Digital Signal Processing

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Lecture - 20

Digital Processing of Continuous Time Signals

This is the 20th lecture. We are approximately half way in the course; the usual number of lectures we project in a semester is 40. It may vary from 38 to 45 depending on the time-table. And today we will look more closely at digital processing of continuous time signals. We have been talking about this off and on; repetition is good so long as you repeat the truth, and so long as you consider a very important topic. Since continuous time signals are the ones that we encounter in practice, it is important to have a close examination of them and their digital processing. But before that, let us review what we did in the last lecture. We talked about the stability triangle for second order systems and we also gave, with proof, a general procedure for testing the stability of an arbitrary order IIR transfer function. This test was based on all pass filters.

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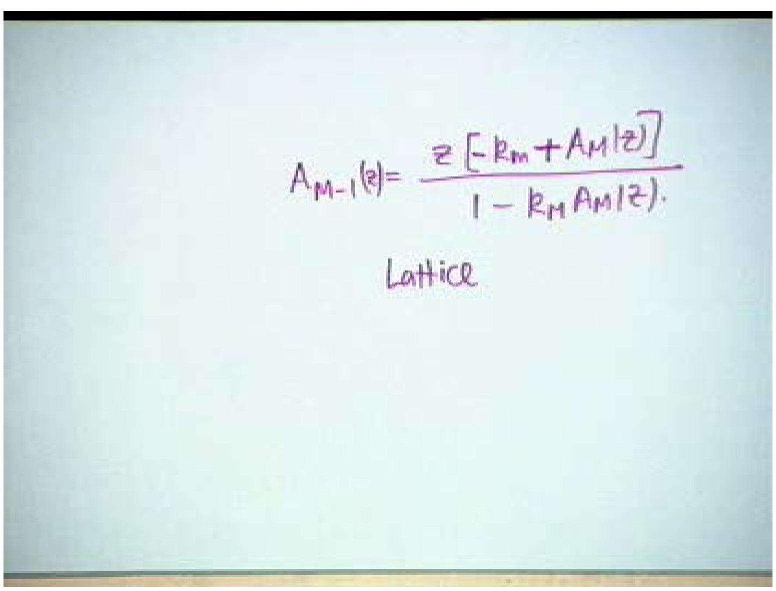

Stability test is a very important topic. From the given $A_M(z)$, we found $A_{M-1}(z)$ by using the algorithm $A_{M-1}(z) = z[-k_M + A_M(z)]/[1 - k_M A_M(z)]$. It is a very important algorithm and we shall come back to this later when we discuss lattice realization of IIR transfer functions. Coming back to analog signals, as I had mentioned in the second lecture, an analog signal is also preferred now-a-days to be processed digitally because of the advantages of DSP and because DSP chips and hardware are very easily and economically available. For this, the analog input first has to be sampled and held.

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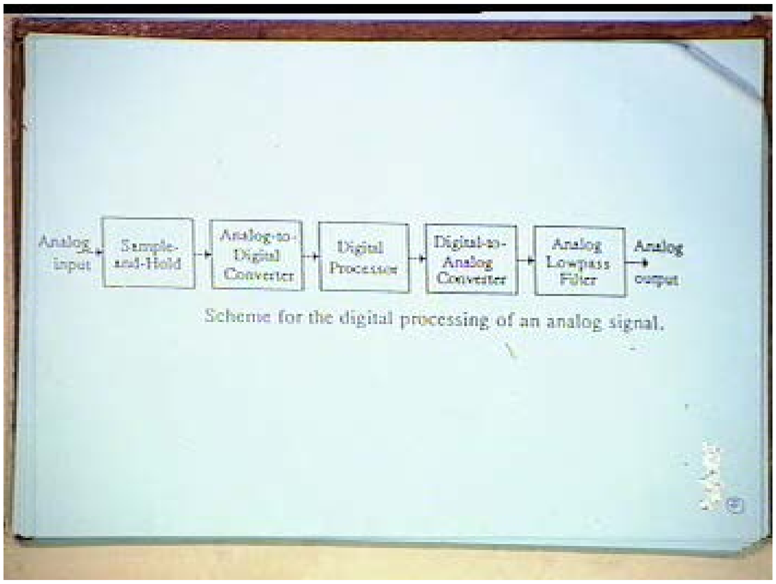

The purpose of holding, I repeat, is to allow time for A to D conversion. Each sample, in order to be converted to a binary number, requires time and during the time of conversion, the sample must be held at the constant value. Then the output of the A to D converter is the digital signal x(n) which is processed by the digital processor, consisting of multipliers, delays and adders; what you get at the output is a desired signal y(n) which is then to be converted back to analog through the D to A converter. The D to A converter produces a staircase like waveform and therefore it contains high frequencies. So at the output end, you require an analog low pass filter which is also sometimes called a reconstruction filter. It reconstructs the analog signal and that is what you get at the output. These are the basic blocks of Digital Signal Processing of an analog signal.

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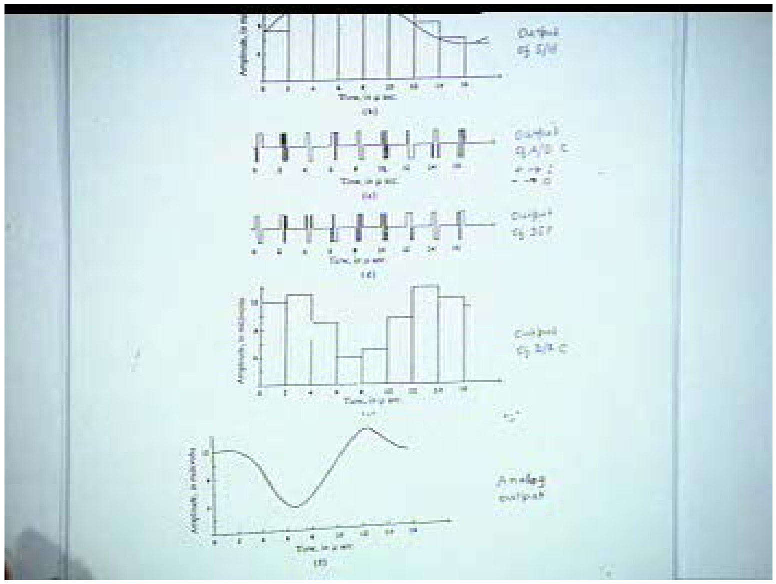

I had also projected this slide in the 2nd lecture showing an analog input and its sampled version, where the holding is obvious, and during this holding, the signal is converted to a digital signal. I use positive pulses for 1 and negative pulses for 0. There are many other ways of representation. After the processing, a typical output waveform is shown in the figure, which also consists of 1's and 0's. In order to convert it back to an analog signal, you require D to A converter where output is a staircase waveform, like the one you get after sample and hold of the input. Therefore you require to cutoff the high frequencies higher than a particular value. You require a low pass filter which limits the high frequencies and produces a desired analog output waveform. Now in both A to D converter and D to A converter, the cost and the accuracy of the converter depends on the number of bits that you use. Ideally, we would have loved to have an infinite number of bits, but in practice we have to limit ourselves to a finite number of bits, and this causes errors which I have not shown in that diagram. The errors are quantization errors and they have to be given the respect that is due to them; respect because we cannot avoid them, they are necessary evils. We must show them respect and contain them; we cannot completely eliminate them. This would not be our part of discussion in this course, this is taken up in the second course on DSP that we offer.

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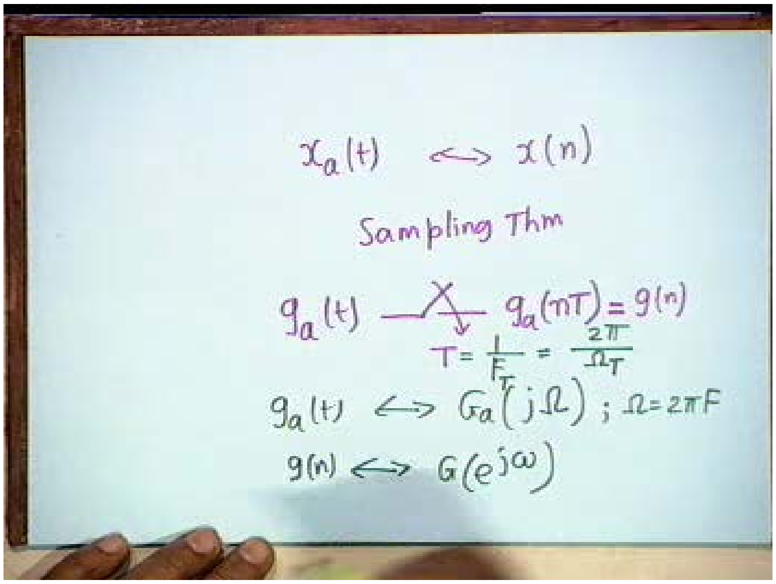

Now, $x_a(t)$, the analog signal, and the corresponding signal x(n) are shown to be connected by a double sided arrow. What does it mean? That means that the conversion should be one to one, i.e. if x(n) is given you can construct x_a; if x_a is given, of course, we can find x(n). In order that this is true we require the famous sampling theorem which states that the sampling must be done with constraint on the sampling frequency. We shall have a brief look at the sampling theorem. In fact we shall prove it in a very simple manner. In order that this is a one to one correspondence, we must sample $x_a(t)$ in an appropriate manner. Let an analog signal $g_a(t)$ be sampled at regular intervals of T. What you get at the output is $g_a(nt)$ which after quantization, i.e. after A to D conversion, becomes g(n). We said that since the mathematics of discrete time signals and digital signals is the same, we shall use them interchangeably. Therefore $g_a(nt)$ can be written as g(n), without any harm, which means that there is no quantization error. If we have to consider quantization error we shall distinguish between them in a statistical manner. That is, we cannot treat this as a deterministic phenomenon; we have to use statistical methods. But that, as I said, will form a topic in a second course on DSP.

Now let $g_a(t)$ have the Fourier Transform $G_a(j\Omega)$; Ω denotes the analog radian frequency, that is radians/s and $\Omega = 2\pi x F$ where F is the frequency in Hz, T is the sampling interval and therefore this is $= 1/F_s$ where F_s is the sampling frequency. We must reserve ω for the main obsession of ours, namely Digital Signal Processing. This ω shall be reserved for normalized digital frequency. We have already said how it comes through. We shall say it again, because truth should be repeated as many times as possible; only then it gets imprinted in the mind. And we also know the g(n) can be represented in the frequency domain by its Fourier transform $G(e^{j\omega})$ where this ω is now the normalized digital frequency. So we are interested in the interrelationship between $G_a(j\Omega)$ and $G(e^{j\omega})$.

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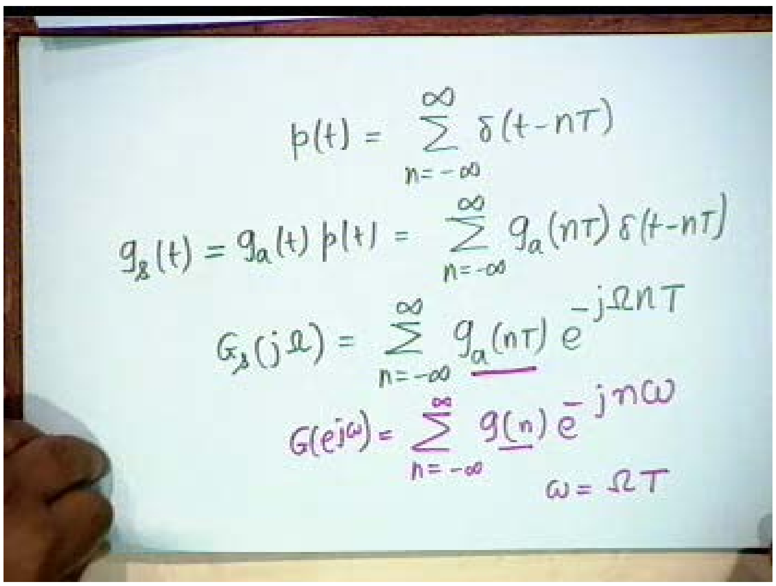

In order to do that we idealize the sampling waveform to a sequence of impulses. Actually, it is done by narrow rectangular pulses, pulses of short duration and large height. We can treat impulse as the limit of such waveforms. We cannot generate an impulse in the laboratory. All we can generate is a high amplitude pulse with short duration. The only distortion that occurs using rectangular pulses in place of impulses is that the spectrum is amplitude modulated by the spectrum of the rectangular pulse. And if it is small enough, then that modulation can be ignored and therefore there is not much of a deviation from practice.

But when you go from rectangular pulse to impulse, the mathematics is drastically simplified. For an engineer, whatever works is good enough a tool. So we idealize the sampling signal as summation $\delta(t - nT)$, where n goes from $-\infty$ to $+\infty$. This is the waveform by which we multiply $g_a(t)$ to get the sampled signal. Let us call the sampled signal as $g_s(t)$. And this you can write as $\sum(n = -\infty \text{ to } \infty)\ g_a(nT)\ \delta(t - nT)$. This becomes the sampled waveform. In other words, the sampled waveform is a sequence of impulses whose strengths are $g_a(nT)$. Strength means area of the impulse. And if I take the Fourier Transform of this, then I get $G_s(j\Omega)$ = summation ($n = -\infty$ to ∞) $g_a(nT)\ e^{-j\Omega nT}$. This is not a very useful form except for establishing the relationship between ω and Ω. You know $G(e^{j\omega})$ is summation ($n = -\infty$ to ∞) $g(n)e^{-jn\omega}$ and these two must be identical. If we take $g_a(nT) = g(n)$, that is we ignore quantization error, then it means that $\omega = \Omega T$, as we have said earlier.

In other words, the normalized digital frequency is $\omega = \Omega/F_s$ and you can write this $2\pi\ F/F_s$. I can also write this as $2\pi\ \Omega/\Omega_s$. So ω is a normalized digital frequency and is dimensionless; it is expressed in radian. Radian is a dimensionless quantity because it is a ratio of length to length, arc/radius. Ω is expressed in radians per second and because of the sampling requirement, by the sampling theorem, Ω should not exceed $\Omega_s/2$. In other words, Ω lies between zero and $\Omega_s/2$. If that is true, then ω must be $< \pi$. In other words, the range of ω is $-\pi$ to $+\pi$. In analog, we had to go from $-\infty$ to $+\infty$, a doubly infinite scale; that length has been compressed, to our advantage, to $-\pi$ to $+\pi$. So in a sense, DSP is a much simpler concept than ASP but it also has its own disadvantages. Now, in order to see how $G_s(e^{j\omega})$ and $G_a(j\Omega)$ are related to each other, we make an alternative representation of the impulse train which samples the waveform.

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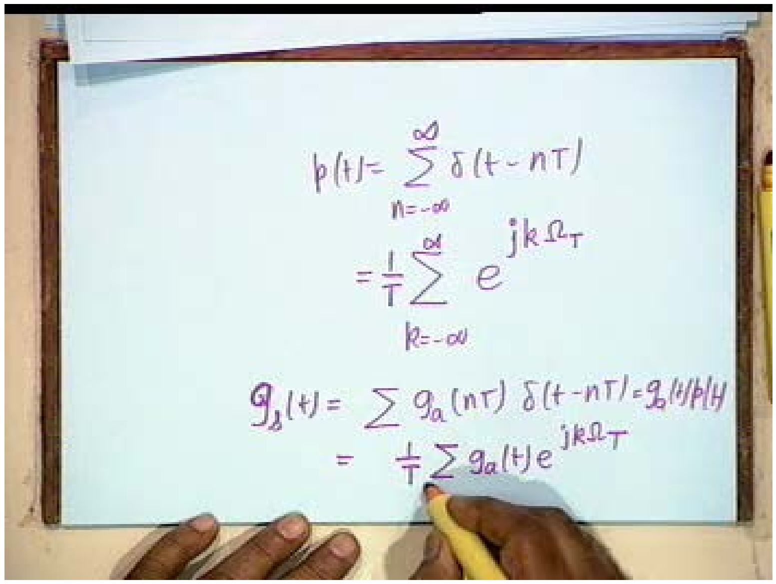

Let p(t) = ∑ δ(t – nT), n = – ∞ to + ∞. It is a periodic waveform and therefore can be expanded in Fourier series. So it can be expanded as summation a_k $e^{jk\Omega_s t}$, k = – infinity to + infinity, where the Fourier coefficient a_k turns out very simply as = 1/T. Now if I use this representation, then my $g_s(t)$, the sampled waveform, becomes $g_s(t) = g_a(nT) \times p(t)$. When I represent p(t) by Fourier series, then my representation becomes (1/T) summation $g_a(nT) \times e^{jk\Omega_s t}$, k goes from – ∞ to + ∞. I wrote $g_a(nT)$ because δ(t – nT) exists at t = nT only. I can also write this as $g_a(t)$ without any harm.

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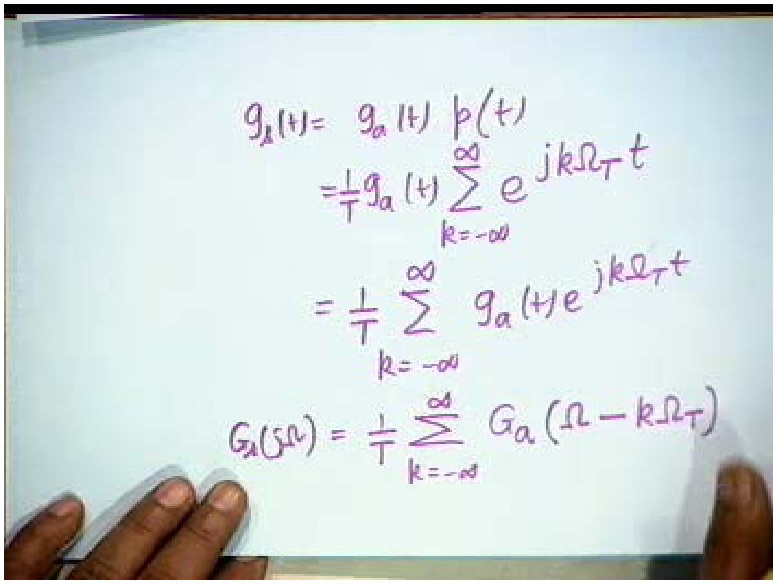

Originally $g_s(t) = g_a(t) \times p(t)$ and I write this as $g_a(t)$ (1/T) ∑ (k = – ∞ to + ∞) $e^{jk\Omega_s t}$. I can take $g_a(t)$ inside the summation. Now if I take Fourier Transform of both sides then I get $G_s(j\Omega)$ = (1/T) ∑ (k = – ∞ to + ∞) $G_a(\Omega - k\Omega_s)$. Therefore this spectrum is a superposition of the base spectrum which occurs for k = 0 and its delayed versions occurring with centers at $k\Omega_s$. That is, the same spectrum has shifted + Ω_s on the right, – Ω_s on the left, + $2\Omega_s$ on the right and – $2\Omega_s$ on the left, and so on. Can I write the argument of G_a as $\Omega + k\Omega_s$? Yes, we can, without changing anything.

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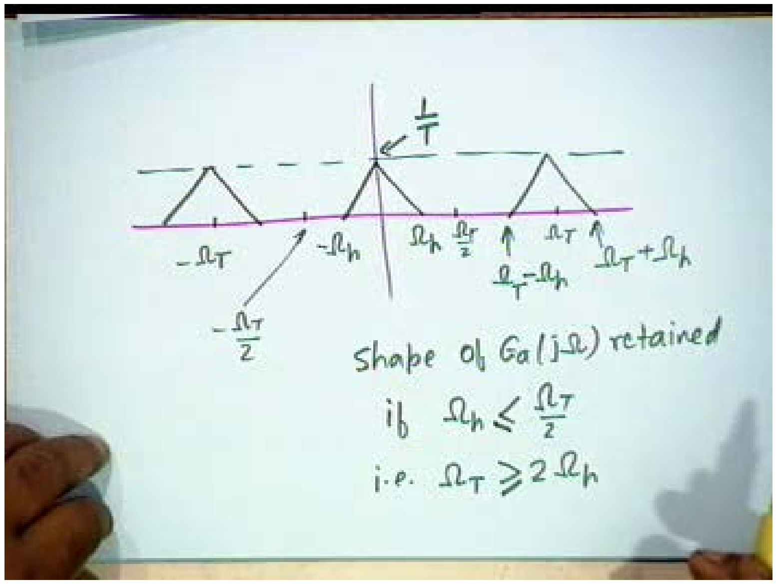

Consider a hypothetical base band spectrum of the shape of a triangle which goes from – Ω_h to + Ω_h, Ω_h being < $\Omega_s/2$. In $G_s(j\Omega)$ we shall have repetition of this with centers at $k\Omega_s$, k = ± 1, ± 2,… as shown in the figure. The amplitude now becomes 1/T, because of the Fourier series expansion of p(t). You understand that the shape of the base band spectrum is retained only if Ω_h is less than $\Omega_s/2$. So shape of $G_a(j\Omega)$ is retained if Ω_h is < $\Omega_s/2$; in other words, $\Omega_s > 2\Omega_h$. Ω_s can also be equal to $2\Omega_h$. The shape shall be retained and this is the famous sampling theorem. Since spectrum and time domain signal are in one to one correspondence, if you can retain the spectrum you also retain the original signal. It can be retained only if the sampling frequency is at least twice the highest frequency content of the signal; this is the sampling theorem.

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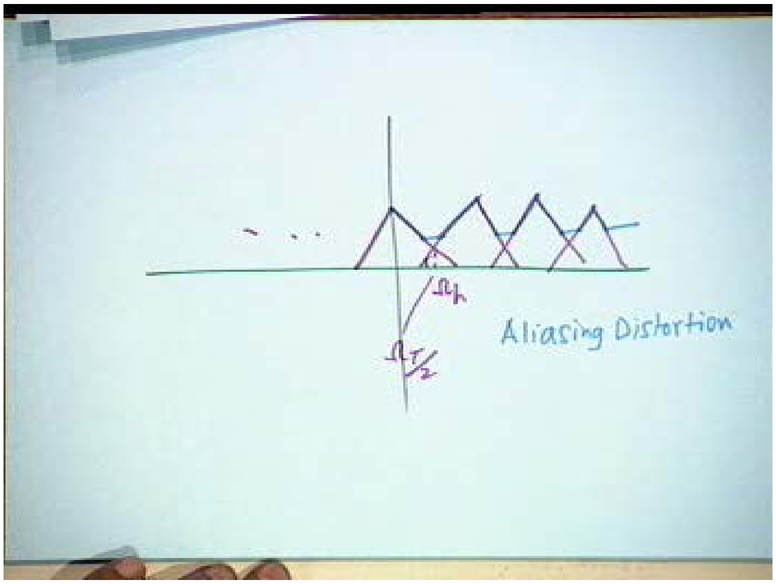

On the other hand, if this is not the case, e.g. if we have Ω_h greater $\Omega_s/2$, then we shall have overlaps, as shown in the figure. The composite spectrum would be distorted. There is no way that you can recover the original spectrum. This is the case of aliasing, and the resulting distortion is called aliasing distortion. We have taken a numerical example early in this course to show how aliasing distortion occurs, how a high frequency like 13 Hz poses as 7 Hz, and that is why the name aliasing was given.

The fact of the matter is that aliasing distortion cannot be completely removed because no signal in practice is band limited. That is why you require analog low pass filter at the front end. The input analog signal is first filtered through a low pass filter and the purpose of that filter is to constrain the bandwidth, that is, you retain only the essential part of the spectrum, up to 3.4 KHz in digital telephone application, for example. Well, you are free to take it to even 20 KHz, but you also have to be prepared to pay the cost of a higher bandwidth. Essential bandwidth for intelligible speech is 3.4 KHz and therefore the low pass filter may have a cutoff at 4 KHz. But no low pass filter is ideal. You know, an ideal low pass filter is non-realizable and also unstable. Therefore some high frequencies, may be of very small magnitude, shall be retained. As long as the aliasing distortion does not hamper with the main objective of recovering our original signal, as long as it does not distort to a large extent, it is acceptable.

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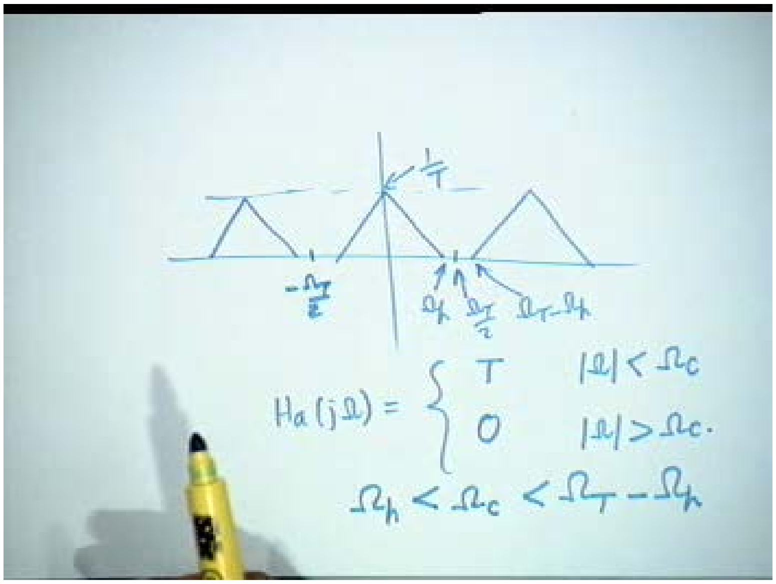

In order to recover the original signal from the sampled one, we require a low pass filter whose magnitude response, $|H_a(j\Omega)|$, should be = T for $\Omega < \Omega_c$, the cutoff frequency; it should be 0 ideally for $\Omega > \Omega_c$, where Ω_c itself must be restricted to be anywhere between Ω_h and $\Omega_s - \Omega_h$. In practice, mathematics becomes simpler if Ω_c is chosen $\Omega_s/2$, that is half the sampling frequency. Half the sampling frequency is also called the Nyquist frequency. This band $-\Omega_s/2$ to $+\Omega_s/2$ is also called the Nyquist band or base band. We shall refer to it by various names although they mean the same. Now in the time domain we take up a discussion on the characterization of the reconstruction filter.

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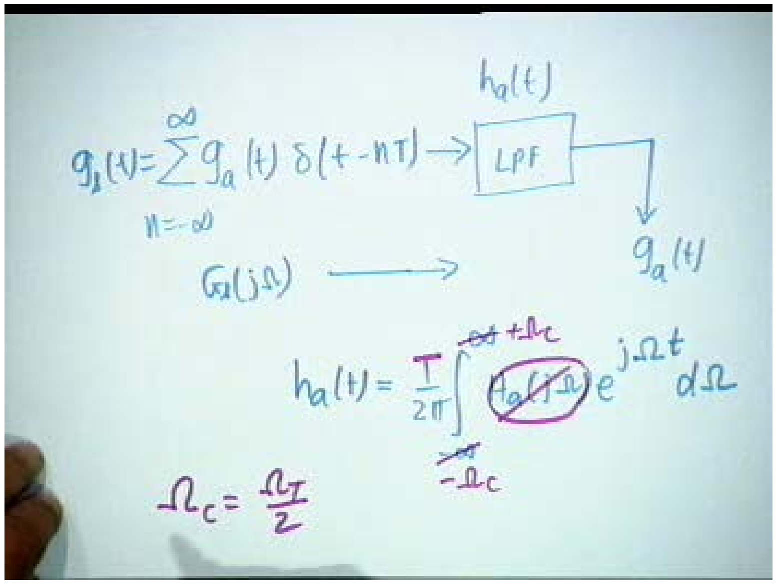

What we have is $g_s(t) = \sum g_a(t)\, \delta(t - nT)$, n = – ∞ to + ∞. What should be the impulse response $h_a(t)$ of the LPF which will recover $g_a(t)$ from $g_s(t)$? For the ideal filter, $H_a(j\Omega) = T$ for mod $\Omega < \Omega_c$ and Ω_c itself is restricted to lie between Ω_h and $\Omega_s/2$. Now $h_a(t)$ is obtained by the inverse Fourier Transform that is, $[1/(2\pi)]$ integral (– ∞ to + ∞) $H_a(j\Omega)\, e^{j\Omega t}\, d\Omega$. Instead of – ∞ to + ∞, we have to go from $-\Omega_c$ to $+\Omega_c$. We have done this earlier, and we know that the resulting $h_a(t)$ is non-causal and unstable. Look at the problem from another point of view. We have the analog signal $g_a(t)$ whose Fourier Transform is $G_a(j\Omega)$. Thus $g_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_a(j\Omega)\, e^{j\Omega t}\, d\Omega$. Now $G_s(j\Omega)$, the spectrum of the sampled signal is given by $G_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G_a[j(\Omega - k\Omega_s)]$; $G_a(j\Omega) = H_a(j\Omega)\, G_s(j\Omega)$ where $H_a(j\Omega) = T$ for $-\Omega_s/2 \le \Omega \le \Omega_s/2$ and 0 otherwise. Also, $\Omega c = \Omega_s/2$. Thus $g_a(t) = \frac{T}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} G_s(j\Omega)\, e^{j\Omega t}\, d\Omega = \frac{T}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} \sum_{n=-\infty}^{\infty} g_a(nT)\, e^{-j\Omega nT}\, e^{j\Omega t}\, d\Omega$. Now interchange integration and summation to get $g_a(t) = \sum_{n=-\infty}^{\infty} g_a(nT) \frac{T}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} e^{j\Omega(t - nT)}\, d\Omega$. Now change the variable of integration from Ω to $\varpi = \Omega T$. Then the limits will change to $-\pi$ and $+\pi$. Carrying out this integration and simplifying we get the required formula $g_a(t) = \sum_{n=-\infty}^{\infty} g(n) \sin \frac{\pi(t-nT)}{T}/$

$[\frac{\pi(t-nT)}{T}]$ where we have replaced $g_a(nT)$ by g(n). Unfortunately, n has to go from – ∞ to + ∞. Not only it is not hardware realizable, it is not software computable either because we have to go from – ∞ to + ∞.

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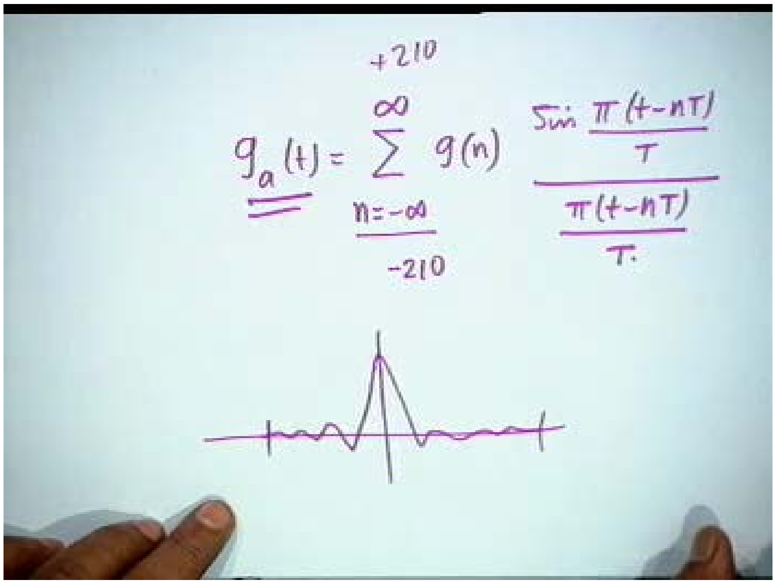

If you are interested in the original signal as a function of time, usually one restricts this to a large enough value of n, may be – 300 to + 300. And you assume that this is a reasonable approximation to the given function. Is it logical? Yes, it is, because you know that $h_a(t)$ shall look like a sinc function; it has a main lobe and then goes on decreasing. Therefore if you go for a sufficiently large interval including the main lobe in the middle then you are reasonably sure that it would be a reasonable approximation.

Reconstruction formula is an important formula and you should remember this. Now therefore our concern is with these two low pass filters; one is band constraining, and the other is a reconstruction filter after D to A conversion. How good these low pass filters are shall determine how good your DSP is for the processing of an analog signal. And in very simple industrial situations, for example in digital telephony or the digital stereo, you use simple RC low pass filters, to keep down the cost. But then there is a problem; the capacitance required is large, it cannot be integrated. Integration of a very large capacitor requires a lot of silicon area and therefore you have a problem there. We would like to investigate how very good analog filters can be designed; the cost will be higher but we shall be able to avoid as much distortion as possible.

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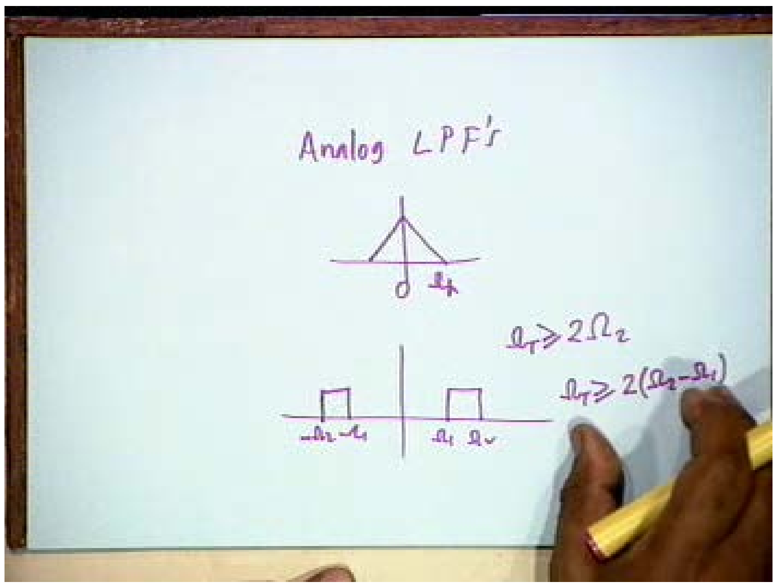

So, our next topic for study would be analog low pass filters. But before we do that, I would like to have a brief discussion about sampling of a band pass signal. So far we have talked about low pass signal that is, a signal having frequency from dc to high frequency Ω_h. But suppose you have a band pass signal having frequencies from Ω_1 (>0) to Ω_2 (<∞). In reality, you cannot get a strictly band limited signal, but we assume this ideal situation to keep mathematics simple.

Suppose we have a band pass signal having a flat envelope, so assumed for simplicity. How do we sample this? By sampling theorem, we require $\Omega_s > = 2\Omega_2$. it turns out that $\Omega_s \geq$ twice the bandwidth is good enough, that is $\Omega_s \geq 2(\Omega_2 - \Omega_1)$ is good enough. The underlying reason is that although this may not satisfy the sampling theorem, sampling at this rate makes a repetition of the spectra in such a manner that one of these repetitions shall come within the base band. And since the shape remains the same, we can do further processing in the base band only. We would not like to do a rigorous mathematical analysis but let me illustrate with a very simple example.

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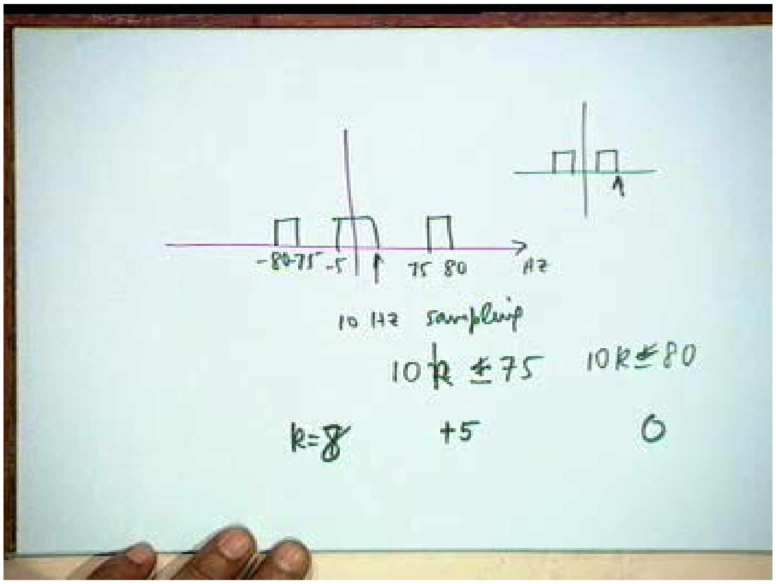

Suppose we have a band pass signal from 75 Hz to 80 Hz then we shall have the signal between – 80 Hz and – 75 Hz also. And let us sample this waveform at exactly 10 Hz. Obviously we should have required 160 Hz as the minimum by sampling theorem, but twice the bandwidth is 10 Hz. For example, if we do the sampling at 10 Hz, then this same shape shall be repeated infinite number of times on the right and also on the left. There shall be bands between 10k $\pm$ 75 and 10k $\pm$ 80, where k is an integer, positive or negative. Suppose k = 8, then 10k – 75 and 10k – 80 become 5 and 0Hz, respectively.

Similarly, if you take k = 7, then you get – 5 to 0 and therefore this same spectrum comes within the base band. The shape remains the same; hence we do not have to process the signal beyond 5 Hz. What would be the cut off frequency required for the low pass filter? It is 5 Hz and therefore you can do with a much less costly low pass filter. All processing is done at the base band, i.e. – 5 Hz to + 5 Hz. We do not have to go to high frequencies if the sampling frequency is correct and distortion is contained.

Distortion will definitely occur if Ω_s is less than twice the bandwidth. But if it is higher than twice the bandwidth, distortion should not occur. You draw the waveforms and verify for yourselves.

The basic point that I am making here is that for a low pass signal, sampling theorem constrains the sampling frequency to be at least twice the highest frequency content. For a band pass signal the sampling frequency required is at least twice the bandwidth, which can be a much smaller frequency. Instead of 75 – 80 Hz, if we have 75 - 80 MHz, then you require 10 MHz as the sampling frequency. You do not have to go to 160 MHz. The higher the sampling frequency, the more complex is the hardware. In fact beyond 10 MHz sampling frequency, the present state of the art encounters difficult problems. The hardware is limited; also at such high frequencies, the parasitic capacitance plays nuisance.

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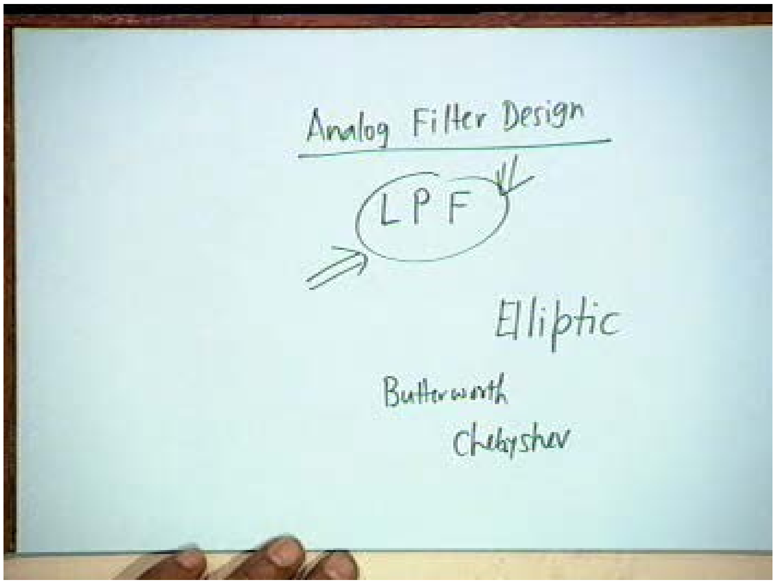

Tomorrow we will have a problem session. Then the next few lectures would be on analog filter design. This will be a review; also I will not discuss everything in analog filter design. I will only talk about approximation of analog filters. And what is the motivation for discussing analog filter design in a DSP course? One is that we require analog filters at the transmitting end and also the receiving end; therefore we should know about analog filters. More importantly, analog filter design forms the basis of IIR digital filter design. The techniques for analog filter design are well established. They are available in handbooks on filter design. Everything is known about analog filters. So we make that as the base and make a transformation to get a digital filter. These are the two reasons why we discuss analog filter design in some details.

Now for the DSP of analog signals we require only two low pass filters; so it should suffice for us to consider only low pass filters. But if you want to design IIR digital filters of various kinds, like band stop, band pass or multi-band pass, you require other types of analog filters as well. It turns out that if you know how to design a low pass filter, then you know how to design any other kind of filter because low pass filter can be transformed to any other kind. Therefore, our focus shall be on analog low pass filter design only.

In analog low pass filter design, also there are various techniques. The ones that are most important in practice are generated by two considerations; one is that the filter design must be simple. If you spend many hours in designing a filter, well, you have already hiked up the price of the product and it may not sell. For example, if you put 80 man hours, let us say 20 engineers for 4 hours each, then we have already spent money on their salary, the infrastructure and everything, therefore the product that you are going to design must incorporate this expenditure. There will be competitors who will design simpler filters in less time; so there has to be a compromise between simplicity of the filter design and satisfying the specs. The same specs can be satisfied by many different kinds of filters. And the filter that is the best or the optimum is the elliptic filter. But then elliptic filters are very difficult to design. They require much effort in design as compared to other filters.

As compared to elliptic filters, Butterworth and Chebyshev types are much simpler to design. They can be designed from analytical formulas whereas elliptic filters, of necessity, require computational efforts; no generalization as in Butterworth and Chebyshev types are possible for elliptic filters. Even the tables that are given are given at discrete intervals of tolerances. On the other hand, Butterworth and Chebyshev can be designed analytically. That is, we have analytical formulas for computing the transfer functions of Butterworth and Chebyshev. And therefore starting our Monday's lecture, we shall discuss Butterworth and Chebyshev low pass filters only. We would make a kind of a review, because I assume that you have done this topic in some other course.