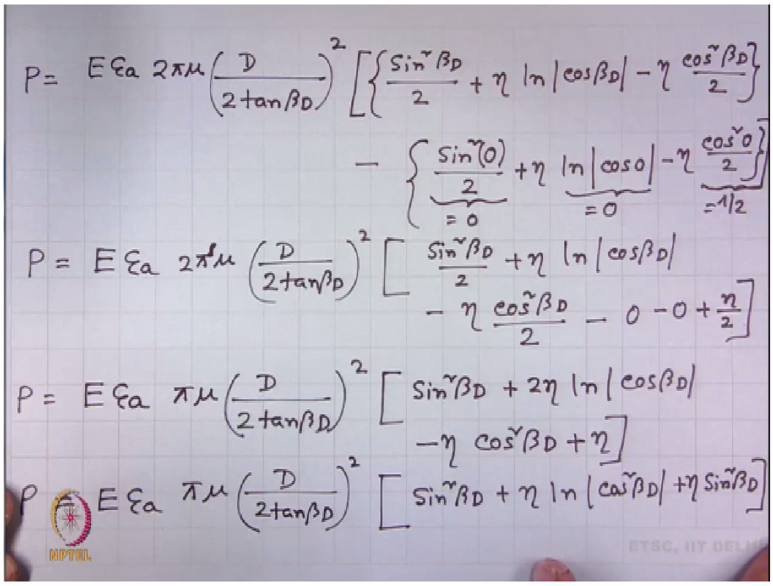

So P = E epsilon a 2 pi mu D/2 tangent beta d square, sin square beta D/2 + ln cos beta D – eta cos square beta d/2. This is the first part - second part, sin squared 0 by 2 eta ln cos 0 – eta cos square 0/2. So what it becomes? P E epsilon a 2 pi mu D/2 tangent beta d squared sin square beta D/2 + eta ln cos beta D – eta cos square/D -, this is = 0, this is = 0, this is = 1/2. So 0 – 0 + eta/2.

What do we do now? these 2 and these 2 we used to cancel So E epsilon a pi mu beta D squared sin square beta D + 2 cos beta D – eta cos capital D + ETA okay. Further we write E epsilon a pi mu squared sin square beta D + these two we will shift here because of logarithmic function, eta ln cos square beta D + eta – eta cos square beta D. So + eta * 1-cos square beta D, that means + eta * sin square beta D. Sin square beta D + eta * sin square beta D. So 1+ eta sin square beta D, that will be the next step.

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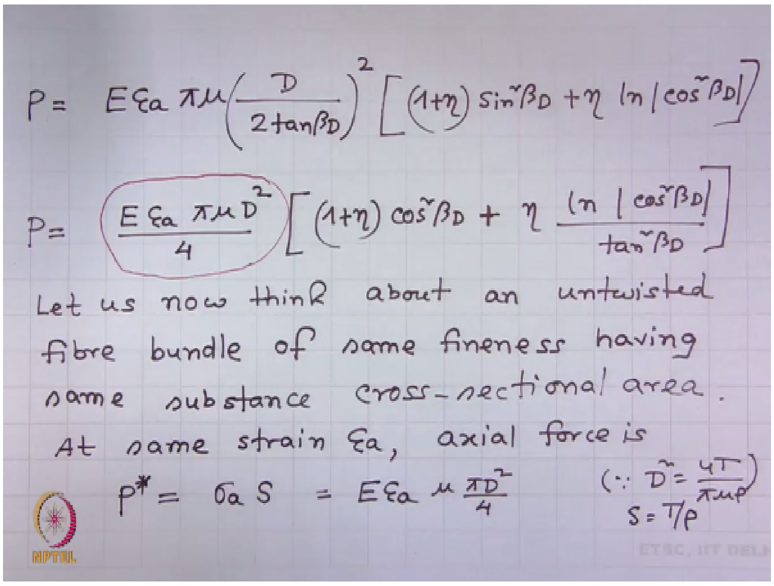

So P E epsilon a pi * mu D/2 tangent beta D square 1 + eta sin square beta D + eta ln cos square beta D. This tan square beta D we will divide. So what will become E epsilon a pi mu D square/4 * 1+ eta, sin square beta D/tan square beta D, cos square beta D + eta ln. So this is the expression for axial force in yarn. Now this quantity talks about something what is that? Suppose we think about an untwisted fiber bundle of same count and same substance cross-sectional area.

Let us now think about an untwisted fiber bundle that means parallel fiber bundle of same fineness having same substance cross-sectional area. Let us now think about an untwisted parallel fiber bundle of same count and having same substance cross sectional area. So at same strain? What is the strain? epsilon a, what will be the axial force? This axial force we denote by P star.

P star will be equal to sigma a * substance cross sectional area. Force is = stress * area. What is sigma a for such parallel fiber bundle? E * epsilon a and what is s? s is mu times pi d square/4. Since d square is = 4T/pi mu rho and S = T/rho right. This we learned in module 2. So you see this now. E epsilon a mu pi d square 4. So this quantity is = the axial force of a parallel fiber bundle having same fineness and same substance cross sectional area.

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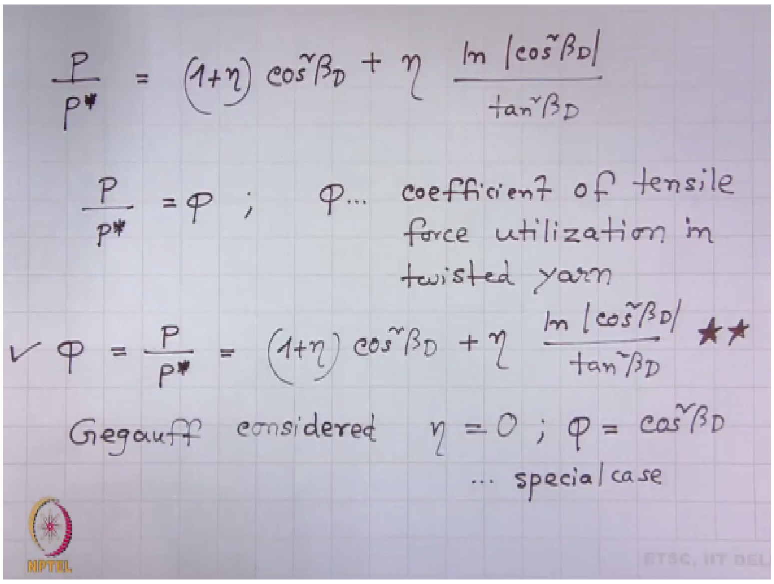

So if we write now P/P star this will be = 1 + eta cos square beta D + ETA ln cos square beta D/tan square beta D. So P/P star is = 1 + eta cos square beta D + eta ln cos square beta d/tan square beta d, same here. Now something about this quantity P/P star. Yarn axial force divided by axial force on a parallel fiber bundle having same fineness and same substance cross sectional area like yarn.

So this quantity we denote as phi, where phi is known as coefficient of tensile force utilization in twisted yarn. The material remains same, fiber material remains same; however, this is yarn, this is parallel fiber bundles so there is an influence of actually obliquity. So this quantity characterizes structural effect in the yarn. So coefficient of tensile force utilization in twisted yarn is = P/P star which is = 1 + eta cos square beta D + eta ln cos square beta D/tan square beta D.

This relation is very important because it characterizes the relation between yarn stress and fiber stress, yarn force and fiber force. Now Gegauff considered contraction = 0, there is no radial contraction in yarn, so is cos square beta D. So this is a very special case; however, this is a very generalized expression.

Now how does this expression relates to reality? What is the comparison between this expression? this theoretical result with experimental findings. Let us now learn about that.

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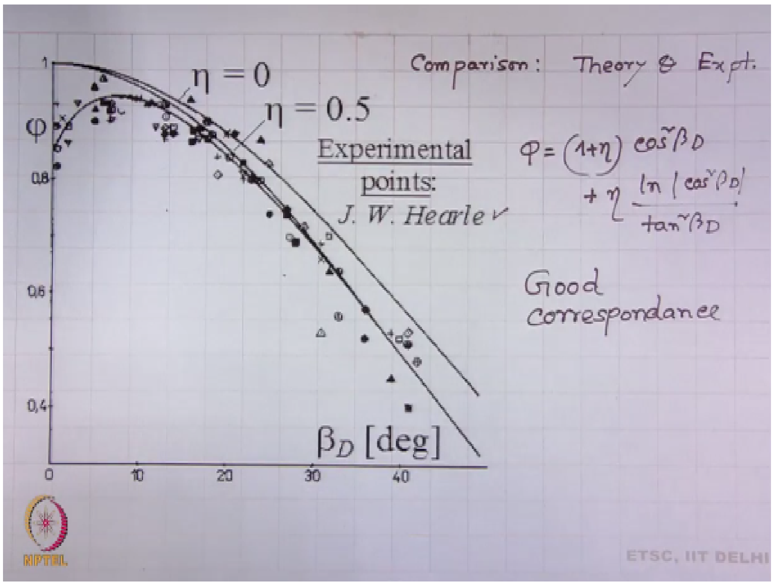

This compares theory and experiment. When eta if we put eta is = 0 here then we obtain the first curve, this curve. When eta is = 0.5 here for different angles of beta D, we obtain the next curve and this curve is something different, which is the result of another theory where (()) (49:09) variability was considered. What we see is that from this region onwards except from 0 to 10 degree to surface twist angle, all the experimental results are tending to fall onto the theoretical line.

In this part there was little deviation was there and that could be probably explained by some other phenomenon, it is related to (()) (49:44) variability in fibers in yarn. So in this particular part there is a very good correspondence obtained between theory and experimental results. Now we would like to comment on a very well-known aspect in yarn.

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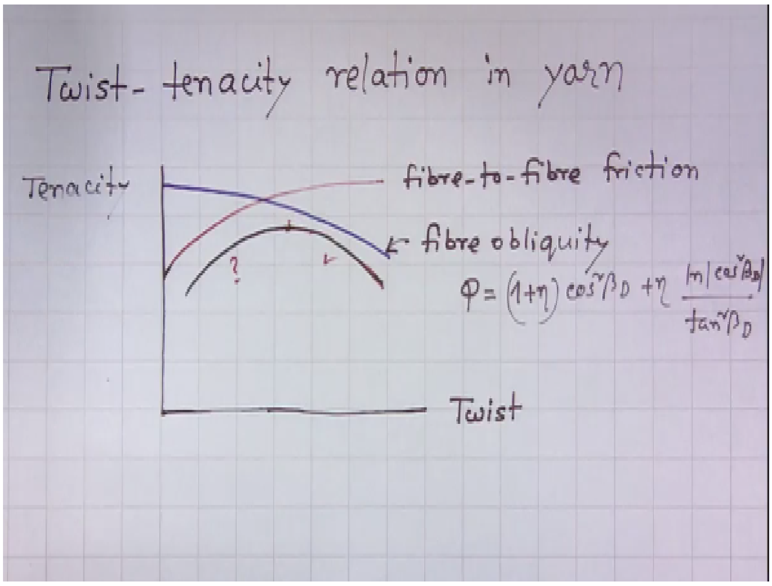

Twist tenacity relation in yarn, as it is well known that it increases and then decreases. This particular curve is the result of 2 influences, one influence is this which is the result of fiber to fiber friction in yarn. The other influence we have just now learned is the effect of fiber obliquity. So in a twist tenacity curve this part of the curve the reason is understood; however, fiber to fiber friction and its effect on tenacity of yarn still an unsolved problem in the theory of yarn.

In future one can attempt to solve this part of this curve theoretically; however, this part is theoretically solved. So now we will proceed to solve numerical problems related to this module. The very first numerical problem what we would like to solve is related to fiber strain and yarn strain. The second numerical problem will be related to fiber stress and yarn stress.

So there were earlier 4 numerical problems related to Hamburger's model. So this is the fifth problem in this module.

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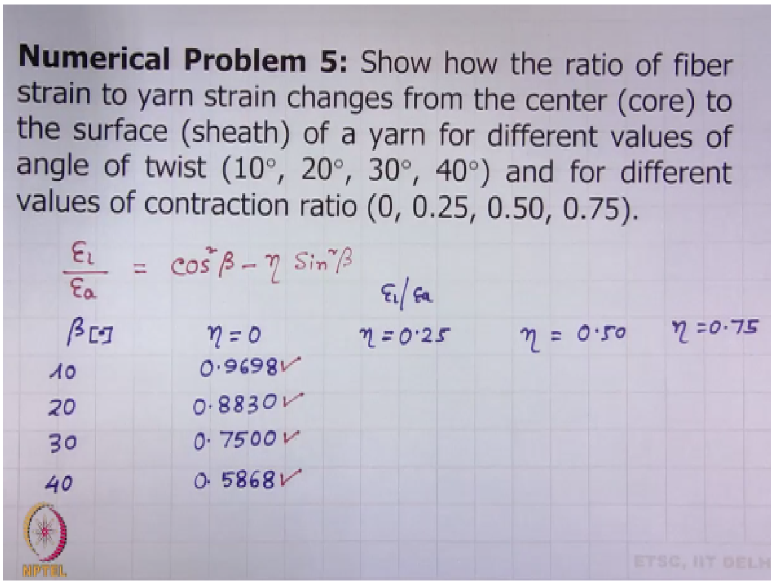

So how the ratio of fiber strain to yarn strain changes from centre to surface of a yarn for different values of angle of twist 10 degree, 20 degree, 30 degree, 40 degree and for different values of contracts and ratio 0, 0.25, 0.5, 0.75. So this is a problem. Fiber strain to yarn strain. this is your fiber strain to yarn strain. Cos square beta * sin square beta. So we have to use this formula to solve this problem.

See beta is = 10 degree, 20 degree, 30 degree and 40 degree okay. We will talk about this ratio here. It is a dimensionless ratio for different value 0, 0.25, 0.50, 0.75 right. Now we substitute beta 10 degree and eta 0. So you will obtain this value 9698. Then we substitute beta 20-degree here and eta 0. So this component is 0. So then you will see the value 0.8830. Now eta 0, so this component vanishes 30 degree beta.

So beta 30 degree means 0.7500, then 40 degree beta, eta is 0 so this component vanishes. Cos square 40 degree, what will be the value? 5868 okay. So when eta = 0, so this component vanishes, so this column is obtained cos square 10 degree is this value, cos square 20 degree this value, cos square 30 degree is this value, cos square 40 degree is this value.

What we see is that as the beta increases this ratio decreases that means if we apply more twist because of the obliquity this ratio is reducing, similar manner we will be able to solve what is this ratio when eta = 0.25 for different angles of beta and for different values of contraction ratio. We will solve it in the next class. Thank you. Thank you for your attention.