Digital Signal Processing

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Lecture - 24

Analog Chebyshev LPF Design

This is the 24th lecture on DSP and our topic today is Analog Chebyshev Low Pass Filter Design. In the last lecture, the 23rd, we talked about the Butterworth filter. We first talked about the motivation for Analog Filter Design in a course on DSP and then we said we only discuss Low Pass Filter because all other kinds of filters can be obtained by transformation of a low pass filter. Then we discussed the characteristics of Butterworth. Butterworth is monotonic; it is maximally flat at omega = 0. Its asymptotic slope is 6N decibels per octave and 20N decibels per decade; this is true about any all pole low pass filter.

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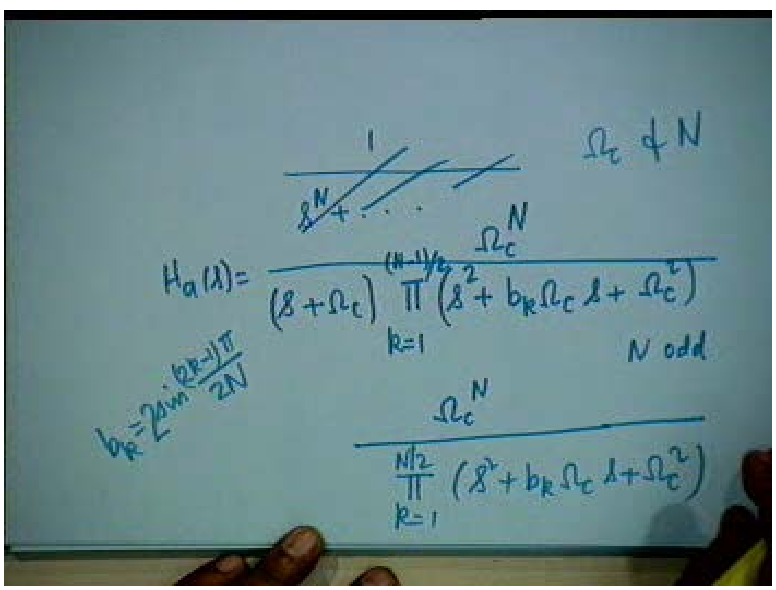

Then we discussed the pole locations. The pole locations were on a circle of radius capital omega$_c$ in the s plane and the pole factors were s + omega$_c$, and a continued product of (s^2 + b_k

omega_c s + omega_c^2) if the order N is odd, with k = 1 to (N – 1)/2. Then the transfer function would be of the form omega_c^N divided by the pole factors. If N is even, then we shall have omega_c^N divided by continued product (k = 1 to (N/2)) (s^2 + b_k omega_c s + omega_c^2) where this constant b_k = 2 sin (2k – 1) pi/(2N). All that you have to determine for Butterworth are omega_c and N.

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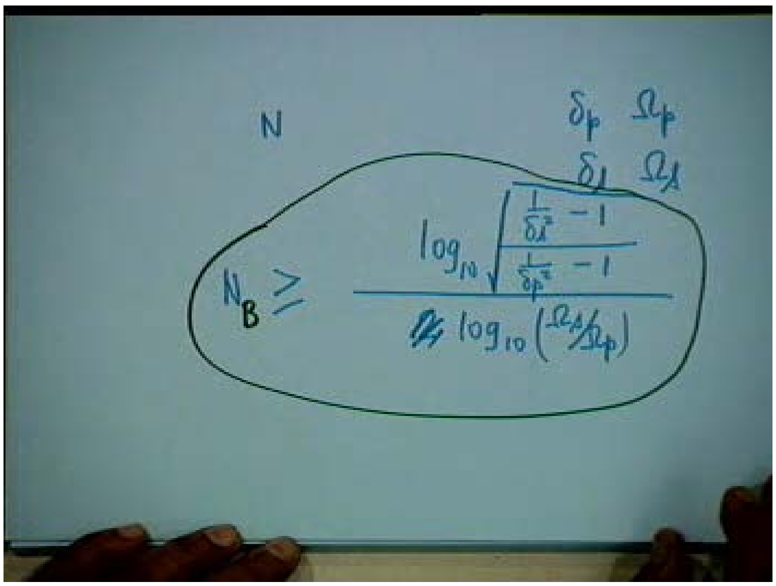

In general, N shall be determined from delta p, omega_p, delta s and omega_s. And N is greater than or equal to the formula that we derived, which is 2 $\log_{10}$ ($\text{omega}_s/\text{omega}_p$) in the denominator and in the numerator, we have $\log_{10}$[((1/delta s)^2 – 1)/((1/delta p)^2 – 1)]. For reasons to be made clear a little later, we can absorb the factor 2 in the denominator by introducing a square root sign after $\log_{10}$ in the numerator. This is N_B. We shall refer to it later also. It is a ratio of two logs and one of them contains the tolerances in the Pass Band and Stop Band and the other contains the Stop Band to Pass Band edge ratio.

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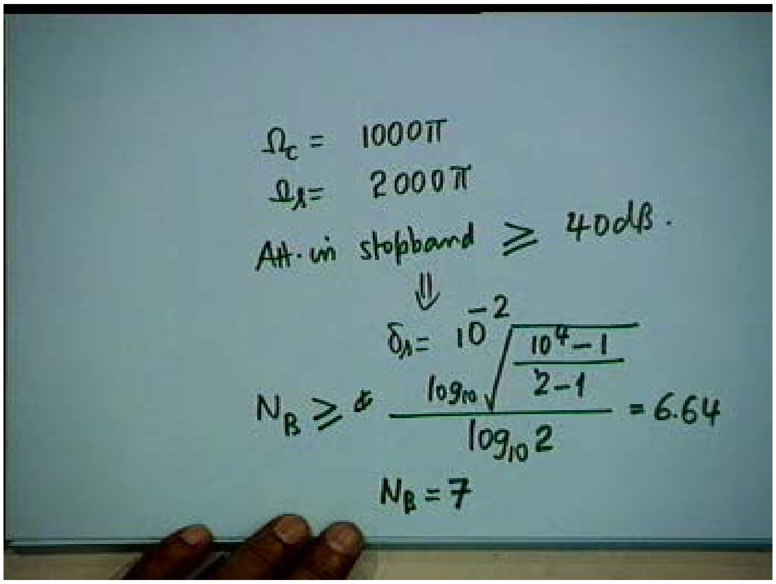

Let us take an example. Let $omega_c$ = 1000 pi and $omega_s$ = 2000 pi. Attenuation in Stop Band should be greater than or equal to 40dB, which gives rise to delta s = 10^{-2}. As calculated from the formula, N_B is greater than or equal to 6.64. Therefore, the order of the Butterworth filter needed is N_B = 7. With this, obviously $omega_s$ realized would be lower than that specified. We do not have to find $omega_c$. $Omega_c$ is already given. Otherwise, if $omega_c$ was not given, then what would you have done? You will have to find $omega_c$. Why do you find $omega_c$, what is the need? The transfer function is expressed in terms of $omega_c$; hence $omega_c$ has to be found out.

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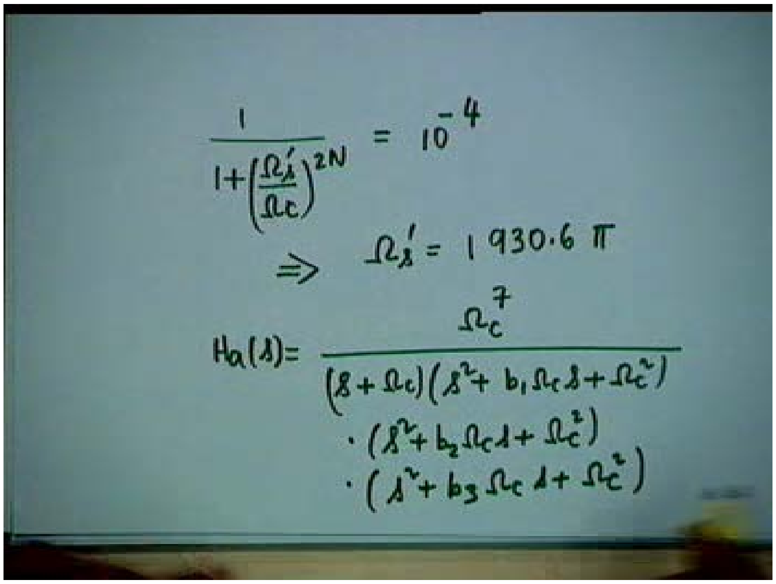

If N_B = 7 then obviously the realized omega_s, that is the new edge of the Stop Band, omega_s prime, would be given by 1/(1 + omega_s prime divided by $\text{omega}_c)^{2N}$ equal to (delta s$)^2$ = (10^{-4}) and if you solve this, then the omega_s prime comes out to be 1930.6 pi instead of 2000 pi. So it occurs earlier and one should be happy about it because you have over satisfied the Stop Band. Now you can find $H_a(s)$ as omega_c^7/[(s + omega_c) (s^2 + b_1 omega_cs + omega_c^2) (s^2 + b_2 omega $_c$ s + omega_c^2) (s^2 + b_3 omega $_c$ s + omega_c^2)]; you have to find out b_1, b_2 and b_3.

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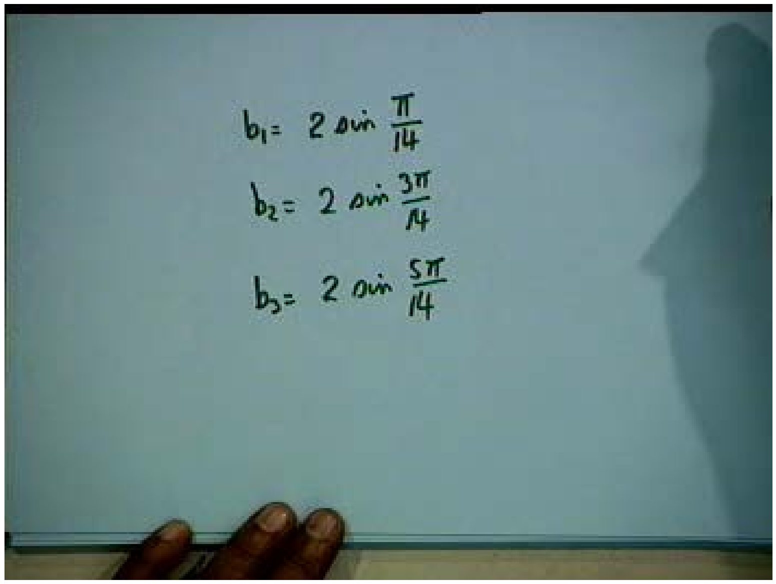

Here b_1 = 2 sin (pi/14), b_2 = 2 sin (3 pi/14) and b_3 = 2sin (5 pi/14). After a while you will be able to write the transfer function almost blindly. It can be very easily programmed, given the order and omega_c. Now let us go to Chebyshev.

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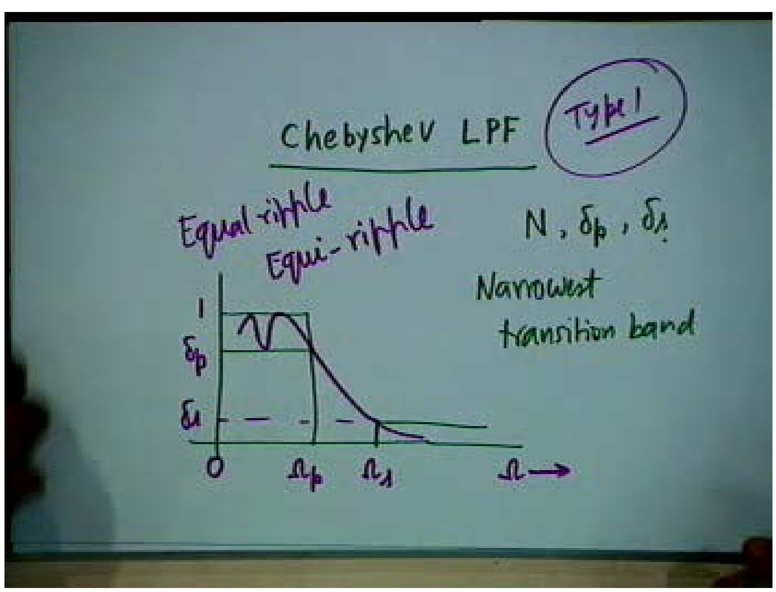

The actual spelling of Chebyshev is quite complicated; this form, I guess is the American simplification. Actual spelling is T sc h y b e sc h e f f. Chebyshev LPF has the distinctive characteristic that if elliptic filters are excluded, then Chebyshev LPF is the Optimum One. Chebyshev LPF is optimum in the sense that for given order, $omega_p$, and tolerances, the Chebyshev LPF, of all all-pole filters that are possible, will give the narrowest transition band. It can also be qualified or characterized in alternative ways.

For example, if N, transition band and delta p are specified it will give the lowest delta s. The characteristic is that instead of monotonicity as in Butterworth filter, it ripples in the Pass Band. That is, the characteristic is something like the one shown in the figure. I have intentionally not shown the value at omega = 0 because it differs according to the evenness or oddness of the order. Not only it ripples but also it executes equal ripples in the Pass Band; it is monotonic in the Stop Band. So unlike Butterworth, it is not a monotonic filter; it has maxima and minima but between two limits. Equal ripple is sometimes abbreviated as Equi-ripple. Thus Chebyshev LPF is Equi-ripple in the Pass Band and monotonic in the Stop Band.

There can also be another type of Chebyshev, that is, we can reverse the position of the ripples; we can have monotonic in the Pass Band and equal ripple in the Stop Band. This type is called the inverse Chebyshev filter. It is also sometimes called a type two Filter, while the previous type is called type one Chebyshev. Obviously type one is the preferred filter because once we have specified delta s and satisfied this, one should not bother whether there are ripples or not in the stop band. I repeat: Type one Chebyshev is the most favored optimum Low Pass Filter provided you exclude Elliptic Filters from your field of view.

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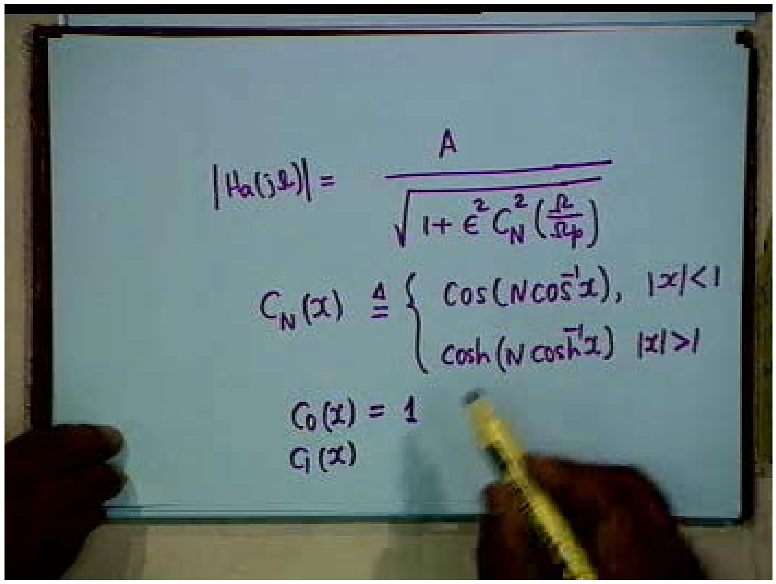

The magnitude function of a Chebyshev filter is given by A/square root of 1 + ϵ^2(epsilon square, epsilon is a constant, A is a constant) C_N^2 (N is the order) (omega/omega$_p$). This is the form of the magnitude function. $C_N(x)$ where x = omega/omega$_p$ is the Chebyshev polynomial and is defined as cosine(N cosine^{-1} x) for mod x less than one, that is x between – 1 and + 1. And when $|x|$ exceeds 1 cosine loses its meaning and is replaced by cosh; this is a natural transition. Cos(N cos^{-1}x) becomes Cosh (N cosh $^{-1}$x) for mod x greater than 1.

Now you know that cosine function is oscillatory and this is what gives rise to oscillations or ripples in the Pass Band. For mod x less than 1, that is omega between – omega$_p$ and + omega$_p$ there shall be oscillations. Since the magnitude function is even, it suffices to consider the range 0 to omega$_p$. For mod x greater than 1, that is omega greater than omega$_p$ cosh function is monotonic and therefore the magnitude characteristic shall be monotonic beyond the Pass Band. In the Pass Band it will be oscillatory; beyond the Pass Band, it shall be monotonic which is the characteristic of the Chebyshev filter. Now let us get familiarized with Chebyshev polynomials. Notice that C_0 (x) is simply = 1. C_1(x), cosine of cosine inverse x, obviously is = x.

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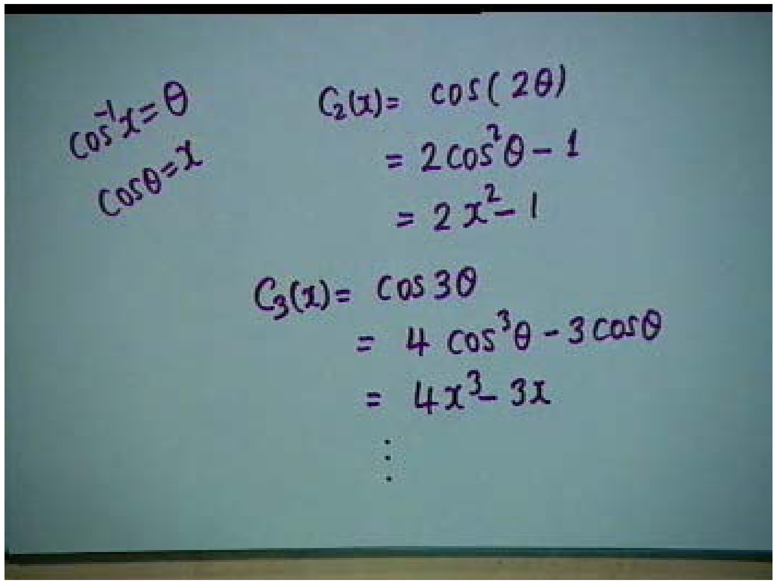

To find $C_2(x)$, let us put cosine^{-1} x = theta. Then cosine theta = x and therefore $C_2(x)$ would be cosine (2theta), that is equal to $2x^2 - 1$. Similarly $C_3(x)$ which is cosine of 3 theta is = 4 cosine^3 theta – 3 cosine theta and therefore $C_3(x) = 4x^3 - 3x$. Does that mean that we shall have to remember all the multiple angle formulas? No, we can find a recurrence relationship.

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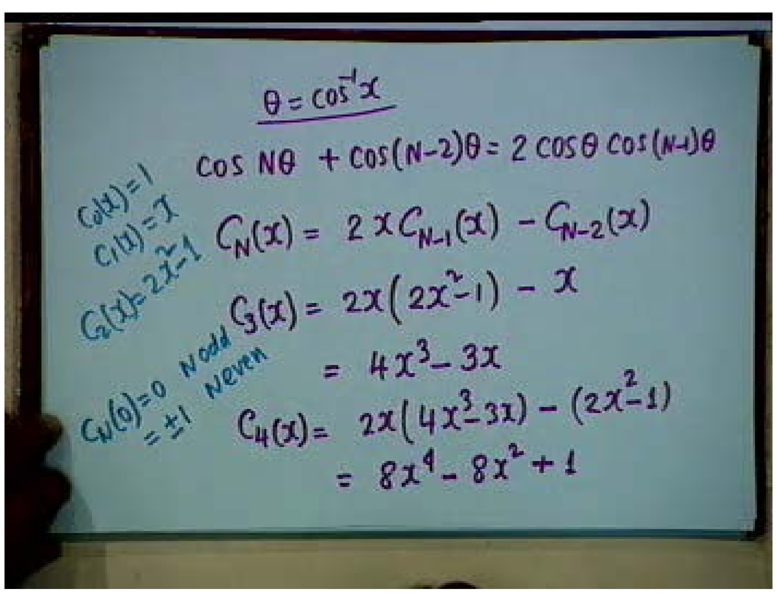

The recurrence relationship comes from this: cosine (N theta) + cosine (N – 2 theta) = 2 cos theta cos(N – 1) theta. Since cosine N theta is $C_N(x)$, we have $C_N(x) = 2x\, C_{N-1}(x) - C_{N-2}(x)$. For example, if N is 3, you get $C_3(x) = 2x\, C_2(x) - C_1(x) = 4x^3 - 3x$. This agrees with the previous result. In a similar way you can find out $C_4(x)$ as $2x(4x^3 - 3x) - (2x^2 - 1)$, that is equal to $8x^4 - 8x^2 + 1$. Also, as already indicated, $C_0(x) = 1$, $C_1(x) = x$, and $C_2(x) = 2x^2 - 1$.

You notice a pattern: if the order is odd then the polynomial itself is odd. $C_1(x) = x$,$C_3(x) = 4x^3 - 3x$ and so on. On the other hand, if the order is even, then the polynomial is also even. So it is 1, $2x^2 - 1$, and $8x^4 - 8x^2 + 1$ for N = 0, 2 and 4 respectively. And you notice that $C_N(0)$ has to be equal to 0 if N is odd because $C_N(x)$ is odd polynomial. And $C_N(0)$ is equal to + or – 1 if N is even. $C_2(0) = -1$ but $C_4(0)$ is + 1. So it is either – 1 or + 1. So we learnt two things: one is, if N is even the polynomial is even and if N is odd, the polynomial is odd and because of even or odd characteristic the value at x = 0 is either 0 or $\pm$ 1 depending on whether N is odd or even.

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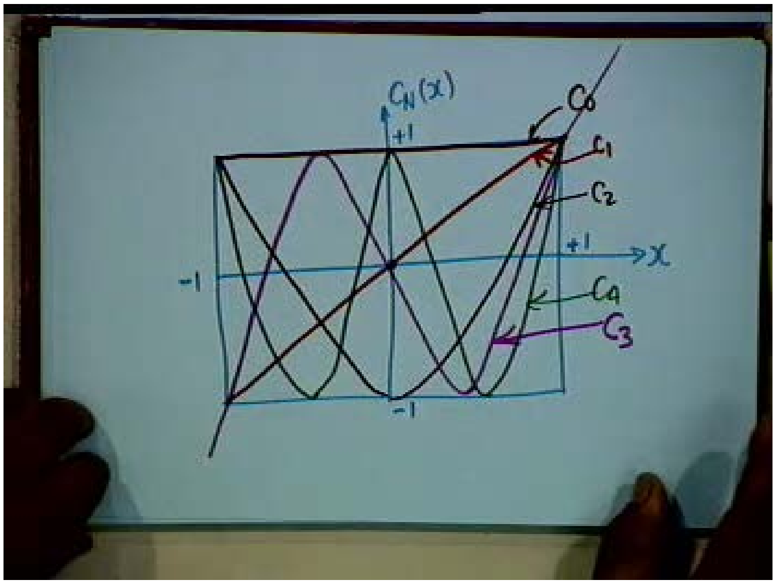

Now a picture is a better indicator of these properties. Draw this picture. Draw a square and divide into four equal parts. We plot $C_N(x)$ in the vertical direction and x in the horizontal direction.

The plots of C_0, C_1, C_2, C_3 and C_4 are shown. When x exceeds 1 or x goes below – 1, it is monotonic. Now, the equal ripple oscillation is obvious from x = – 1 to + 1. Our concern is between 0 and + 1. We only consider positive frequencies; and 0 for x shall correspond to capital omega = 0; + 1 for x shall correspond to capital omega = omega_p.

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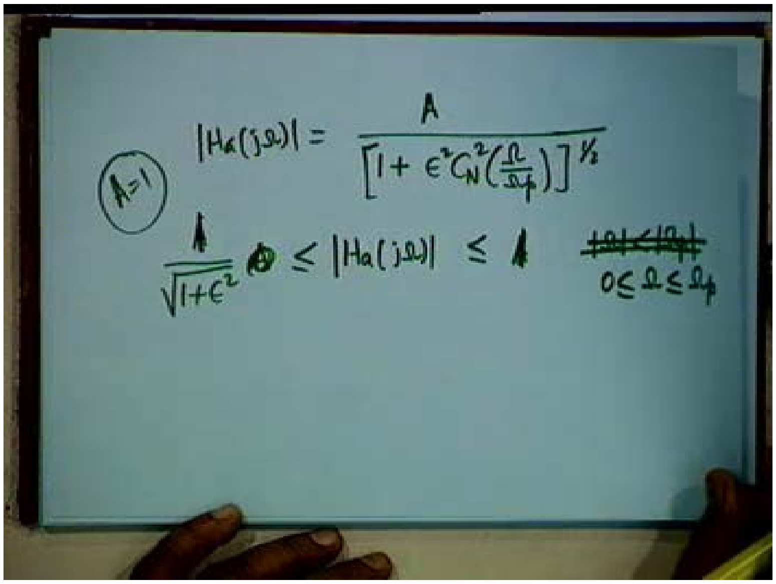

Therefore if we consider the transfer function, then H_a (j omega) magnitude = A/square root of [1 + ϵ^2 C_N^2(omega/omega_p)]. Thus the maximum H_a(j omega) magnitude shall obviously lie between two limits for mod omega less than omega_p. We are considering only positive frequencies: 0 less than or equal to omega less than or equal to omega_p. In this range because C_N is oscillating, the minimum value of magnitude shall be A/square root (1 + ϵ^2). When $C_N = \pm 1$ and if A is normalized to 1, then the magnitude shall lie between 1 when C_N = 0 and this value. Now, let us draw a few typical magnitude characteristics starting with N = 1.

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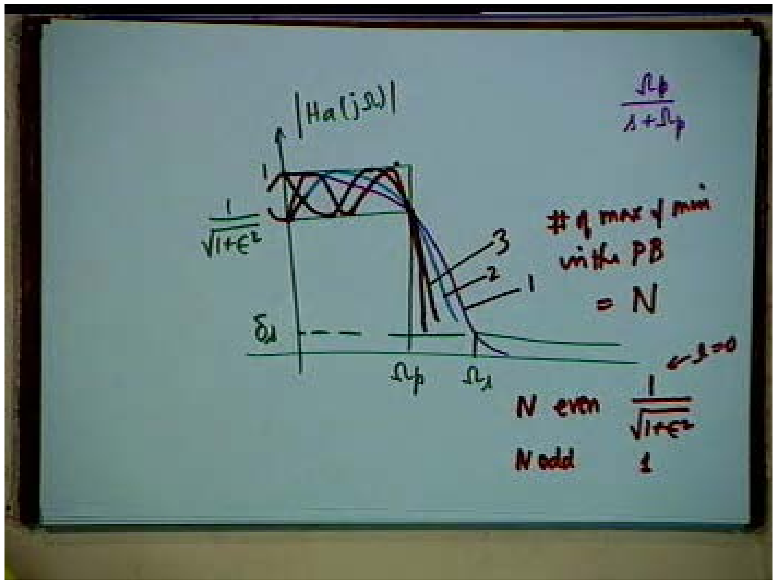

Clearly, if N is odd, then at omega = 0 the magnitude shall be 1. If N is even, then the magnitude is 1/square root of (1 + ϵ^2) at omega = 0. For the first order filter, Butterworth and Chebyshev are the same because the transfer function is $\text{omega}_p/(s + \text{omega}_p)$.

Notice that at omega = omega_p, $C_N^2(1) = 1$. And therefore, irrespective of the order, all characteristics must pass through the point [omega_p, 1/square root of (1 + ϵ^2)] exactly like Butterworth filters. Omega_p is the edge of the Pass Band. Unlike Butterworth, the value at omega = omega_p is not 1/root 2. It is equal to 1/root 2 if $\epsilon = 1$. The characteristics clearly show that the number of peaks and dips in the pass band is exactly equal to the order; considering omega = 0 as either a peak or a dip. If the function is even, then at omega = 0, there is a minimum; on the other hand, if N is odd, then at omega = 0, there is a maximum.

I repeat: the number of maxima and minima in the Pass Band is equal to N, the order of the filter. And if N is even then starting point is 1/square root of (1 + ϵ^2). If N is odd then the starting point is 1. Starting point means at omega = 0. If you remember this then we can proceed further. If someone asks you to find out the 3dB frequency of a Chebyshev filter, you can; 3dB frequency is a very important factor in Butterworth filter, it is not so in Chebyshev. In Chebyshev, $omega_p$ and ϵ are arbitrary.

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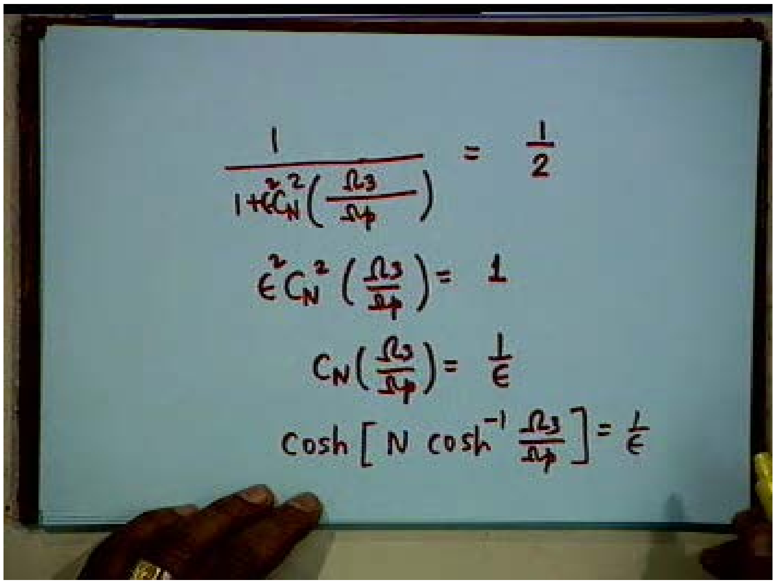

To find out the 3dB frequency, $omega_3$, then obviously the defining equation would be $1 + \epsilon^2 C_N^2$ ($omega_3/omega_p$) = 2 and therefore ϵ^2 C_N^2 ($omega_3/omega_p$) = 1. Therefore C_N ($omega_3/omega_p$) = $1/\epsilon$. I can take plus or minus sign before $1/\epsilon$; then I will find + $omega_3$ and – $omega_3$. Let us take plus sign; now $1/\epsilon$ shall be greater than 1. Epsilon normally is a very small quantity, $1/\epsilon$ is greater than 1 and therefore for C_N I should use cosh.

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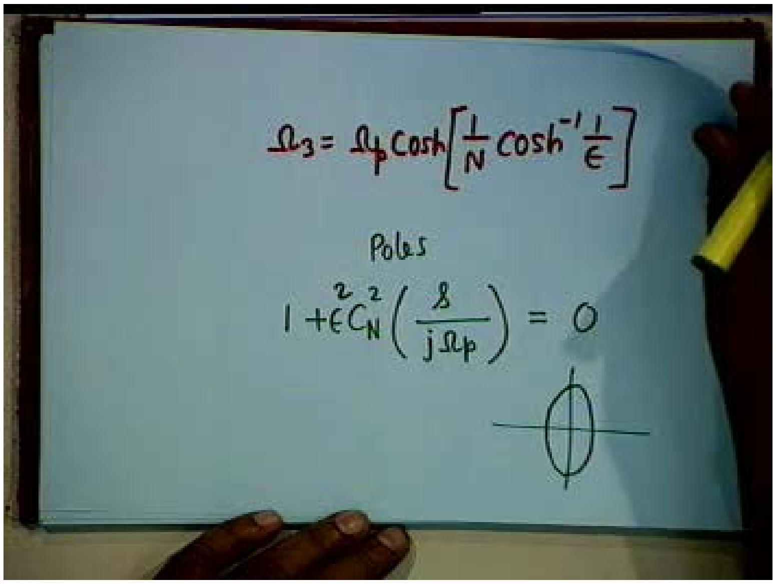

So cosh[N $\cosh^{-1}$ $omega_3/omega_p$] = 1/ϵ from which you can find out the expression for $omega_3$; that should be equal to $omega_p$ [(1/N)$Cosh^{-1}$ 1/ϵ]; this is the expression for the 3dB frequency. Obviously this will be greater than $omega_p$. if $\epsilon < 1$. Next, the question of Poles and Pole Factors: obviously, the poles will satisfy 1 + C_N^2 [(s/j) $omega_p$] = 0 and this turns out to be a lot of complex algebra which we shall not go into but the end result is that the poles are located on an ellipse, instead of a circle as in the Butterworth case.

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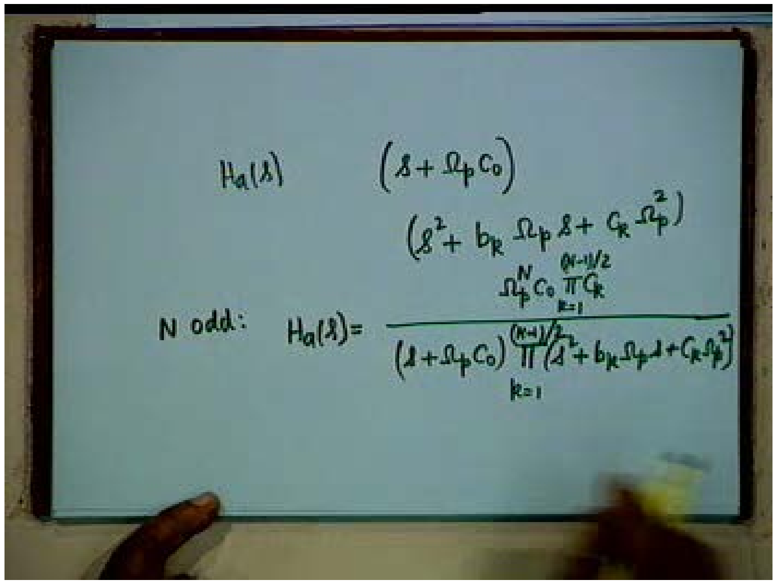

The pole locations are such that finally the denominator of the transfer function $H_a(s)$ has a factor of s + omega_p C_0 (in the Butterworth case we had simply s + omega_c) when the order is odd. And the other factors are different from Butterworth case; we shall illustrate them a little later) omega_ps + C_k omega_p^2). These are the factors in the denominator. In other words, when N is odd, $H_a(s)$ denominator is (s + omega_p C_0) multiplied by continued product (s^2 + b_k omega_p s + C_k omega_p^2), k = 1 to (N – 1)/2 and the numerator shall be omega_p^N C_0 × continued product of C_k, k = 1 to (N – 1)/2. We have brought all the denominator constants in the numerator. So the dc value of this transfer function would be = 1.

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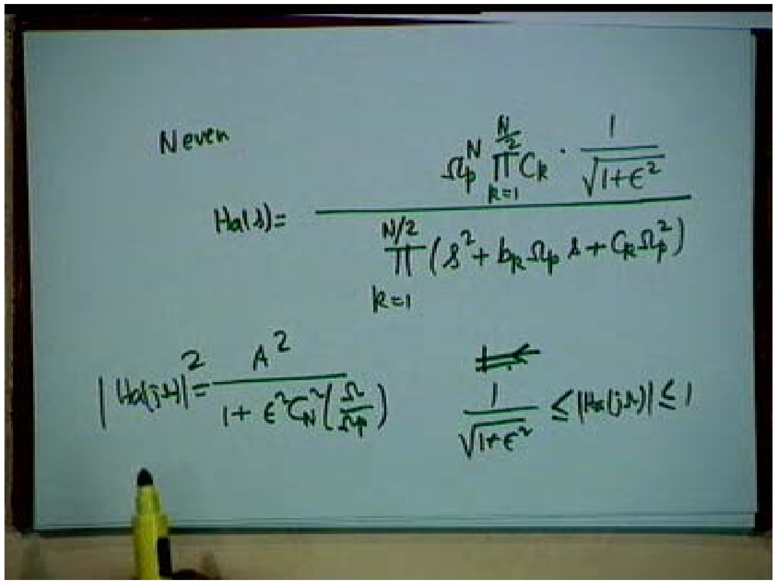

If the order N is even, then $H_a(s)$ denominator shall be continued product (s^2 + b_k $omega_p s$ + C_k $omega_p^2$), k = 1 to N/2 and in the numerator we shall get $omega_p^N$, then continued product C_k, k = 1 to N/2, multiplied by 1/square root of (1 + ϵ^2), to make the d.c. value equal to the last term. If the order is odd, the magnitude starts from 1. If the order is even, the magnitude starts from 1/square root of (1 + ϵ^2).

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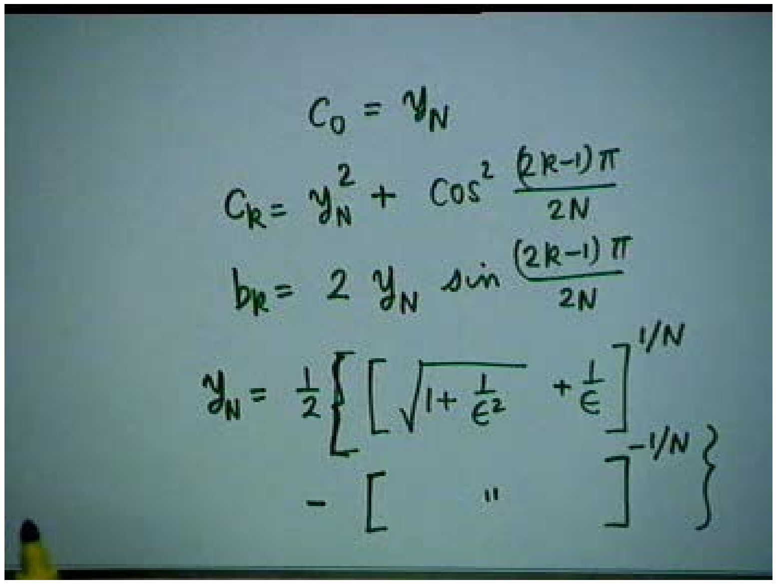

Now, it is time to tell you what these b_k's and C_k's are. We rename C_0 as y_N, to indicate that it depends on the order of the filter. Then $C_K = y_N^2$ + cosine^2[(2k – 1) π/(2N)]. Recall that this angle, (2k – 1)pi/(2N) is the Butterworth angle. b_K in Butterworth case was 2sine [(2k – 1)pi/(2N)]. In the Chebyshev case, it is multiplied by y_N. The factor y_N comes because of the shrinking of the circle into an ellipse. It is as if you press the circle from two sides and it elongates and shrinks in the middle. Finally what is y_N? This requires a little bit more calculation but it is not too difficult. y_N = ½ {[square root of $(1 + 1/\epsilon^2)$ + $(1/\epsilon)$] to the power 1/N – ½ [square root of $(1 + 1/\epsilon^2) + 1/\epsilon]^{-1/N}$}. So given N and ϵ you can find out the transfer function. N and epsilon (ϵ) are the only two parameters of a Chebyshev filter. Once you know them you can find out the transfer function.

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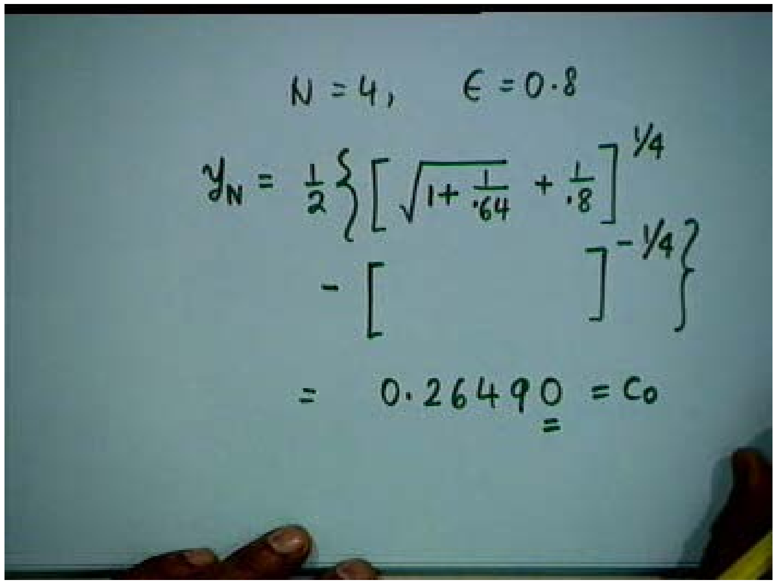

As an example, suppose capital N = 4 and ϵ = 0.8; then the calculation proceeds like this (you can also make a small program). First you find out y_N, y_N = ½{[square root of (1 + (1/0.64)) + (1/0.8)]$^{1/4}$ – [square root of (1 + (1/0.64)) + (1/0.8)]$^{-1/4}$}. This works out to 0.26490. This 0 at the end is significant. I could have omitted this but it shows that I have calculated it correct up to five places of decimals.

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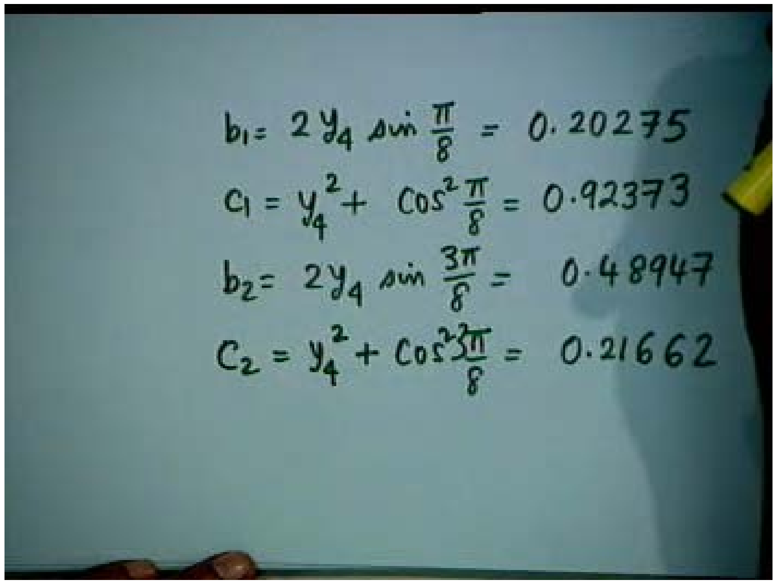

This is y_N, and y_N is the same as C_0. Now you have to find out 2 b's and 2C's. $b_1 = 2y_4$ (y_4 is already calculated) sine of pi/8 and that comes out as 0.20275, $C_1 = y_4^2$ + cosine^2 pi/8 and that calculates as 0.92373. At least go up to five places of decimals. $b_2 = 2y_4$ sine 3pi/8 and that comes out as 0.48947 and $C_2 = y_4^2$ + cosine^2 3pi/8 and that comes out as 0.21662.

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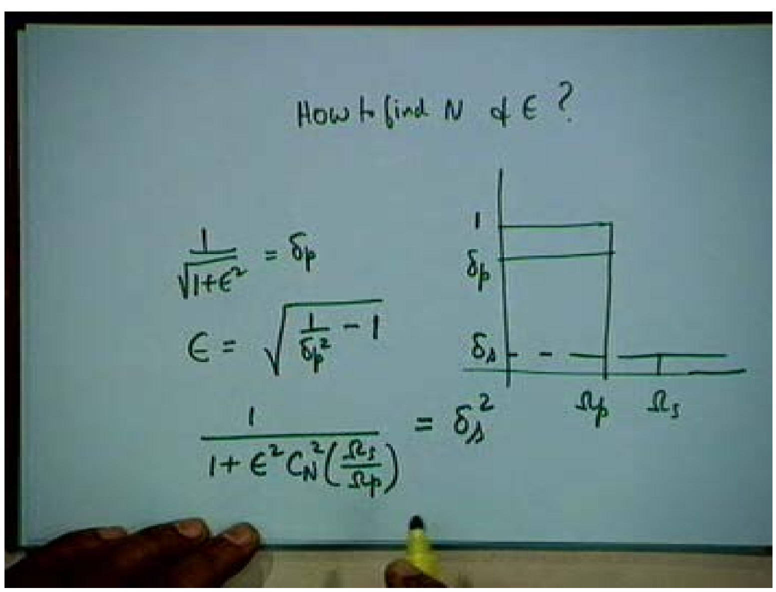

So the problem now is to find N and ϵ from the given specs. Well, the specs are: delta p, $omega_p$, $omega_s$ and delta s. If delta p is known then you know what is epsilon because, 1/square root of $(1 + \epsilon^2)$ = delta p. Therefore ϵ = square root of [(delta p)$^{-2}$ – 1]. The Stop Band specification says that $1 + \epsilon^2\ C_N{}^2(omega_s/omega_p)$ should be equal to (delta s)$^{-2}$.

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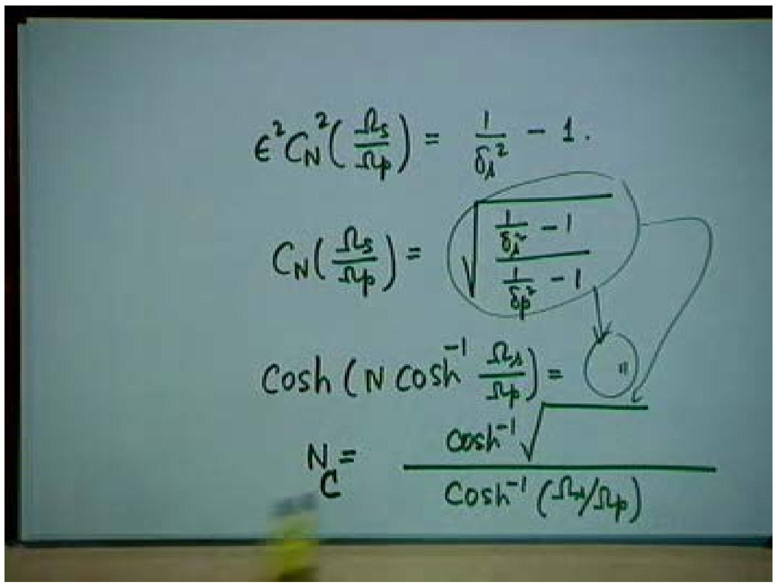

Therefore $\epsilon^2\ C_N{}^2\ (omega_s/omega_p)$ should be equal to (1/(delta s)^2) – 1 or $C_N\ (omega_s/omega_p)$ should be equal to 1/ϵ times square root of [(1/(delta s)^2) – 1]. If I put the value of epsilon, namely, square root of [(1/(delta p)^2) – 1], I get the expression for $C_N(omega_s/omega_p)$ = square root of [1/delta s)^2 – 1]/[(1/delta p)^2 – 1]. Let us denote this by delta. Because $omega_s$ is greater than $omega_p$, C_N becomes Cosh (N $\cosh^{-1} omega_s/omega_p$).

Therefore N = $\cosh^{-1}$ delta divided by $\cosh^{-1}$ $(omega_s/omega_p)$. And if you compare this with the Butterworth case, all that changes is that $\cosh^{-1}$ replaces $\log_{10}$.

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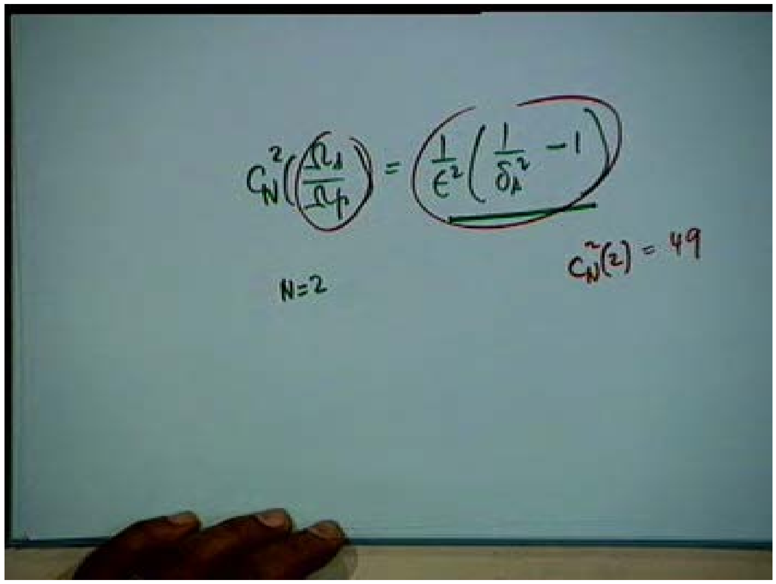

If you do not have excess to $\cosh^{-1}$, then use the intermediate expression C_N^2(omega$_s$/omega$_p$) = $(1/\epsilon^2)$ [(1/(delta s)^2) – 1]. You have numerical value for the right hand side and the argument of C_N^2. So try values of N. Try from N = 2. If the right hand side is less, then you increment N by one. This can also be programmed, increment N to N + 1 and then go on doing this till the left hand side exceeds the right hand side. There is no guarantee that they would be equal. In other words, this formula for N_C should also be written with greater than or equal to sign. Try values of N till the left hand side exceeds the right hand side. It may exceed drastically, but there's no other way; the positive point is that you are over satisfying the stop band specs. Next time we will take a few examples and also we shall consider how Low Pass Filters can be transformed to any other kind of filter.