Digital Signal Processing

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Lecture - 21

Problem Solving Session: FT, DFT and Z Transform

This is the 21st lecture and as promised this will be a problem solving session, with problems on Fourier transforms, Discrete Fourier transforms and Z transforms.

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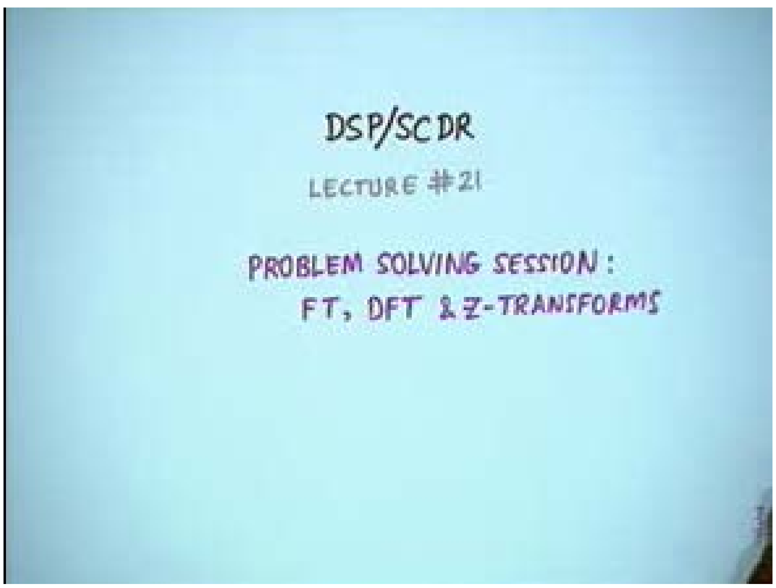

I have selected some problems from Mitra. The first problem that we take is 3.3(a). The problem is to derive the Fourier Transform of u(n). The book usage is DTFT but as I said I will not use DT but use simply Fourier Transform. One way is to use the method adopted by Oppenheim, Willsky and Young. The other method that I find more convenient is to breakup u(n) into its even part and odd part. The even part is ½ [u(n) + u(–n)]. If we plot it then at n = 0 the amplitude would be 1 and at all other points it shall be ½. I can write this as ½ + ½ δ(n); then it takes care of the amplitude 1 at n = 0.

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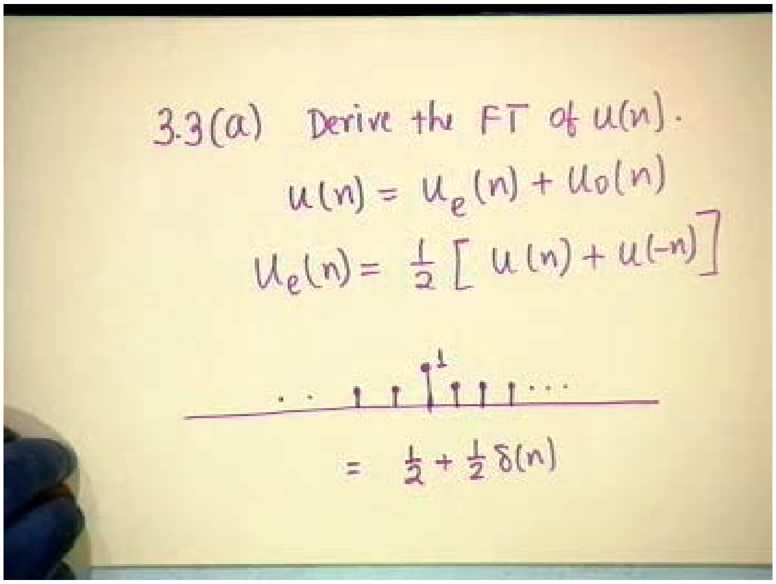

If I take the Fourier Transform of this, I get $U_e(e^{j\omega}) = ½ + \frac{1}{2}$ [2 $\pi\sum\delta(\omega + 2\pi k)$], where k goes from – ∞ to + ∞. We have obtained the Fourier transform of the even part and now let us look at the odd part, i.e. $u_o(n)$ = ½ [u(n) – u(–n)]. Obviously at n = 0 the sample would be 0 and at n = 1, 2 etc the value would be ½. At n = – 1, – 2 etc, it would be – ½ because of u(–n). This waveform can be written as u(n) – ½ – ½ δ(n). Once you have guessed this, the solution is over. Therefore $u_0(n)$ = u(n) – ½ – ½ δ(n), and $u_0(n – 1)$ = u(n – 1) –½ – ½ δ(n – 1). In the latter, I have replaced n with n – 1. Then I subtract $u_0(n – 1)$ from $u_0(n)$, and get $u_0(n) – u_0(n – 1)$ = u(n) – u(n – 1) – ½[δ(n) – δ(n–1)].

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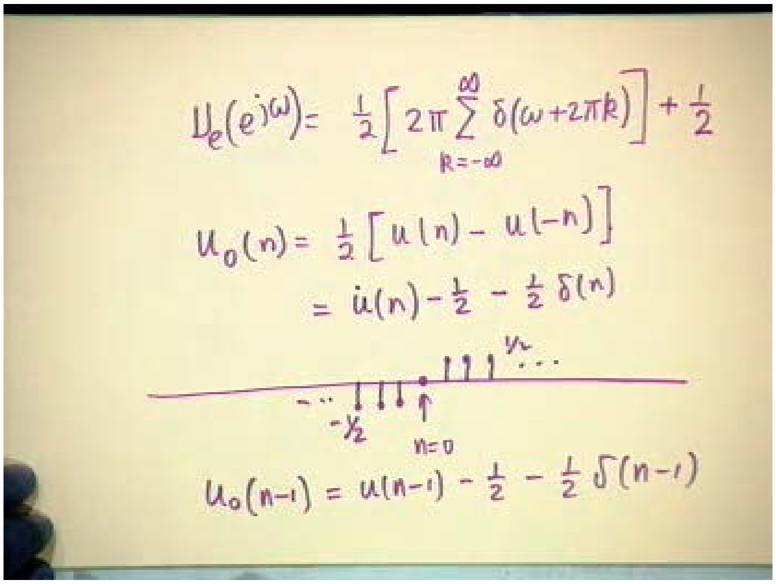

Now u(n) – u(n – 1) is δ(n); therefore $u_o(n) - u_0(n-1) = \delta(n) - \frac{1}{2}\delta(n) + \frac{1}{2}\delta(n-1) = \frac{1}{2}[\delta(n) + \delta(n-1)]$. Now I take the Fourier Transform, so I get $U_0(e^{j\omega})\,[1 - e^{-j\omega}] = \frac{1}{2}(1 + e^{-j\omega})$. Thus $U_0(e^{j\omega}) = \frac{1}{2}(1+e^{-j\omega})/(1-e^{-j\omega})$. Now I add U_e and U_0.

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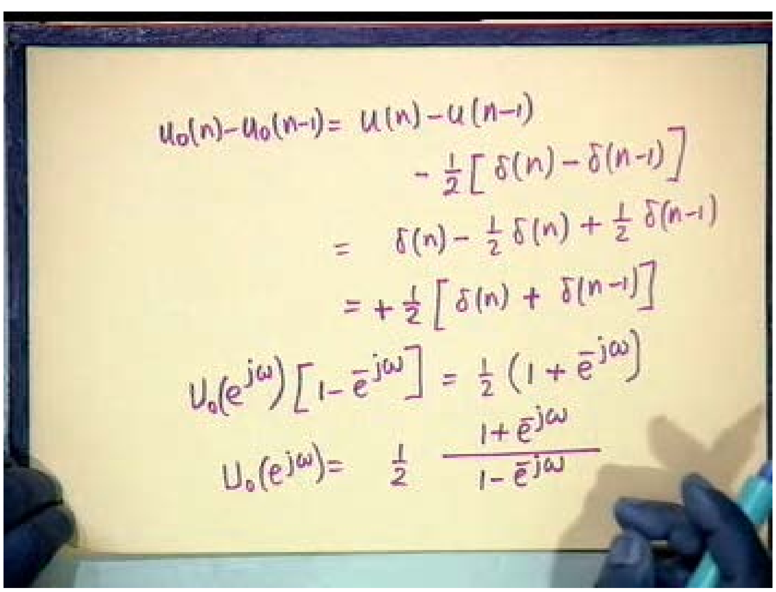

Therefore $U(e^{j\omega})$ which is the Fourier Transform of $u(n)$ becomes $\pi \sum \delta(\omega + 2\pi k)$ ($k = -\infty$ to $+\infty$) $+ \frac{1}{2} + \frac{1}{2}$ $(1 + e^{-j\omega})/(1 - e^{-j\omega})$. Then if I sum this up I get $\sum \delta(\omega + 2\pi k)$ ($k = -\infty$ to $+\infty$) $+ 1/(1 - e^{-j\omega})$. Conceptually and step-wise, it is a much simpler procedure rather than bringing this signum function which Oppenheim has done.

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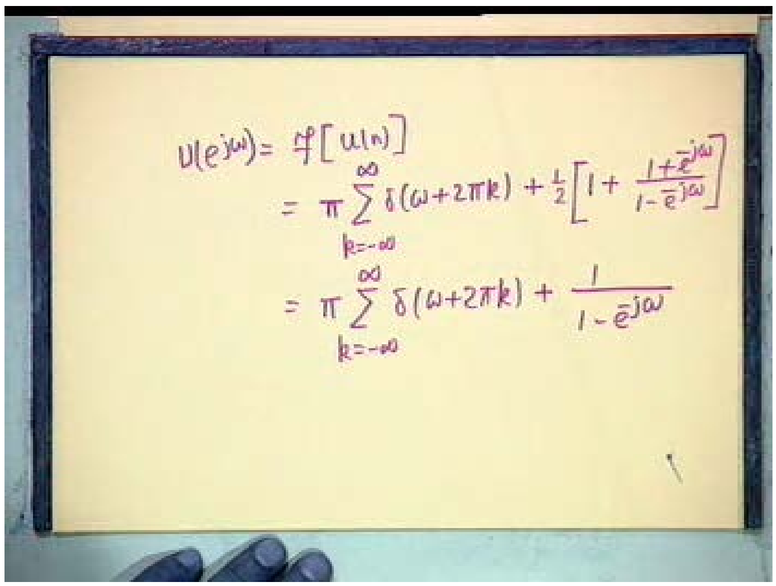

The next problem I choose is 3.17. Problem 3.17 says: let $x(n)$ be real and have a Fourier Transform $X(e^{j\omega})$. I have chosen this problem to illustrate a particular point which went unnoticed in minor one answer. The question is: find $y(n)$ such that its F T $Y(e^{j\omega})$ is $X(e^{j3\omega})$. The point that I want to illustrate is the following. The solution to this obviously is $X(e^{j3\omega}) = \sum x(n) e^{-j3n\omega}$, by definition where n goes from $-\infty$ to $+\infty$. Now, we put $3n = r$; then my summation becomes $\sum x(r/3) e^{-jr\omega}$. Now you cannot write $r = -\infty$ to $+\infty$ because $(r/3)$ may be an integer or may not be an integer. Therefore this exists only for $r = 0, \pm 3, \pm 6$ and so on. Notice that this is a case of an up sampler.

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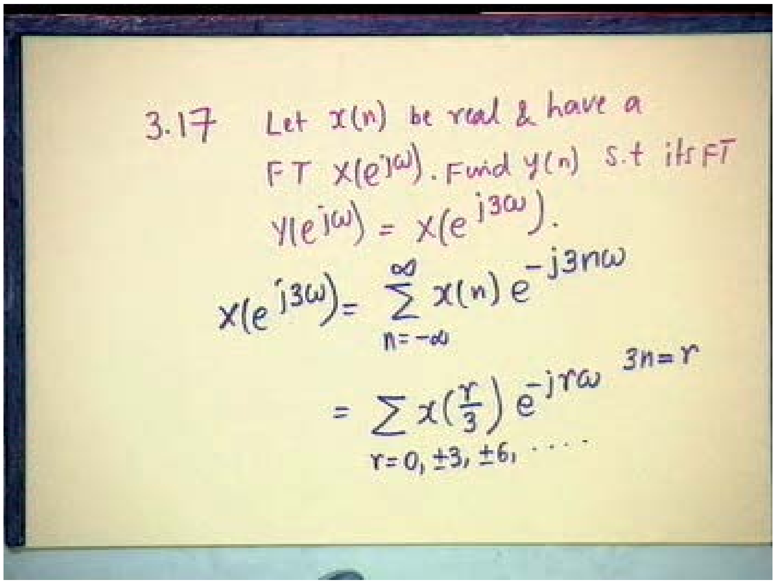

Therefore your answer now shall be y(n) = x(n/3) for n = 0, ± 3, ± 6 and so on and 0 otherwise; it is an up sampler. The intermediate samples are 0. Between 0 and + 3 there are two more 0 valued samples. That was the purpose of this particular problem.

The next one is 3.24. It says: using Parseval's relation for FT evaluate the integrals a) ∫(0 to π) (4/(5 + 4 cosine ω)) dω, b) ∫(0 to π) dω/(3.25 – 3 cosine ω) and c) ∫(0 to π)/[4/(5 – 4cosine ω)^2] dω.

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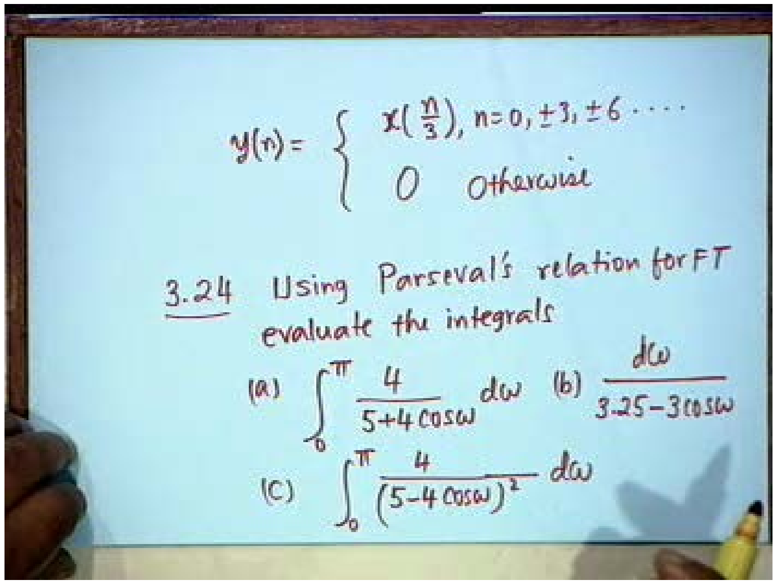

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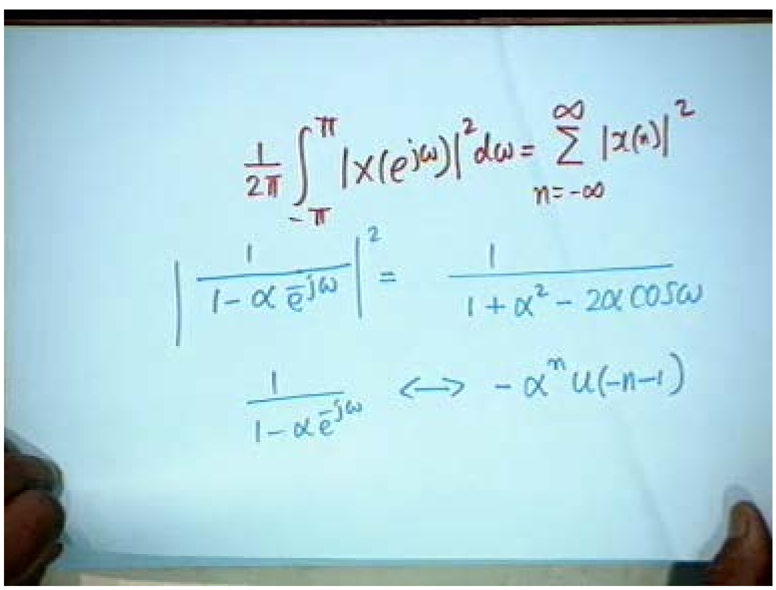

Let us recall Parseval's relation; it is $[1/(2\pi)] \int(-\pi \text{ to } \pi) |X(e^{j\omega})|^2 d\omega = \sum|x(n)|^2$ $n = -\infty$ to $+\infty$. If I have to use this it means that you have to identify the integrand. Let us take the first one, you have to identify the integrand with some $X(e^{j\omega})$ magnitude squared. Once you do that, find the corresponding x(n) and then sum it up. Now integration limits of 0 to π is not a problem because the integrand is an even function and therefore integral 0 to π shall be half of integral $-\pi$ to π. First, note that $(1-\alpha e^{-j\omega})$ magnitude squared = $1 + \alpha^2 - 2\alpha$ cosine ω.

Now this is exactly of the form that is wanted. You see 5 is 1 + 4. So for the first problem α is -2. For the second problem, 3.25 = 1 + 2.25, 2.25 is the square of 1.5. So here α is + 1.5. For the third problem, $5 - 4\cos\omega$ is magnitude squared of $1 - \alpha e^{-j\omega}$ with $\alpha = +2$. So $|\alpha|$ for each of these three cases is greater than unity and therefore the corresponding sequences of which the Fourier Transform is $1/(1 - \alpha e^{-j\omega})$ must be anti causal, they cannot be causal. If alpha magnitude is greater than 1 and the sequence is right sided then the z transform does not exist. The Fourier Transform does not exist because the corresponding series does not converge. Then, when I invert $1/(1 - \alpha e^{-j\omega})$, the inversion shall not give me $\alpha^n u(n)$. You might be in a hurry to write $\alpha^n u(n)$ but that is not correct. It shall be $-\alpha^n u(-n-1)$. Once you recognize these three basic facts, the rest is easy.

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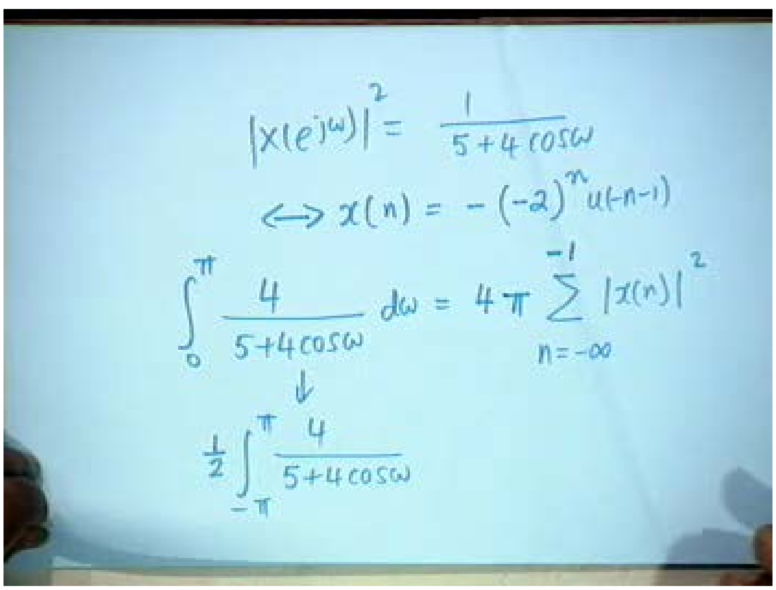

I will work out only the first one I will leave the rest to you. In the first one $|X(e^{j\omega})|^2 = 1/(5 + 4$ cosine$\omega)$. The corresponding $x(n) = -(-2)^n u(-n-1)$; since I have to integrate from 0 to π, and there was a constant factor 4, the integral will be = $4\pi \sum |x(n)|^2$, n = – ∞ to + ∞. Also recognize that x(n) is left sided; it is anti causal. So n goes from – ∞ to – 1.

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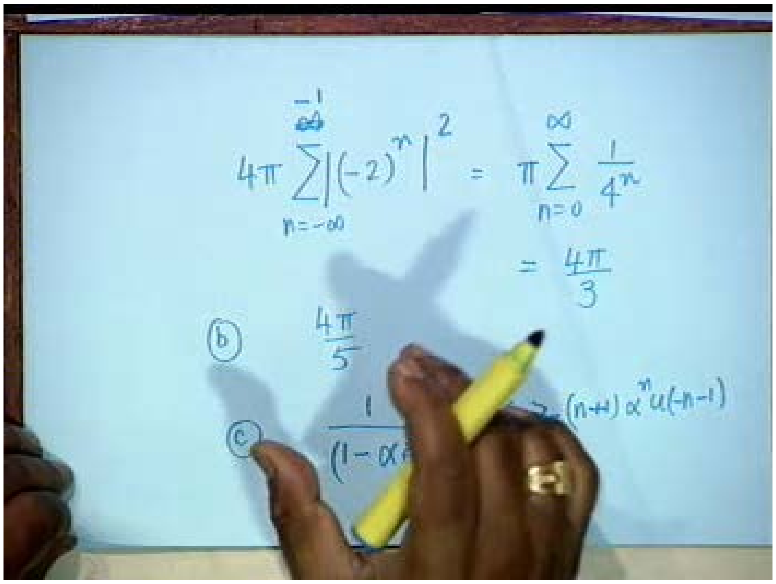

With a little jugglery, you can put this as = $\pi \sum(n = 0 \text{ to } \infty)\ 1/4^n$ and this happens to be 4π/3. In the second problem α = + 1.5 and you workout the same way; the final answer will be 4π/5. In the c) part, our function $(4/(5 - 4\ \text{cosine}\omega)^2)$ corresponds to magnitude of $(1 - \alpha\ e^{-j\omega})^2$ and the inverse transform of this is $-(n + 1)\ \alpha^n\ u(-n - 1)$ where the first term shall be 0. Therefore it can be written in some other form also. Then you have to find $\sum((n + 1)\ \alpha^n)^2$.

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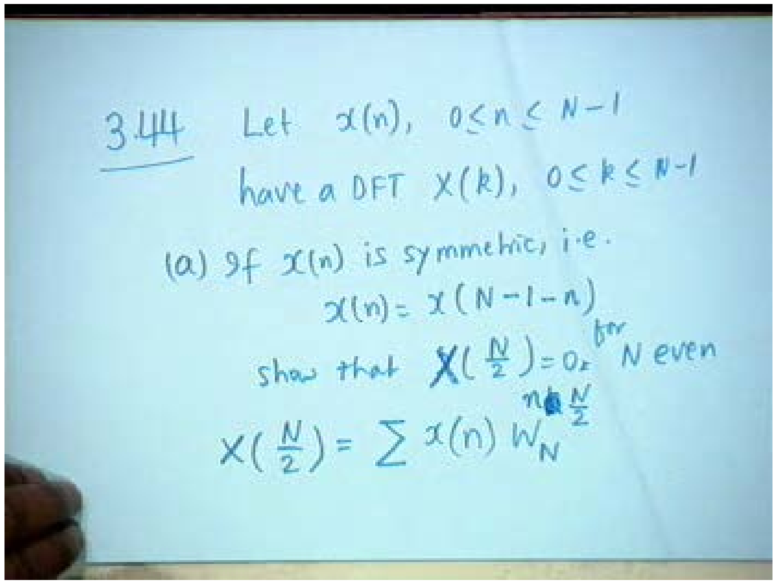

The next problem is 3.44. Let x(n), 0 less than equal to n less than equal to N – 1 be a sequence. That is x(n) is a sequence of length N. Its DFT is X(k), 0 less than equal to k less than equal to N – 1. Then it says if x(n) is symmetric that is x(n) = x(N – 1 – n), then show that X(N/2) = 0 for N even; this is the first part. This is extremely simple; you write X (N/2) = $\sum x(n)\, w_N^{nN/2}$, n = 0 to N – 1.

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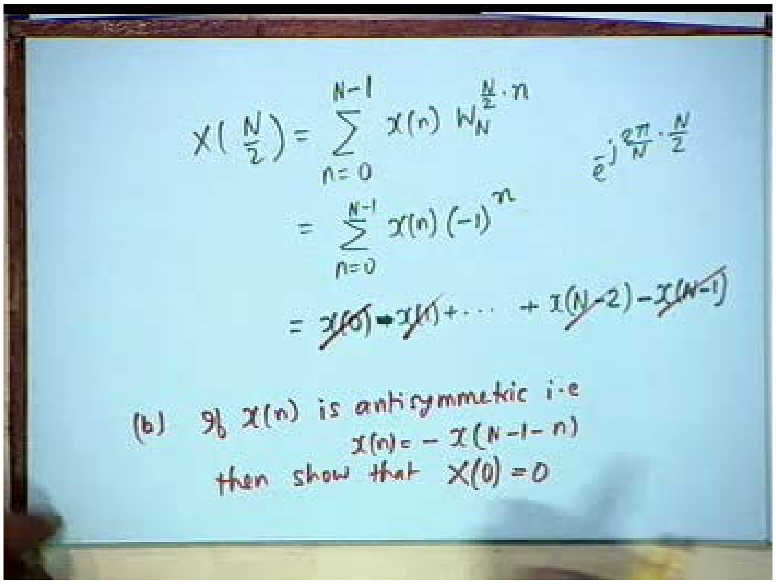

Now $w_N^{N/2}$ is – 1 and therefore I get ∑(n = 0 to N – 1) x(n) $(-1)^n$ that is = x(0) – x(1) +…+ x(N – 2) – x(N – 1); because it is symmetric, x(0) is the same as x(N – 1), x(1) = x(N – 2) and so on so that the total sum will be = 0 because there are even number of terms here. In part b, the statement of the problem is: if x(n) is antisymmetric, that is x(n) = – x(N – 1 – n), then show that X(0) = 0, no condition on N. Let us see how this is satisfied.

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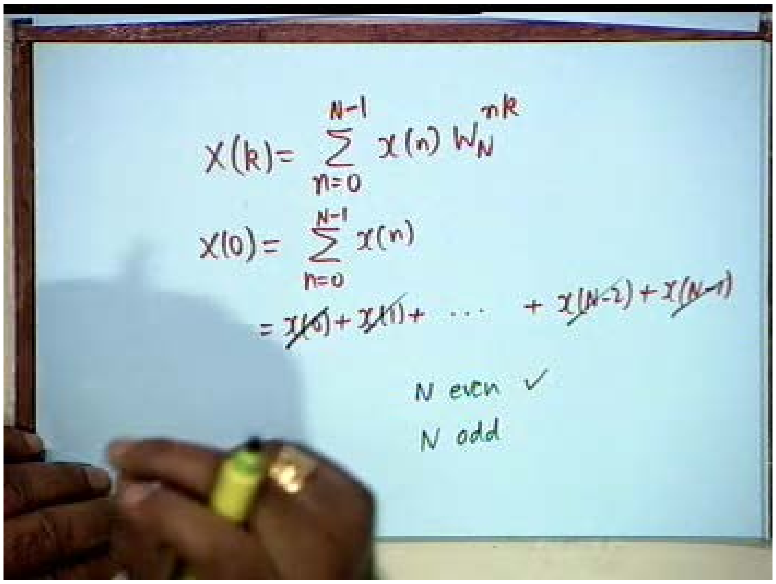

Now since X(k) = ∑x(n) W_N^{nk}, n = 0 to N – 1, X(0) is simply the ∑(n = 0 to N – 1) x(n). That is x(0) + x(1) +…+ x(N – 2) + x(N – 1). Since it is antisymmetric, x(0)cancels with x(N – 1), x(1) cancels with x(N – 2). If N is even I do not have a problem. There are pairs to cancel, but if N is odd then there shall be a loner which is 0 because x(n) is antisymmetric. x((N – 1)/2) must be = – X((N – 1)/2), so it must be = 0.

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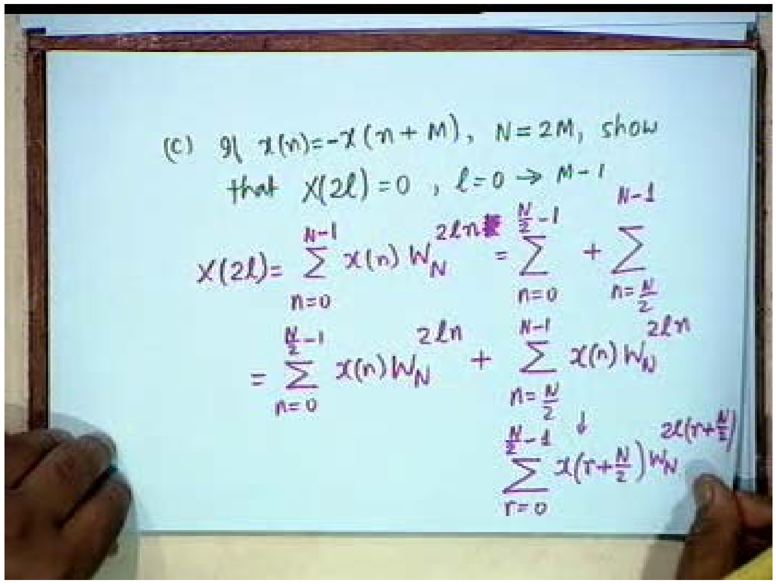

The third part says that let x(n) = – x(n + M) where N = 2M, that is N is even and M is an integer. If x(n) is – x(n + M) show that X(2ℓ) = 0 for ℓ = 0 to M – 1. The proof is quite straightforward but you have to use the Kronekar delta at some point. X(2ℓ) by definition is $\sum x(n)W_N^{2\ell n}$. This goes from n = 0 to N – 1. This summation I can write in two parts ∑(n = 0 to (N/2) – 1) + ∑(n = (N/2) to N – 1). Now in the second summation, put r = n – N/2. Then I can write this summation as ∑(r = 0 to (N/2) – 1) x(r + N/2) $W_N^{2\ell(r+N/2)}$.

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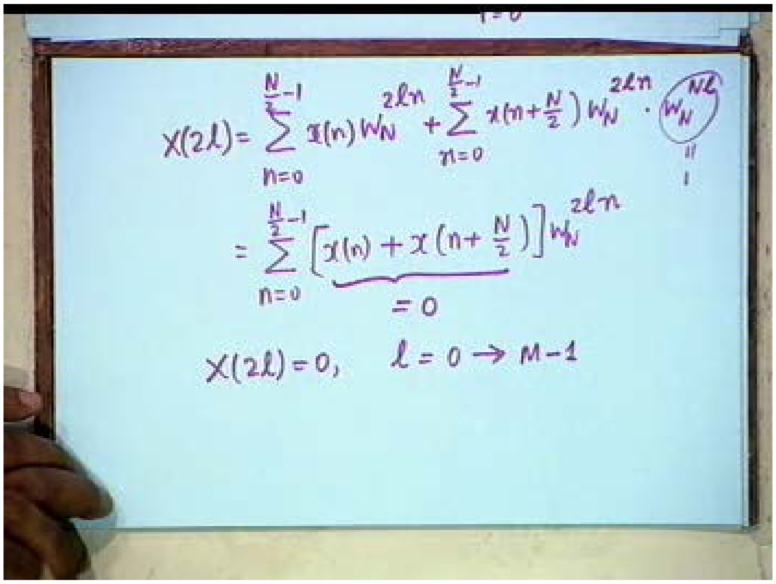

Since $W_N^{\ell N} = 1$, we can bring both of them under one same summation, viz. $\sum(n = 0$ to $(N/2) - 1)\ [x(n) + x(n + N/2)]\ w_N^{2\ell n}$. But $[x(n) + x(n + N/2)] = 0$. Therefore $x(2\ell) = 0$.

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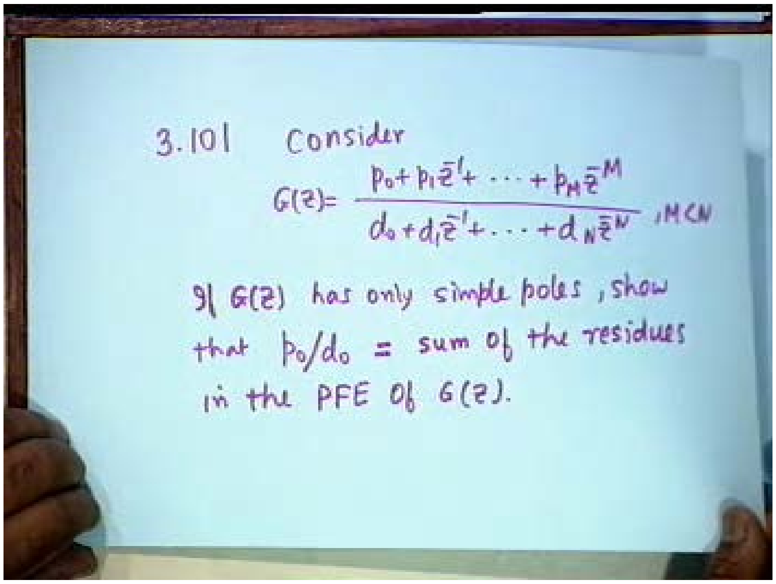

The problem that I now go to is 3.101. The problem is stated in several lines but the solution is very simple. The problem says consider $G(z) = (p_0 + p_1z^{-1} + \ldots + p_Mz^{-M})/(d_0 + d_1z^{-1} + \ldots. + d_N z^{-N})$, M is less than N. If G(z) has only simple poles, show that p_0/d_0 = the sum of the residues in the partial fraction expansion (PFE) of G(z). The solution is extremely simple once we write it. G(z), by hypothesis, has only simple poles and therefore it can be expanded in this form $\sum A_i/(1 - q_iz^{-1})$ i = 1 to N.

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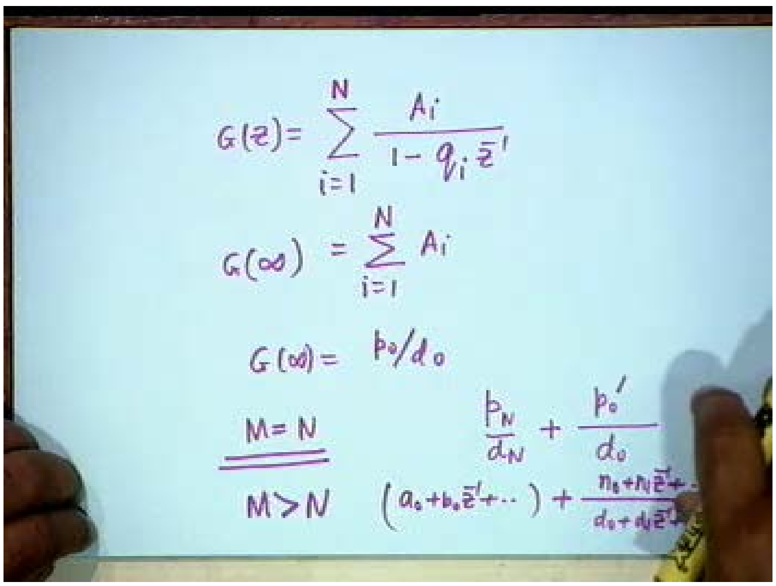

The significance of M less than N is that there is no constant or FIR part in the PFE; if M was greater than N, then you have to take an FIR part out to make it a proper rational function in z^{-1}; therefore M less than N makes life simple. We extract the sum of the residues by using $z = \infty$; this sum equals $G(\infty)$. And if you look at the transfer function, $G(\infty)$ is simply equal to p_0/d_0. If M = N then our result would have been $p_N/d_N + p_0'/d_0$ where p_0' is the constant term in $G(z) - p_N/d_N$. Now if M is greater than N then you have to take out an FIR term. If the FIR term is $(a_0 + b_0z^{-1} + \ldots)$ and the numerator of the proper fraction is $n_0 + n_1z^{-1} + \ldots$, then the sum of the residues would be $a_0 + n_0/d_0$. So M less than N is very significant, if it was not there then you have to modify the problem as per the discussion just made.

The next problem is 3.100. It says: consider $G(z) = P(z)/Q(z)$ let ρ_ℓ be the residue of $G(z)$ at a simple pole $z = \lambda_\ell$. Show that $\rho_\ell = -\lambda_\ell P(z)/Q'(z)$ at $z = \lambda_\ell$ where $Q'(z) = dQ(z)/dz^{-1}$; this is the problem and the solution is very simple.

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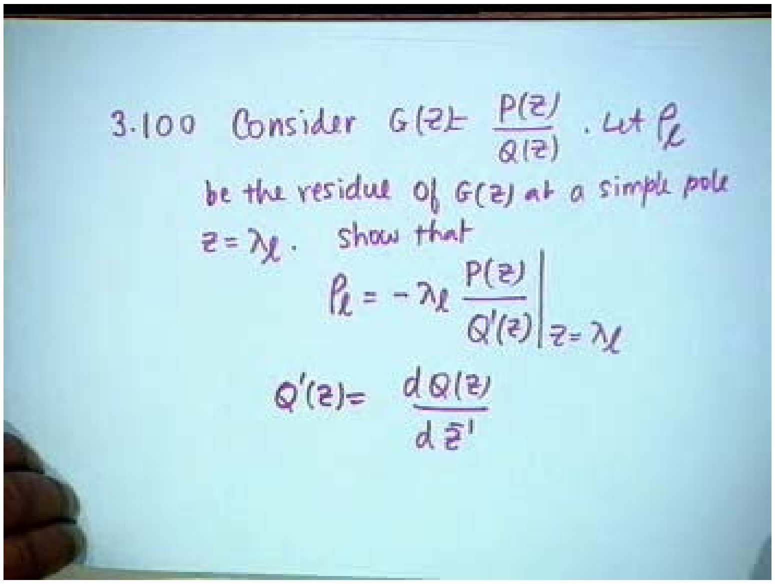

Read the problem carefully; the problem says that ρ_ℓ is the residue of $G(z)$ at a simple pole $z = \lambda_\ell$ and therefore the denominator Q must have a factor $(1 - \lambda_\ell z^{-1})$. The rest of it will be Q_1, some other polynomial. So we will write this as $P(z)/[(1 - \lambda_\ell z^{-1}) \times Q_1(z)]$ and the residue at λ_ℓ is given by $(1 - \lambda_\ell z^{-1}) \times G(z)$ evaluated at $z = \lambda_\ell$ that is equal to $P(z)/Q_1(z)$ evaluated at $z = \lambda_l$. Now we have to establish the relationship between Q_1 and Q'. You notice that $Q(z) = (1 - \lambda_\ell z^{-1}) \times Q_1(z)$. $Q'(z)$, by definition is $dQ(z)/dz^{-1}$. $Q(z)$ is a product of two terms; so $Q'(z)$ is equal to $-\lambda_\ell Q_1(z) + (1 - \lambda_\ell z^{-1}) \times dQ_1(z)/dz^{-1}$.

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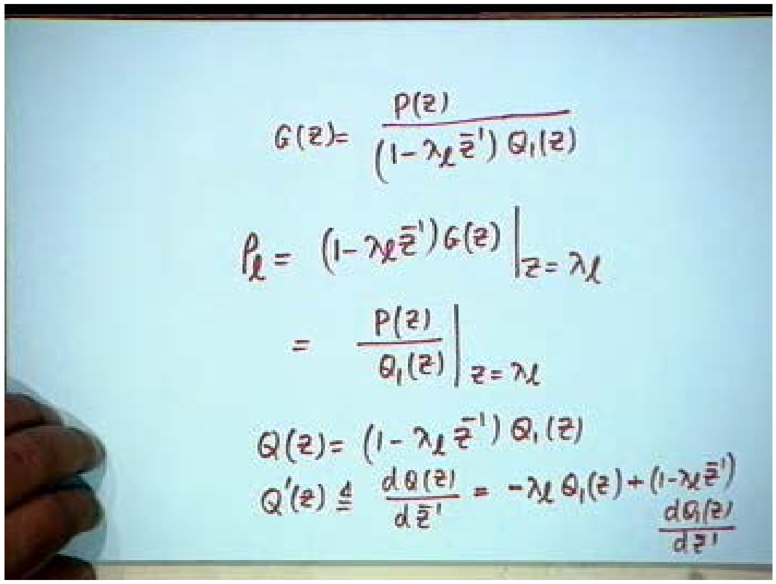

If I put $z = \lambda_\ell$ in the result $Q'(\lambda_\ell)$ simply becomes $-\lambda_\ell Q_1(\lambda_\ell)$. Therefore $\rho_\ell = P(\lambda_\ell)/Q_1(\lambda_\ell) = -\lambda_\ell P(\lambda_\ell)/Q'(\lambda_\ell)$ which is $= -\lambda_\ell P(z)/Q'(z)$ at $z = \lambda_\ell$.

Next problem is 3.93; once again it is a problem of interpolation. It is a problem related to up sampling. Let $X(z) = Z[x(n) = (0.4)^n u(n)]$. There are two parts in the problem; part a) asks you to determine g(n) which is $= Z^{-1}[X(z^2)]$ without computing X(z) and the b) part says determine $y(n) = Z^{-1}[(1+ z^{-1})\, X(z^2)]$, again without computing X(z).

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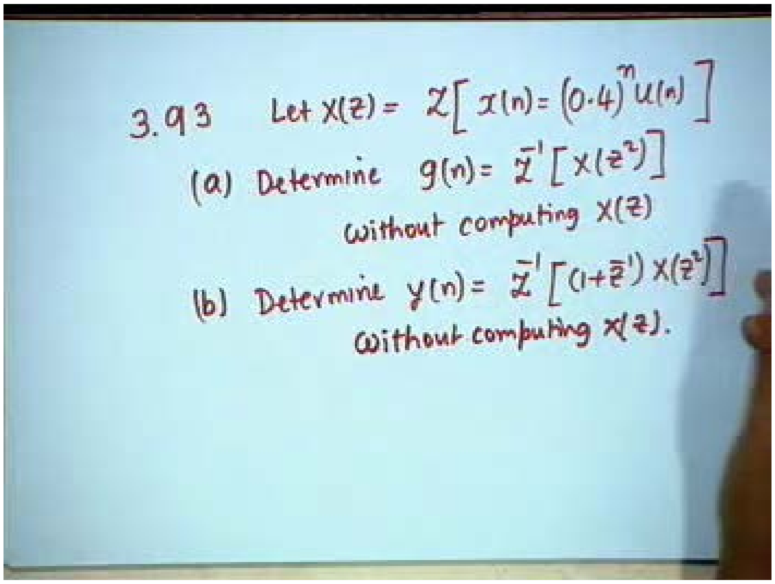

To solve this problem, we write $X(z^2) = \sum x(n)\, z^{-2n}$, n = – ∞ to + ∞, in general, which I can write as $\sum x(n/2)\, z^{-n}$ where n must be 0 and ±2, ±4 …and so on. Therefore g(n) = x(n/2), n = 0, ±2, ±4… and 0 otherwise. In our case x(n) is causal; in this special case of $x(n) = (0.4)^n u(n)$, g(n) should be = $(0.4)^{n/2}$, n = 0, 2, 4,… and 0 otherwise. Let us look at the other part determine the inverse transform of $(1 + z^{-1})\, X(z^2)$.

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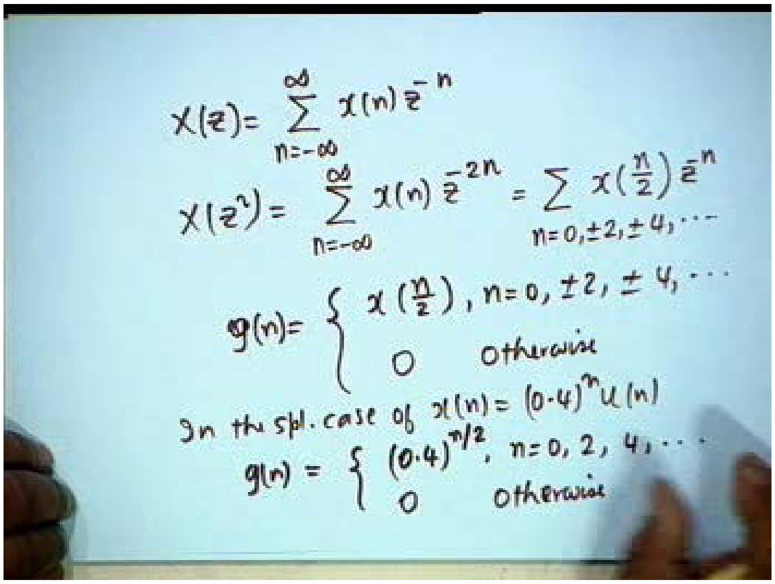

Z^{-1} [(1+ z^{-1}) $X(z^2)$] can be written as Z^{-1} [$X(z^2)$] + Z^{-1} [z^{-1} $X(z^2)$]. Now Z^{-1} [$X(z^2)$] is g(n) of part a) and Z^{-1} [z^{-1} $X(z^2)$] is g(n–1). Now g(n) in general g(n) = {x(n/2) n = 0, ± 2, ± 4,…. and 0 otherwise. g(n – 1) would be = (x(n – 1)/2), values of n should be odd, that is ± 1, ±3 and so on. In the present case, y(n) = {$(0.4)^{n/2}$, n = 0, 2, 4,… etc plus $(0.4)^{(n-1)/2)}$, n = 1, 3, 5…and so on.

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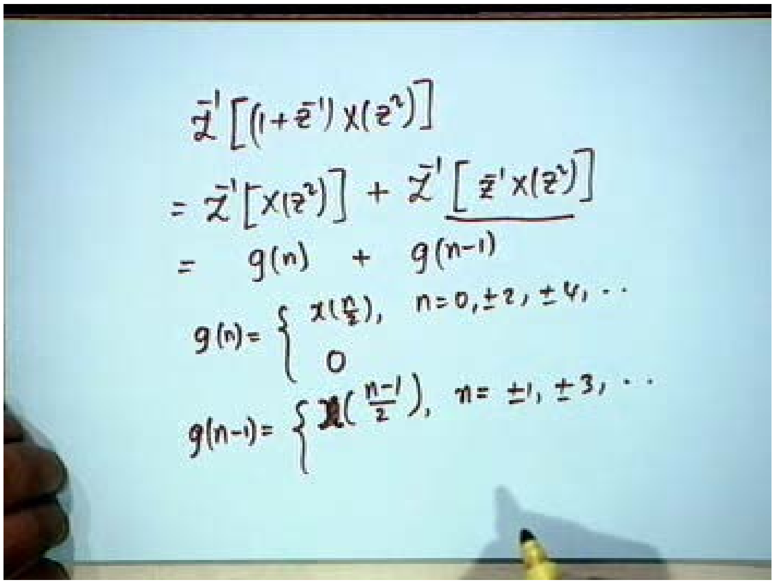

So it exists for all integer values of n. Because the original one is a right sided sequence, this is also a right sided sequence.

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