Digital Signal Processing

Prof: S. C. Dutta Roy

Department of Electrical Engineering

Indian Institute of Technology, Delhi

Lecture - 16

All pass filters, Comb Filters and Linear phase FIR filters

This is the 16th lecture on DSP and we start with all pass filters. We had already started the discussion last time. We will also introduce what are comb filters and then discuss in details linear phase FIR filters.

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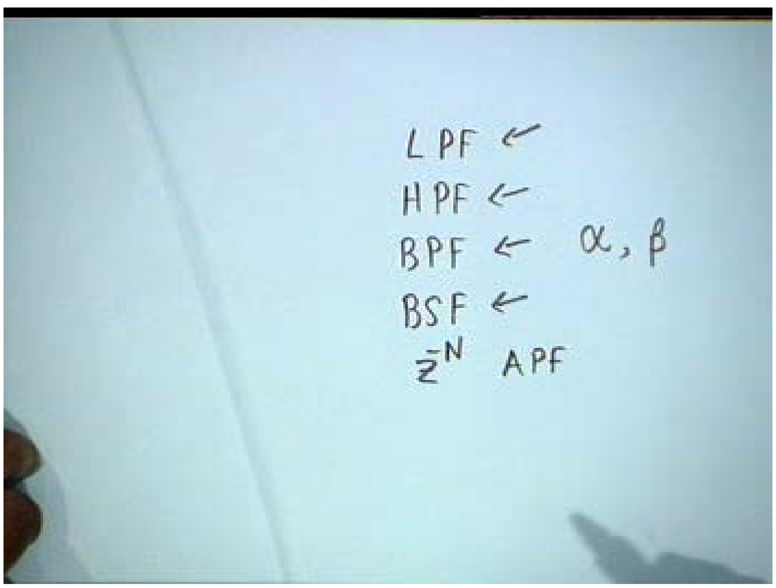

On the last occasion, 15th lecture, we discussed simple FIR filters - low pass, high pass, band pass and band stop. We said that for FIR, the all pass filter is a trivial one. In other words, it is a simple delay. You cannot have a polynomial in z^{-1} whose magnitude will be equal to 1. z^{-N} is the only trivial all pass FIR filter. Then we talked about simple IIR digital filters and we showed how to construct first order low pass and high pass filters. We normalize the maximum magnitude to 1 and that is why we required a multiplying constant (1 + alpha)/2 or (1 – alpha)/2.

Then we introduced the band stop filter; once again the maximum magnitude was normalized by unity. Band pass filter was designed in terms of two parameters - alpha and beta, where beta controls the center frequency and alpha controls the band width. Similarly, for a band stop filter, beta controls the notch frequency $omega_0$ and alpha controls the frequencies at which the magnitude response is 3dB down. There is a difference between bandwidth definitions for band stop filters and low pass, high pass and band pass filters. In all the three latter cases, the 3dB bandwidth is the band of frequencies in which the magnitude remains within 3dB below the maximum magnitude, which was normalized to unity. In a bandstop filter, if w_1 and w_2 ($>w_1$) are the 3-dB frequencies, then the two passbands are 0 to w_1 and w_2 to π. So the pass bandwidths are w_1 and π-w_2. The stop bandwidth is w_{s2}-w_{s1} and w_{s2} and w_{s1} are the frequencies at which the magnitude is $\leq$ some specified value γ ($<<1$). In the case of all pass filters, we discussed several properties. And the simplest all pass filters is the first order one.

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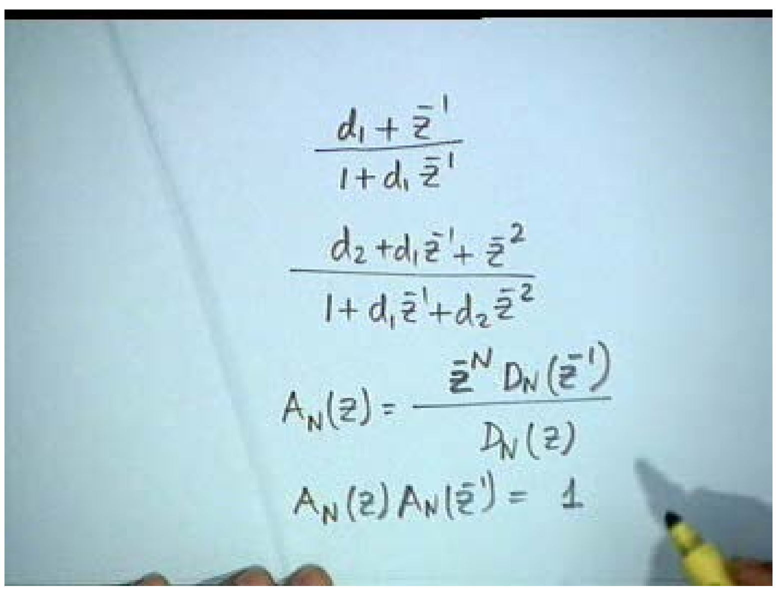

In the first order filter, if you take the denominator as $1 + d_1z^{-1}$, then the numerator is simply the same polynomial with coefficients in reverse order, that is $d_1 + z^{-1}$. Similarly, if I have a second order filter the denominator of which is $1 + d_1z^{-1} + d_2z^{-2}$, then the numerator shall be $d_2 + d_1z^{-1} + z^{-2}$. We also showed that in general, if $A_N(z)$ is the Nth order all pass filter, whose denominator is

$D_N(z)$, then the numerator is simply $z^{-N}D_N(z^{-1})$. This is the general property. Obviously this is written in such a form that the magnitude is normalized to unity. That is, $A_N(z)\,A_N(z^{-1}) = 1$. One question that was asked by one of the students was: where do you apply all pass filters if the magnitude is 1? We apply them in delay equalization.

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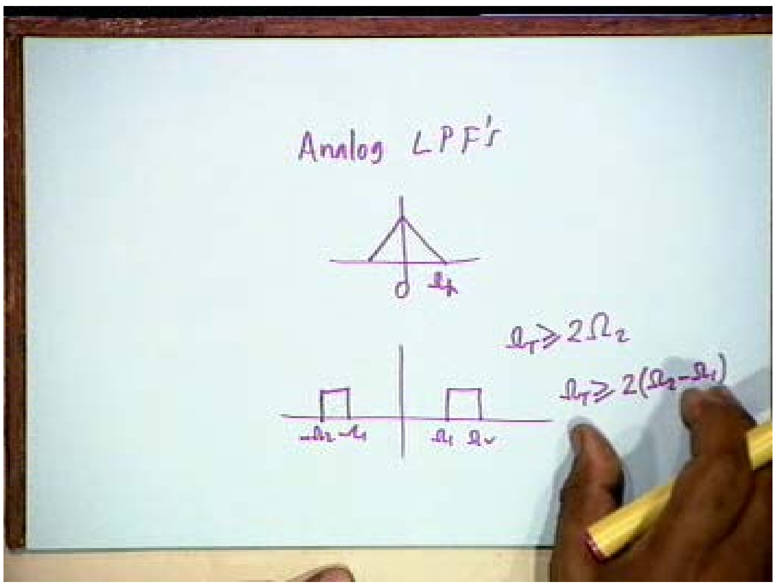

What is delay equalization? I have told you earlier that if a transmission device, which could be a transmission system or a channel, has a nonlinear phase, i.e if the phase does not vary linearly with frequency, then we have a problem of delay distortion. That is, a group of frequencies starts together and if all the frequencies do not reach together, then there shall be distortion in the received signal. This distortion is called delay distortion. Now, in practice, linear phase is possible only with finite impulse response digital systems; with analog systems, it is not possible at all. In digital systems, FIR is possible to design with exact linear phase or constant group delay.

Suppose we have a transmission channel which has dispersive property, in other words, the group delay is not a constant. If the group delay decreases with frequency, then at the receiving end, we can compensate for this by an all pass filter (so that the relative magnitudes do not change) where the phase is so designed that the group delay of the all pass filter increases with frequency in an inverse fashion. Then from the transmitter to the receiver; the group delay shall be a constant. The overall group delay is the sum of the group delay by the channel and that by the all pass filter. So delay equalization is the most important application of all pass filters. But as we go through the course, you will see that all pass filters have many other applications. In fact if you can design all pass filters then you can design any other kind of filter by appropriate combination of all pass filters. We shall see how to design low pass, high pass, band pass, band stop, multi band pass, multi band stop and all other kinds of filters. Some properties of all pass filters require knowledge of what is called a bounded real function.

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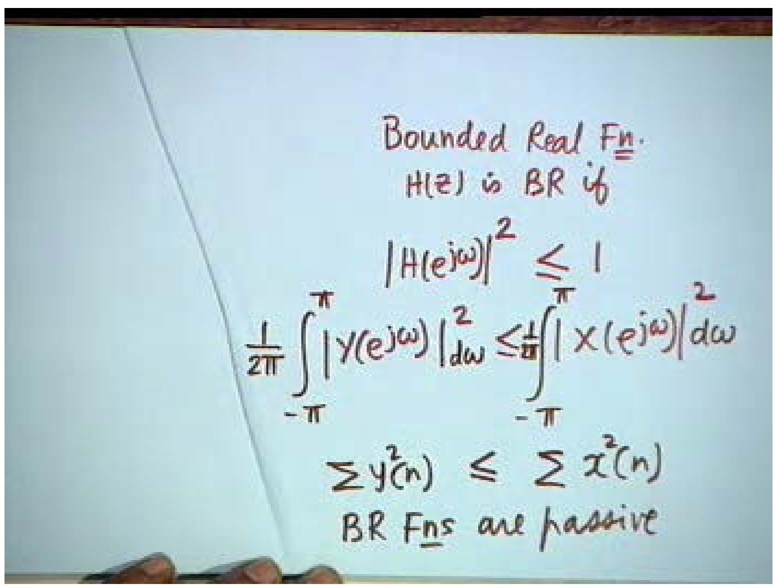

A function H(z) is said to be bounded real (BR) if the magnitude of $H(e^{jomega})$ is less than or equal to 1. Note that we have used all filters such that the magnitude is normalized to unity. In other words, all of the transfer functions that we have derived or discussed so far are bounded real. Bounded real also has an interpretation in terms of energy. If the magnitude is bounded by unity, the square of the magnitude is also bounded by unity. Now what is $H(e^{jomega})$? It is the spectrum, the ratio of the Fourier transform of output to the Fourier transform of input. So bounded real $H(e^{j\omega})$ ensures that $Y(e^{jomega})$ magnitude squared is less than or equal to $X(e^{jomega})$

magnitude squared. By linearity, if we integrate both from – pi to pi and divide it by 2pi, then by Parseval's theorem, these represent the energy of the output and input signals. Therefore it follows that the output energy is less than or equal to the input energy. In other words, bounded real functions are passive. If the inequality is satisfied with the equality sign, then we say the system is lossless.

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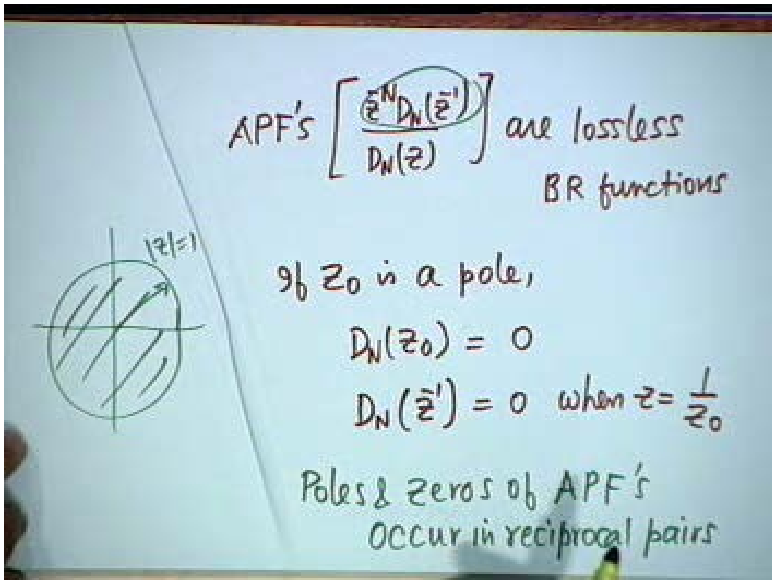

Therefore, we make this statement that APF's as designed by us, that is, of the form $z^{-N}D_N(z^{-1})/D_N(z)$ are lossless bounded real functions. They are not only passive but passivity is taken to the limit, they are lossless; they do not absorb any energy. The other property of APF's is: if p_0 is a pole then $D_N(p_0) = 0$. Therefore $D_N(z^{-1}) = 0$, when $z = 1/p_0$. Now the numerator is $z^{-N}D_N(z^{-1})$, so its zeros shall be those values of z at which $D_N(z^{-1}) = 0$. In other words, the poles and zeros of all pass functions occur in reciprocal pairs. This is a very important property because it says that since the APF is stable (the magnitude cannot be bounded unless it is stable), all poles of it must be inside the unit circle. Naturally all the zeros must be outside the unit circle. Transfer functions, all of the zeros of which are outside the unit circle, are called maximum phase functions.

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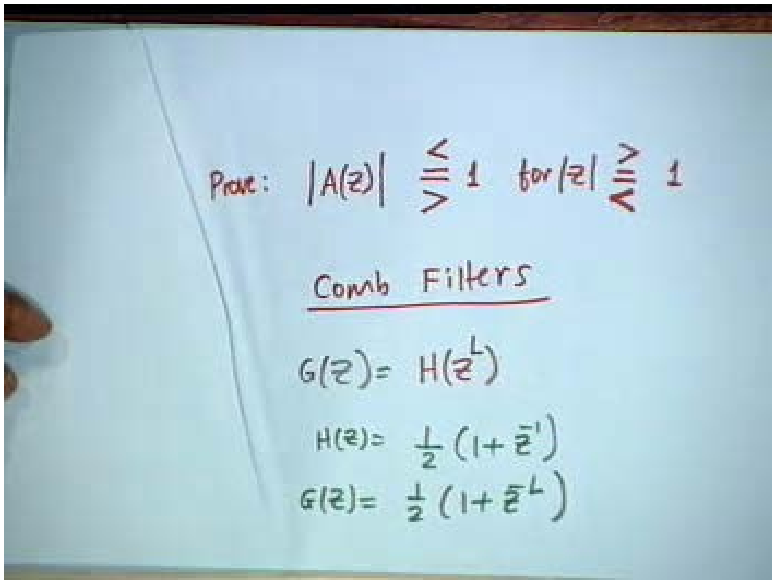

And by logic, it follows that all functions whose zeros are inside the unit circle shall be called minimum phase functions. If a function has zeros inside as well as outside, it is neither minimum nor maximum phase, and is called a mixed phase function. The physical interpretation is that, if you have a zero somewhere inside the unit circle and a zero outside the unit circle at a reciprocal location, then the magnitudes of the corresponding factors are equal. But the phase shift provided by the zero outside the unit circle is greater than that provided by the zero inside the unit circle; that is the interpretation. We shall not go into the details of derivation and other things. It suffices to remember that if zeros are inside the unit circle, then it is a minimum phase. If all zeros are outside then it is a maximum phase function. These are the terminologies; we shall not make much use of them and therefore we will not go into any more details.

Another property which requires proof is cited as a problem in Mitra. If you cannot do it, I shall do it in the class. The property is that the magnitude of A(z), now z is a general complex variable, that is $re^{j \text{ omega}}$, is less than, equal to or greater than 1 for mod(z) greater than, equal to or less than1. That is if mod (z) is greater than 1, i.e. z is outside the unit circle, then mod A(z) is less than unity and so on. This is a property which has to be proved. It is not obvious. Try to prove this; if you cannot do it then I shall do it in one class. That is about all pass, for the present;

we shall come back to all pass again and again. We shall require these properties and therefore make all pass an integral part of your DSP education. It plays a very important role today since the properties and applications of all pass have become extremely important today. No digital filter designers can afford to ignore any of these properties. You shall be able to do intelligent design of digital signal processors if you remember these properties of all pass filters.

We next discuss comb filters. The name is derived from the property that the frequency response looks like a comb. It has gaps and solid lines. The lines are not straight lines, they are frequency responses. I shall take a very simple case, but the definition is, that if you have a H(z) which has one pass band or one stop band and we replace each delay by L number of delays that is in the H(z) if we replace z by z^L then we get what is known as a comb filter. G(z) is a comb filter which is derived from a simple digital filter with each delay replaced by L delays where L is an integer. I shall take only one example: $H(z) = ½[1 + z^{-1}]$. This filter is a low pass filter because when z is 1,it is 1 and when z is – 1, it is 0. We must remember how to test what kind of a filter it is, very quickly. Now if I replace z by z^L then I get $G(z) = ½[1 + z^{-L}]$.

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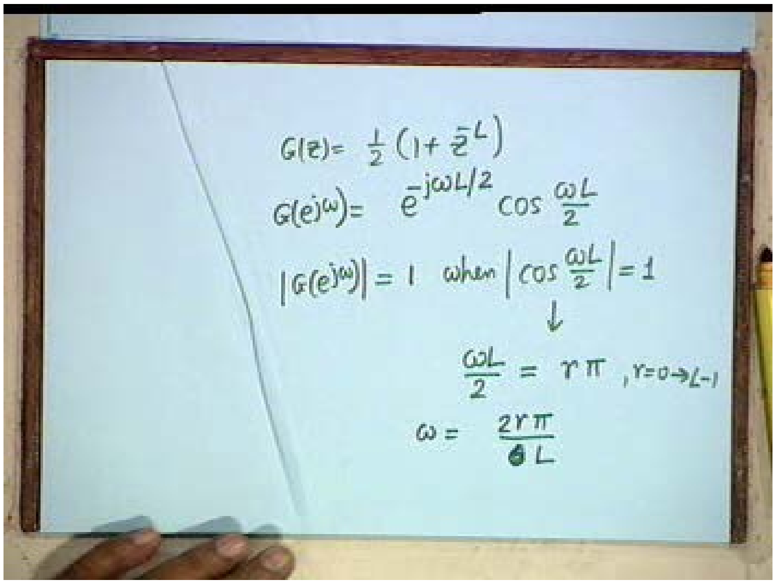

The frequency response of this function is G(e $^{j\ omega}$) = e $^{(-j\ LomegaL}$)/2 cosine(omega L)/2). Therefore │G(e$^{j\ omega}$) │= 1, when magnitude of [cosine(omega L)/2)] = 1. That occurs when (omega L)/2 = r pi, where r is an integer, going from 0 to L – 1. It is a periodic function. In one period, that is 0 to 2pi, the maxima occur at 2r pi/L, where r = 0 to L – 1, i.e. there are L maxima.

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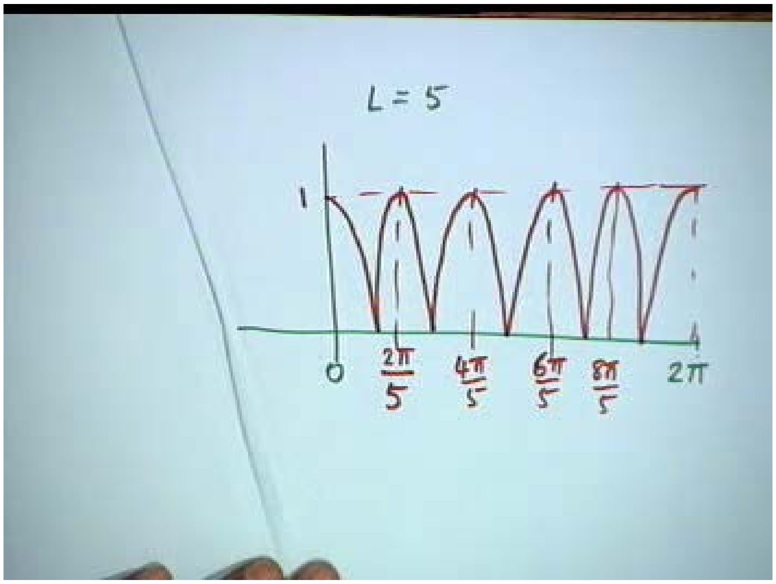

For example, take L = 5, then our frequency response looks like that shown in the figure. It starts from 1 at w = 0; there shall be 5 frequencies at which the magnitude will be 1. This response looks like a comb, that is why it is called a comb filter. Where is it used? Why do you require such a peculiar frequency response? As I told you, one of the simple applications is in a biomedical situation, where the signal is very weak and power line frequency 50Hz will drown the desired signal. You have to get rid of 50Hz. So what you do is that you design a comb filter whose first rejection is at 50Hz. Next one will be at 100Hz and so on. So it gets rid of not only 50Hz it gets rid of all its harmonics also. What you get at the output is a signal devoid of power line pick-up. There are other applications. You can design such a filter starting from FIR high pass filters also, for example ½[1 – z $^{-1}$].

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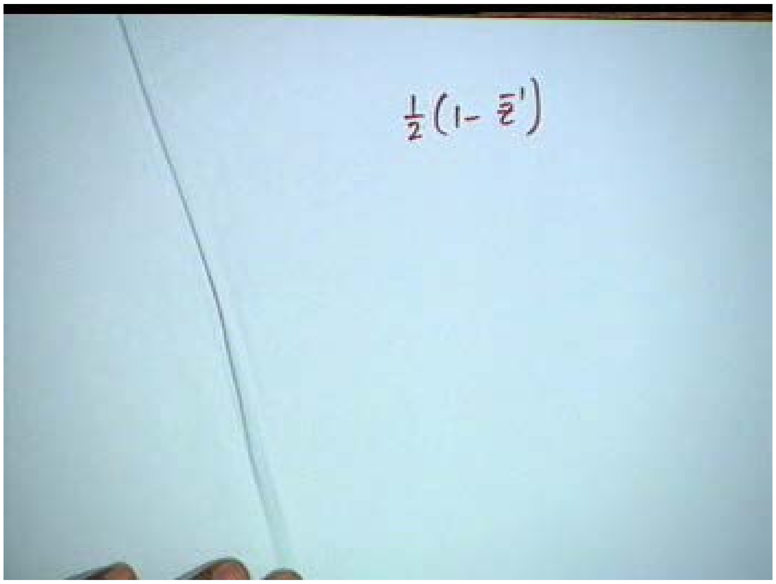

Replace z by z^L; the only difference here would be that the first rejection will be dc. It is useful where dc as well as some other periodic interference are to be rejected. The shape of the frequency response will again look like a comb. There will be maxima and minima. So will be for a band stop filter. How many pass bands are there in a band stop filter? Two, but it has only one stop band. Therefore you can make a comb filter out of that. That will also look like a multiple band stop filter. Similarly, you can make a comb filter from a band pass filter. There are two stop bands but one stop band in the filter.

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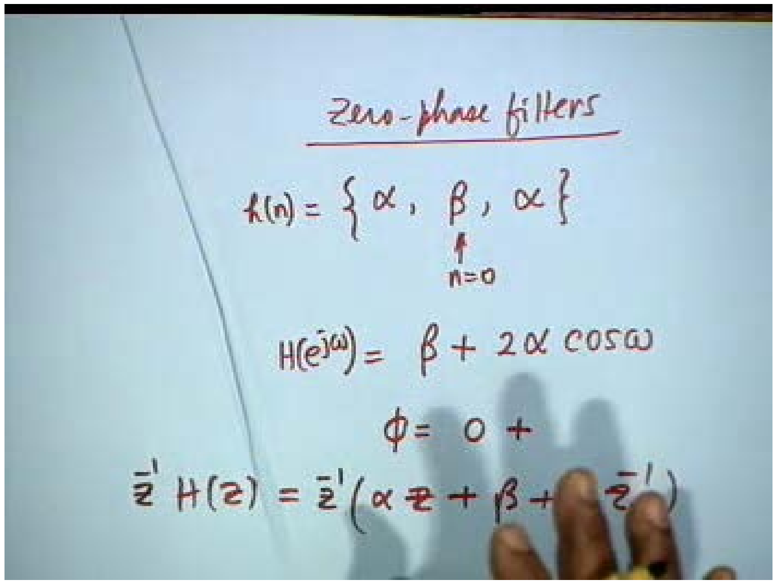

Next we talk of zero phase filters. That is the frequency response shall be a real function. Zero phase filters obviously require FIR. One example is h(n) = {alpha, beta, alpha}, where the sample beta is at n = 0. Frequency response will be beta + (2alpha)cosine(omega). It is a real quantity, but the phase is not 0 necessarily. The phase of this will be 0 + Φ, where Φ could be 0 or pi or 2pi or 3pi, depending on how many sign changes occur in the range 0 to π. Conventionally such filters are called zero phase filters, although the phase may not be 0 for the total frequency range; phase may be 0 or rπ, r being an integer. Obviously zero phase filters are non-causal, that is h(n) is not 0 for n less than 0. In order to make a real frequency response you require h(n) to the right and also h(n) to the left. However, the zero phase filters can be made causal by multiplying by z^{-N}. For the example, here the transfer function will be: H(z) = (alpha)z + beta + (alpha)z^{-1}. I can make it realizable by simply multiplying by z^{-1}, then the non causal filter becomes causal. In other words, physically what I am doing is to push from the left side so that the first sample starts at n = 0. I simply multiply by that number of delays. Some filter designers prefer to keep phase out of the picture in designing the filter and then add the required number of delays. If you get the term zero phase filters, do not get surprised. Ultimately, when you realize it in real time, you shall add appropriate number of delays.

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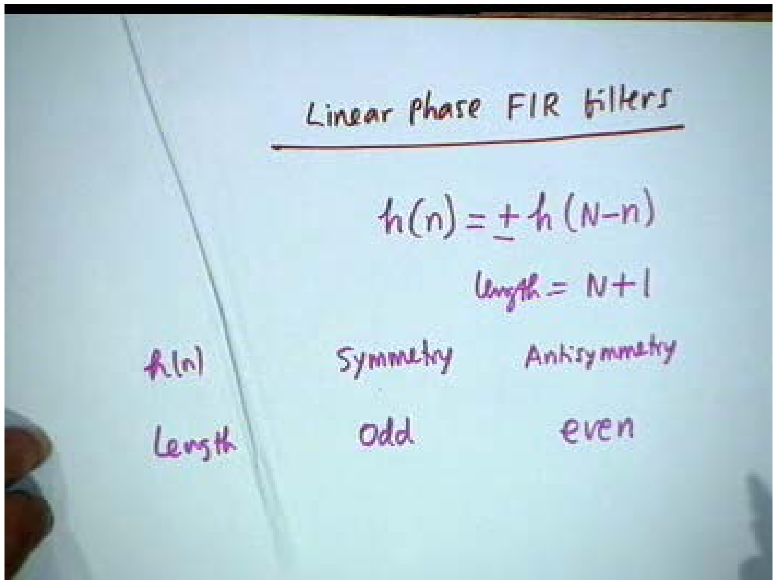

How does one design a linear phase filter? I have told you earlier that exact linear phase is achievable only by FIR digital filters. Linear phase cannot be obtained by an analog filter or an IIR filter. They only achieve approximately linear phase, whereas exact linear phase can be derived only from FIR filters. This is why FIR filters are important. In general, what you can do by a second order IIR filter can also be achieved by an FIR filter, but you will require a very large length. FIR design is costly, but if exact linear phase is a precise requirement, then you have no other alternative. We shall show that linear phase FIR filter must obey symmetry or anti-symmetry of impulse response that is, $h(n) = \pm h(N - n)$. The impulse response must be symmetrical or anti symmetrical in order to achieve linear phase. Now we can have a impulse response h(n), which can have symmetry or anti-symmetry and the length N can be either odd or even. Therefore four possible cases arise, that is h(n) is symmetrical odd length; h(n) is symmetrical even length; h(n) is anti-symmetrical odd length; and anti symmetrical, even length. These are usually designated as types 1, 2, 3 and 4 FIR linear phase filters.

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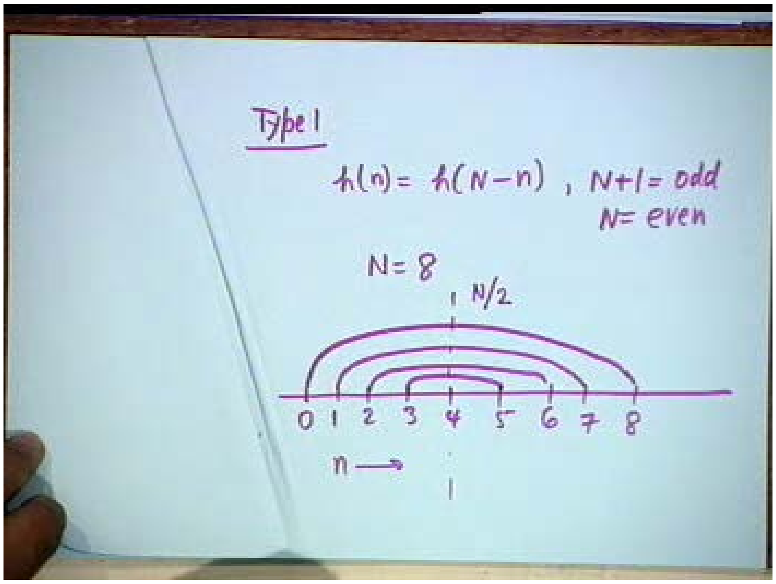

Let us consider type 1: h(n) = h(N – n); symmetrical and N + 1 = odd; therefore N = even. To concretize our concept let us consider a simple example: N = 8. Now look at the symmetry, n has values 0, 1, 2, 3, 4, 5, 6, 7, 8. The length is 9; what we require is that h(0) = h(8), h(1) = h(7); h(2) = h(6); h(3) = h(5). The fourth sample h(4) is a loner; it does not have a pair. The symmetry is therefore around N/2 where N is even. Our transfer function would be $H(z) = h_0 + h_1z^{-1} + h_2z^{-2} + h_3z^{-3} + h_4.z^{-4} + h_3z^{-5} + h_2z^{-6} + h_1z^{-7} + h_0z^{-8}$.

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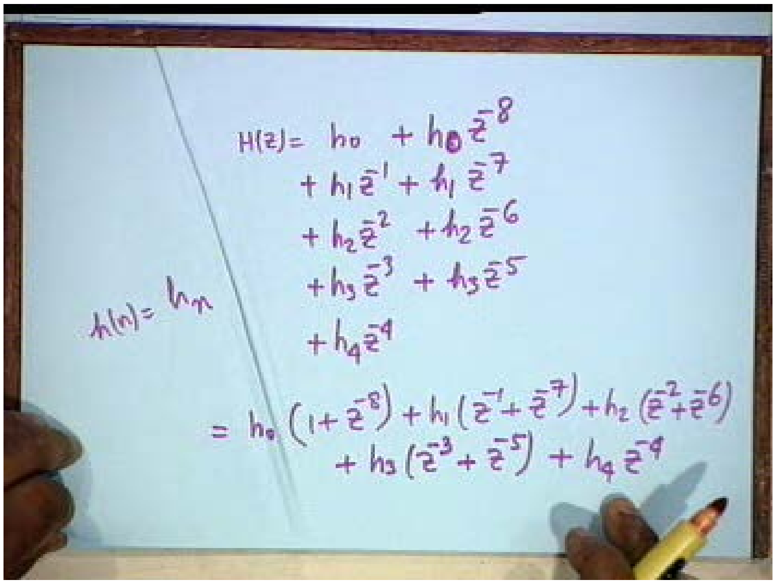

I shall write $h(n) = h_n$; Then the transfer function becomes, by combining terms, $h_0(1 + z^{-8}) + h_1(z^{-1} + z^{-7}) + h_2(z^{-2} + z^{-6}) + h_3(z^{-3} + z^{-5}) + h_4z^{-4}$. If I take z^{-4} out, then h_4 shall be left with no power of z; there are only 4, instead of 8 multipliers.

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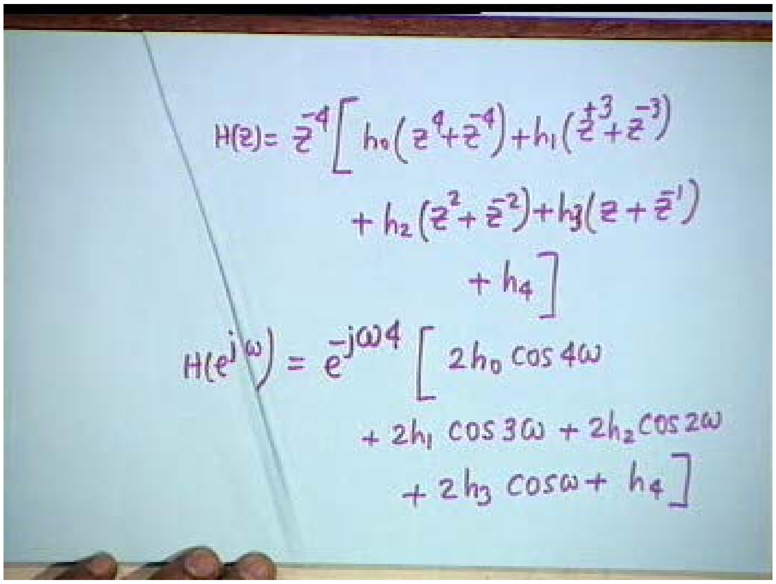

I write $H(z) = z^{-4} [h_0(z^4 + z^{-4}) + h_1(z^3 + z^{-3}) + h_2 (z^2 + z^{-2}) + h_3(z + z^{-1}) +h_4]$. Now find frequency response $H(e^{jomega}) = e^{-jomega\ 4}$ [2 h_0 cosine(4omega) + 2$h_1$cosine(3omega) + 2h_2 cosine(2omega) + 2h_3 cosine(omega)_+ h_4]. You can see that the phase of $H(e^{jomega})$ is – 4omega + constant which can be 0, pi, 2pi or 3pi depending on the sign of the real quantity within square brackets.

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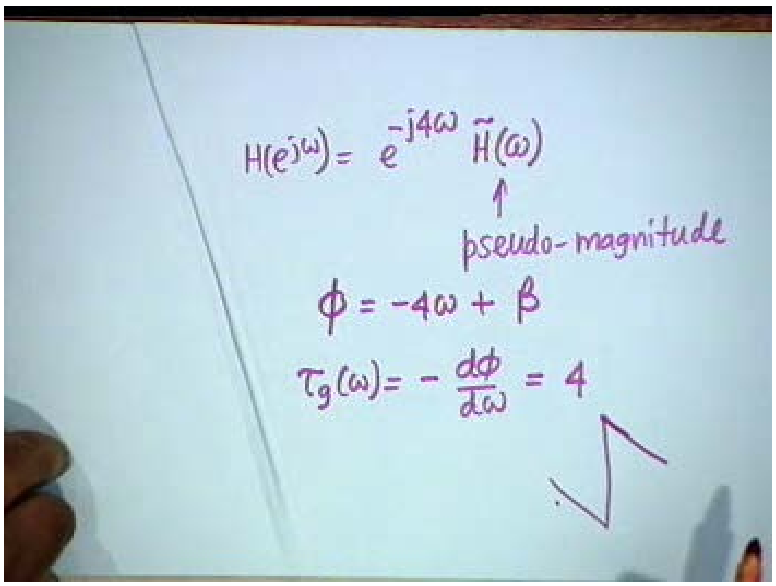

Therefore, in general, we can write $H(e^{jomega}) = e^{-j4omega} \times \tilde{H}$ (omega) where $\tilde{H}$ is real. Since it is real, many times we confuse this with magnitude; this term is given the name pseudo magnitude. It is not magnitude because it can also have a phase. It is called pseudo magnitude. You see the phase phi = – 4omega + beta, where beta can be 0, pi, 2pi, that is multiples of pi. You see that the group delay tau_g(omega) = – d(phi)/d(omega) = 4, so 4 samples of delay will be suffered by all frequencies passing through the filter. Wherever there is phase transition, you cannot differentiate; there will be a delta function there. So the four samples of delay occur in the piecewise linear regions of the phase. Wherever there is a phase change, there will be a delta function. So the group delay is applicable for these linear regions only, not at the frequency at which the pseudo magnitude changes sign from positive to negative or negative to positive.

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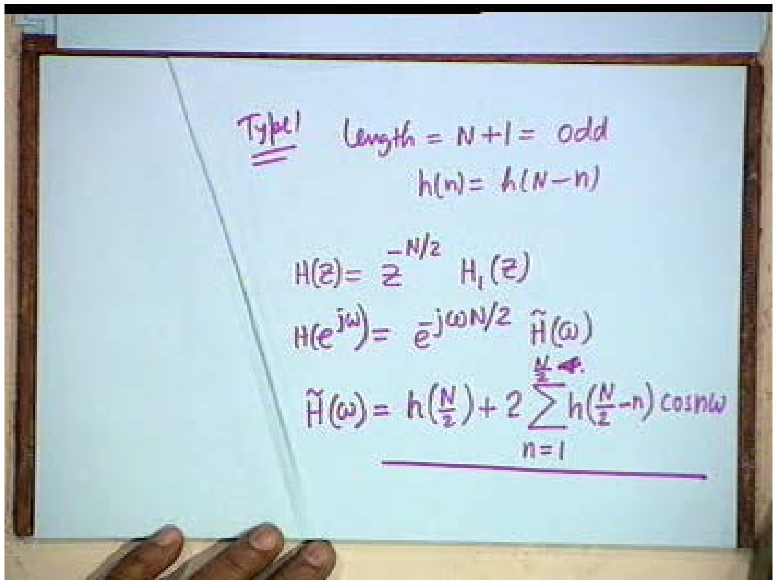

In general if the length of the filter is odd, equal to N + 1, and h(n) = h(N – n) (that is, type 1 filter), then the function is $H(z) = z^{-N/2}H_1(z)$. The phase of $H(e^{jw})$, except for jumps, is determined by the term $e^{(-\text{jomega } N)/2}$. The delay shall be N/2 samples. Also $H_1(e^{j\omega}) \underset{=}{\Delta} H$ (omega) = h(N/2) + 2 summation [h(N/2 – n)cos(n omega)], where n = 1 to N/2. This is the general formula. You must remember that $H(\omega)$ is the pseudo magnitude. It is not magnitude. The other interesting features of this function will be illustrated by taking an example: let $H(z) = h_0 + h_1 z^{-1} + h_0 z^{-2}$.

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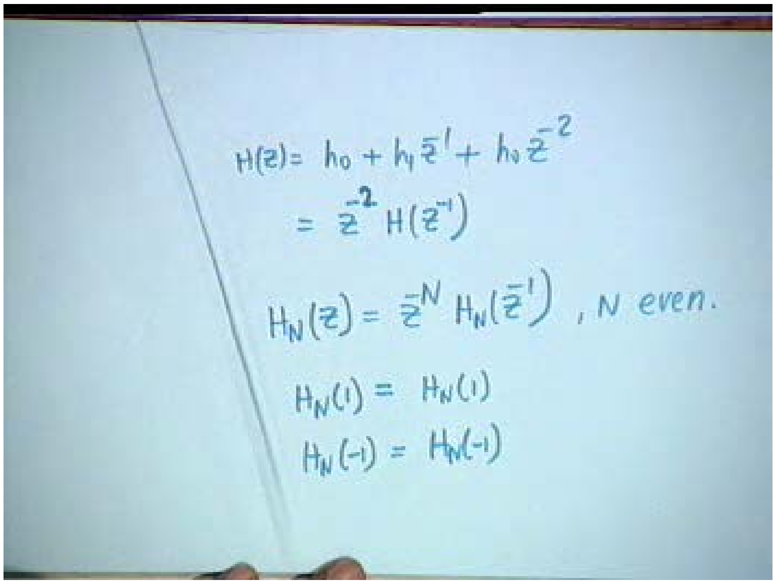

The length is 3 and h_n is symmetric, so this is a type 1 filter. Now I can write this H(z) as: $H(z) = z^{-2} H(z^{-1})$. In general, for type 1 filter, $H_N(z) = z^{-N} H(z^{-1})$;N is the order, not the length of the filter. Length of the filter is N + 1. Do not confuse between the two. N for type 1 filter is even. Put z= ±1 in this relation; you get $H_N(1) = H_N(1)$ and $H_N(-1) = H_N(-1)$. This is trivial, but you will see that they have important effects in the other type of filters. The other important property is that if $z = z_0$ is a zero of H(z), then $1/ z_0$ must also be a zero. The reason?

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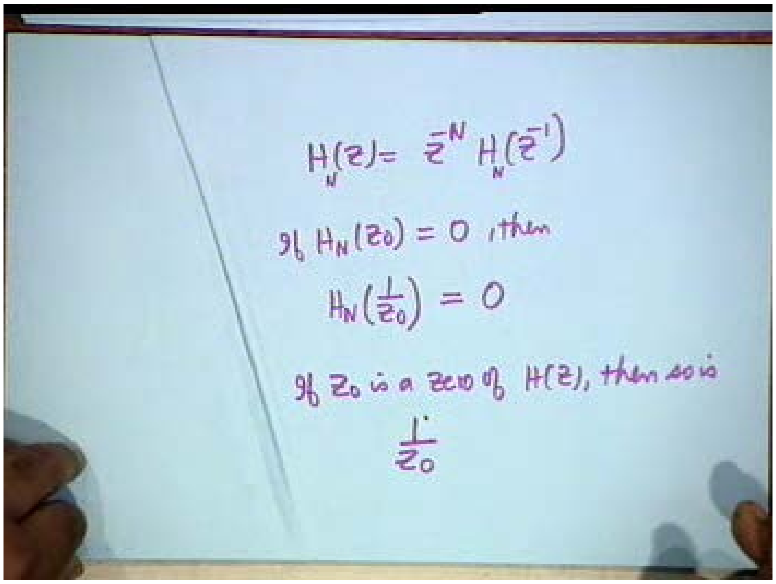

Our relation is $H_N(z) = z^{-N} H_N(z^{-1})$. If $H_N(z_0) = 0$, then obviously $H_N(1/z_0)$ must also be zero, because $H_N(1/z_0) = z_0^N H_N(z_0)=0$. In other words, zeros occur in reciprocal pairs. In all pass filters, the zeros and poles occur in reciprocal pair. Where are the poles in FIR filter? All of them are at the origin, $z = 0$.

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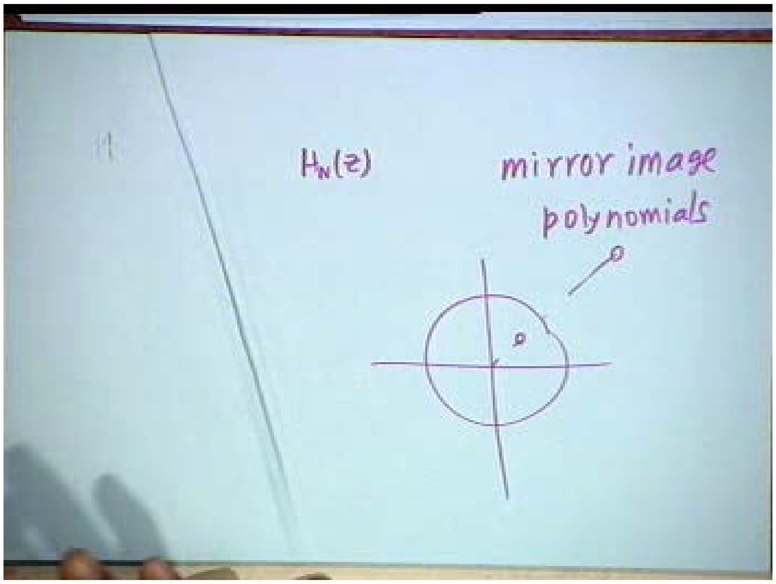

In general, for linear phase FIR filters, $H_N(z)$ is a polynomial whose zeros obeys this relationship, that is, they occur in reciprocal pairs. Such polynomials are called mirror image polynomials. What kind of mirror is it? If z_0, after reflection, goes to $1/ z_0$, what kind of mirror will it be? Obviously not a plain mirror; if it is a plain mirror the distances should have been equal. It is a concave/convex mirror. When you look from inside a unit circle it is concave.

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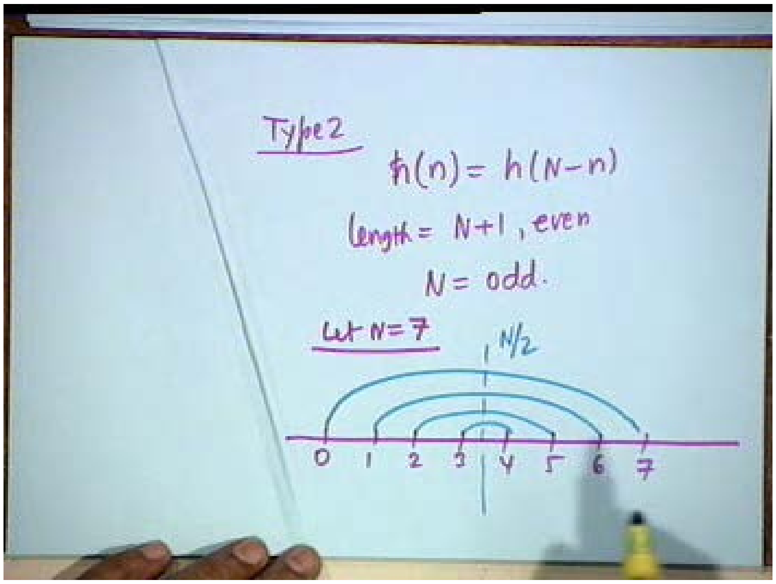

Now let us consider type 2 linear phase FIR: you will see a very interesting feature. Type 2 has symmetry, that is h(n) = h(N – n), and length = N + 1, even; that is, N is odd. Let us consider typical examples. Let N = 7. Now, because of symmetry, h(0) = h(7), h(1)= h(6), h(2)= h(5) and h(3)= h(4). All of them are paired, so the symmetry is around N/2 where no sample exists.

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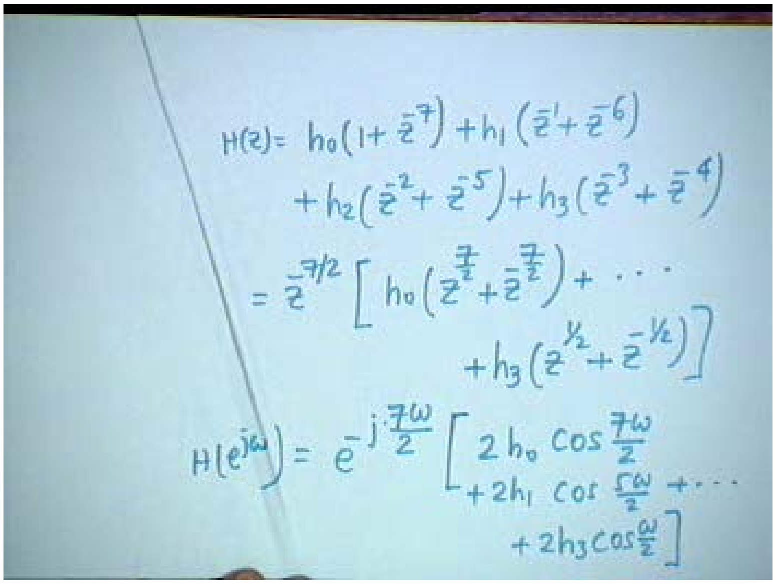

If I write H(z), I shall have h_0 (1 + z^{-7}) + h_1 (z^{-1}+ z^{-6}) + h_2 (z^{-2}+ z^{-5}) + h_3 (z^{-3}+ z^{-4}), which I can write as $z^{-7/2}h_0[(z^{7/2}+ z^{-7/2})$ + …$h_3(z^{1/2}+ z^{-1/2})]$. H($e^{j\ omega}$) = $e^{-(jomega7)/2}$)[2h_0 cos(7omega)/2) + 2$h_1$cos(5omega/2) + …..2$h_3$cos (omega/2)]. What is the group delay now? The group delay is 3.5 samples. Group delay is simply 7/2, not an integer number; it is 3 samples + another half sample. This creates problems in realization because we cannot realize a half sample. In DSP with uniform sampling our time index is an integer but this does produce a delay of 3 and a half samples.

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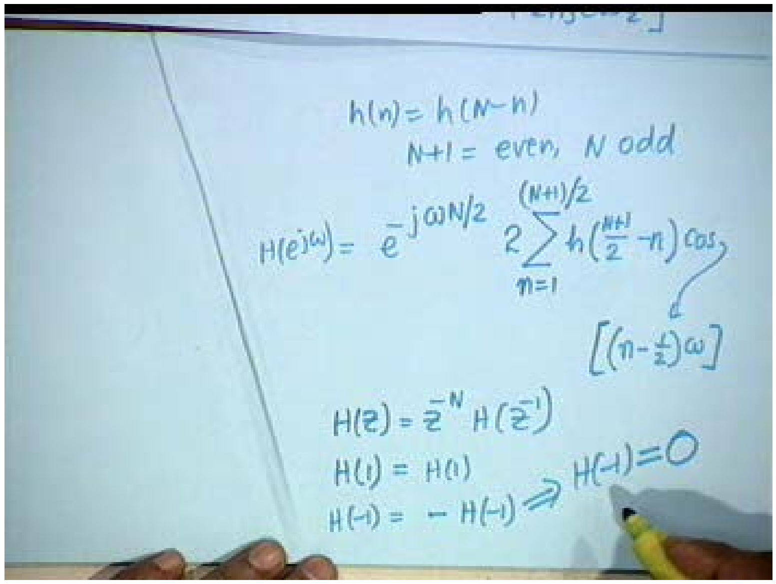

In general if we take h(n) = h(N – n) and N + 1 = even, that is, N is odd, then $H(e^{jomega}) = e^{-j\omega N/2}$ 2summation[h((N+1)/2) – n)cosine((n – ½)omega)]. What should be the limits? We do not have n = 0. We have n = 1 to (N + 1)/2. You must be careful while writing this form because each type has a different form. It is also clear that the relationship $H_N(z)= z^{-N} H(z^{-1})$ holds, as in the previous case of type 1. But there is a different consequence now; here H(1) = H(1), but H(–1) = -H(–1), which means that H(–1) = 0.

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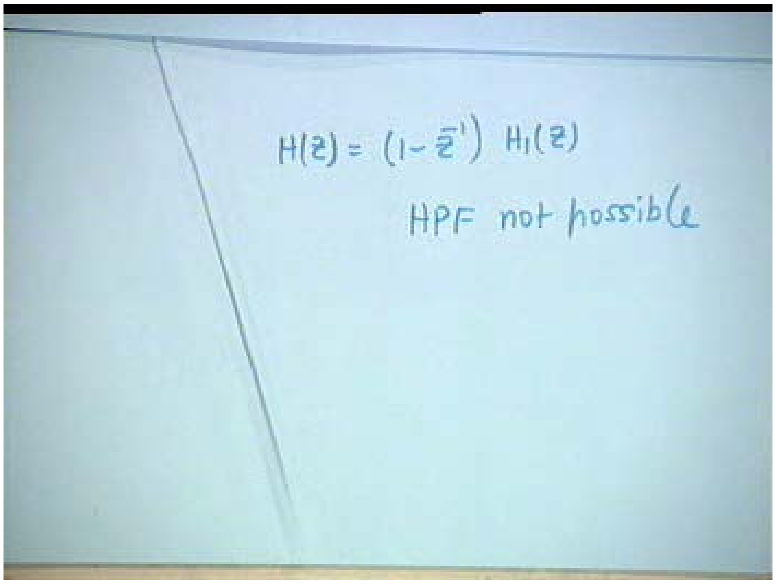

What does this mean? It means that H(z) can be written as $(1 + z^{-1})H_1(z)$, i.e. there is a zero of H(z) at z = – 1, which gives rise to the factor $(1 + z^{-1})$. With type 2, is it possible to get a high pass filter? H(– 1) is identically equal to 0 and therefore HPF is not possible. The type 1 filter has no such restriction. Type 2 filter has the limitation that it cannot realize a high pass filter. If a high pass cannot be realized, can you get a band-stop filter? The answer is: no. This restriction is not there in type 1 filter. We shall see in the next lecture that for type 3 you can neither get a low pass nor a high pass. It can only be used for band pass; band stop is also not realizable. Type 4 is the mirror image of type 2 that is in type 4, LPF is not possible. If LPF is not possible, then we cannot realize a band stop filter either. We will continue in the next lecture.