Digital Signal Processing

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Lecture – 27

Problem Solving Session on Discrete Time System in Transform Domain

This is the 27^{th} session and it is a problem solving session on Discrete Time Systems in the Transform Domain.

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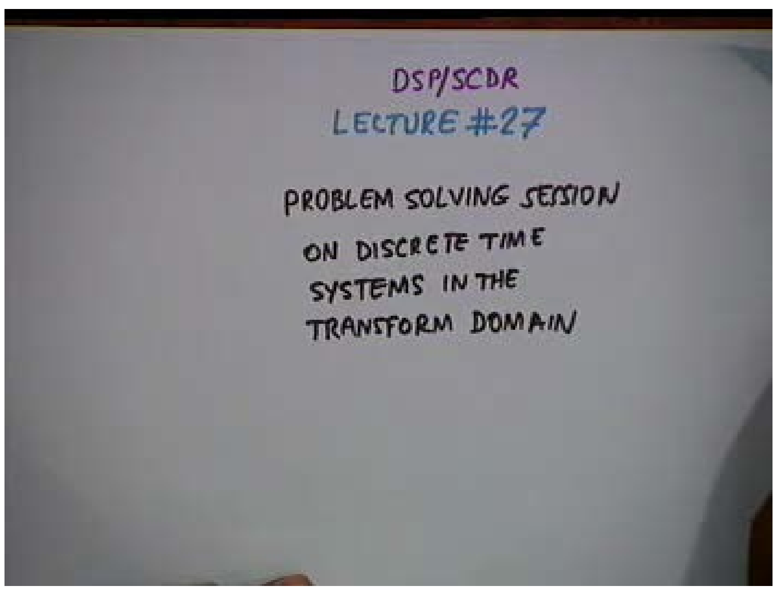

The problems I have chosen were suggested by one of the students and each of them has a small amount of twist which requires a little bit of in-depth thinking. The first problem that I take is 4.14.

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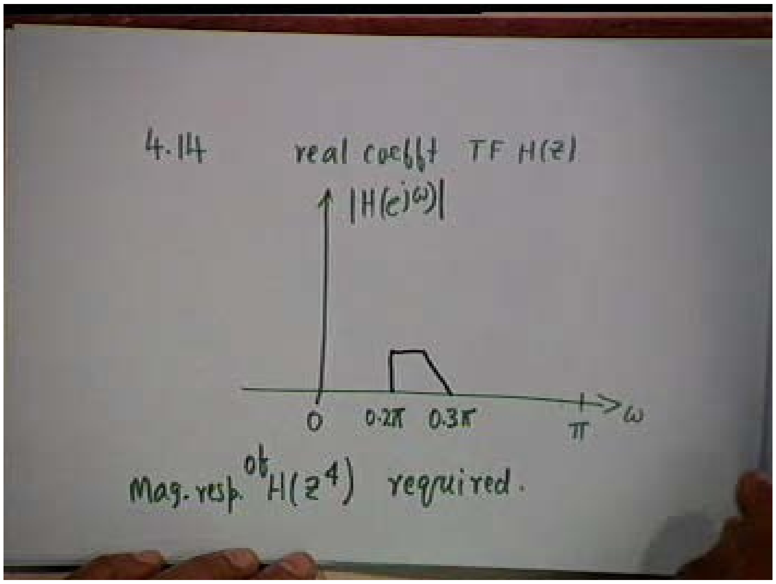

This is relatively simple. The problem states that the magnitude response of a digital filter, with real coefficient transfer function H(z), is as shown in this figure. The response is not to scale, but it does not matter. Plot the magnitude response of $H(z^4)$. z^4 means you will have $e^{j4\omega}$ on the unit circle and therefore π shall be reduced to $\pi/4$ and each point on ω axis shall be divided by 4; there shall be four such repetitions between 0 and π. For drawing the repetitions, you have to be careful. First, you draw the spectrum of $|H(e^{j\omega})|$ for $0 \leq \omega \leq 4\pi$. Then what happens in this range shall give you the picture of what happens between 0 and π for $|H(e^{j4\omega})|$. And between π and 2π the same thing will be repeated but in a flipped fashion and therefore the final diagram shall be like the one shown in the figure.

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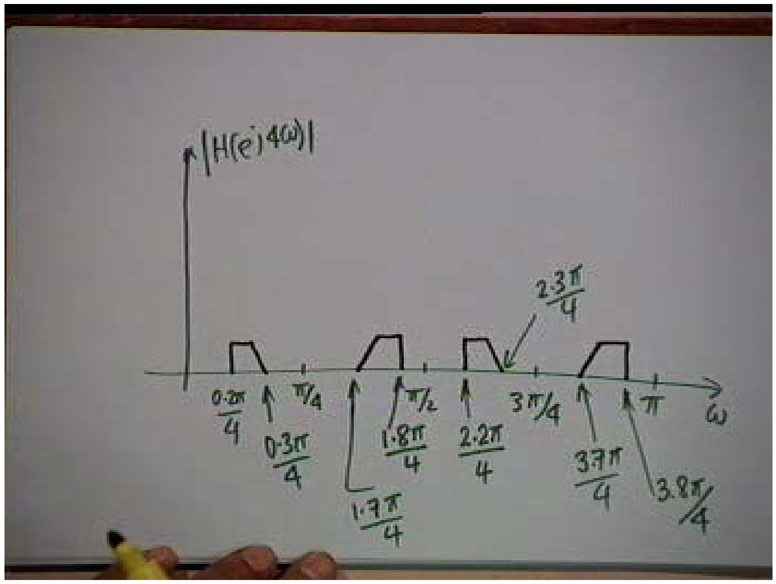

Obviously this is a case of interpolation. That is, three zero valued samples are put in between two successive samples and this is what the spectrum becomes. And if you want to retain only one part of this spectrum then we have use a low pass filter and we shall cut out all others; otherwise these shall arise problems. An interpolator is always succeeded by a low pass filter; otherwise aliasing shall occur.

The next problem is 4.42.

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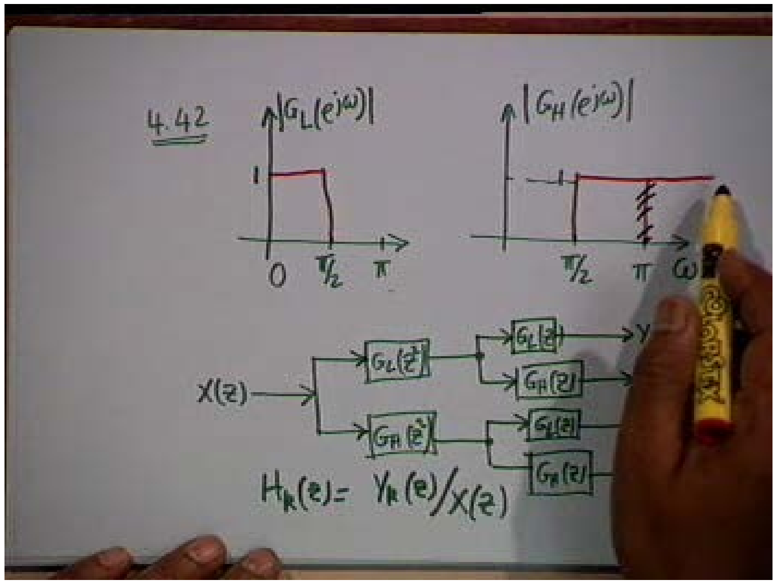

The next problem is also not difficult. Two magnitude responses are given: $|G_l e(^{j\omega})|$ and $|G_h(e^{j\omega})|$ versus ω in the range 0 to π. One is a low pass filter and the other is a high pass filter. The passband/stopband edge is π/2. They are ideal filters. And then you are given a block diagram in which X(z) feeds into two filters $G_l(z^2)$ and $G_h(z^2)$. It is a continuation of the previous problem. Here there shall be two repetitions. Then each channel breaks up into two channels $G_l(z)$ and $G_h(z)$. There are four outputs $Y_k(z)$, $k = 0 \rightarrow 3$.

The problem is to find out first the transfer functions $H_k(z) = Y_k(z)/X(z)$ and then plot their magnitude responses. Note that the LPF response is flat between 3π/2 and 2π and the HPF response is flat from π/2 to 3π/2. Why should ω continue beyond π? We normally do not have to do this but $G(z^2)$ is an interpolator and therefore you require the sketch of G(z) between 0 to 2π. And it is only then that we are able to compress the spectrum to 0 to π. The transfer functions are very simple $H_0 = G_l(z^2) \times G_l(z)$; $H_1 = G_l(z^2) \times G_h(z)$ and so on.

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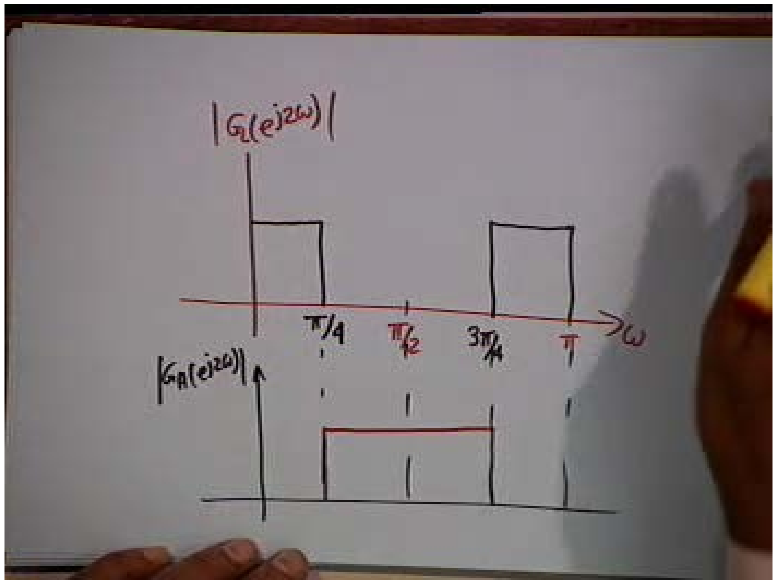

Let us do the plotting. First you have to plot $G_l(e^{j2\omega})$ and $G_h(e^{j2\omega})$ and you shall have to go upto π. Obviously, for $G_l(e^{j2\omega})$, the compression will give rise to the transfer function magnitude which is 1 from $\omega = 0$ to $\pi/4$ and then from $3\pi/4$ to π, as shown in the Figure. You do not have to go beyond this point because π is our highest normalized digital frequency. Similarly, what would be G_h? $G_h(e^{j2\omega})$ magnitude shall be 1 from $\pi/4$ to $3\pi/4$. Once this is known, then it is only the multiplication of two frequency responses, and you can find out the answers. For example, I have plotted the magnitudes of $H_0(e^{j\omega})$ and $H_1(e^{j\omega})$ in the next figure.

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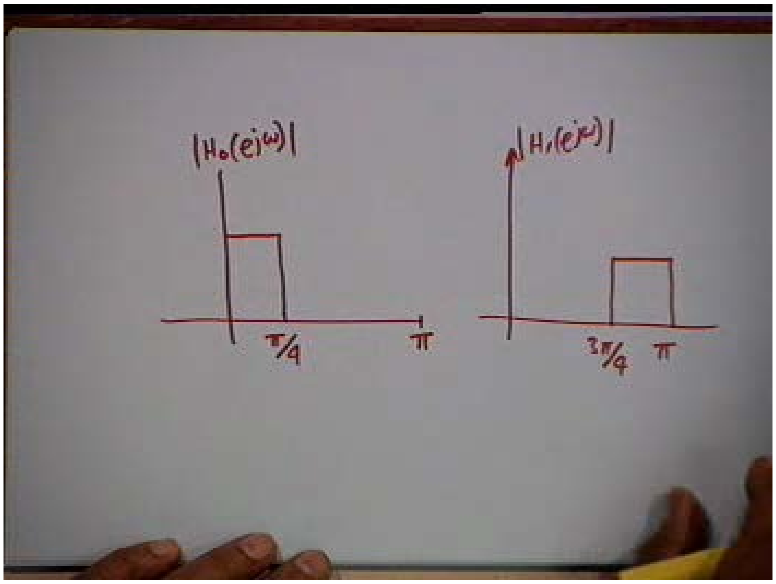

Note that H_0 is low-pass and H_1 is high-pass. You can show that H_2 and H_3 are band pass, with pass bands $\pi/4$ to $\pi/2$, and $\pi/2$ to $3\pi/4$ respectively.

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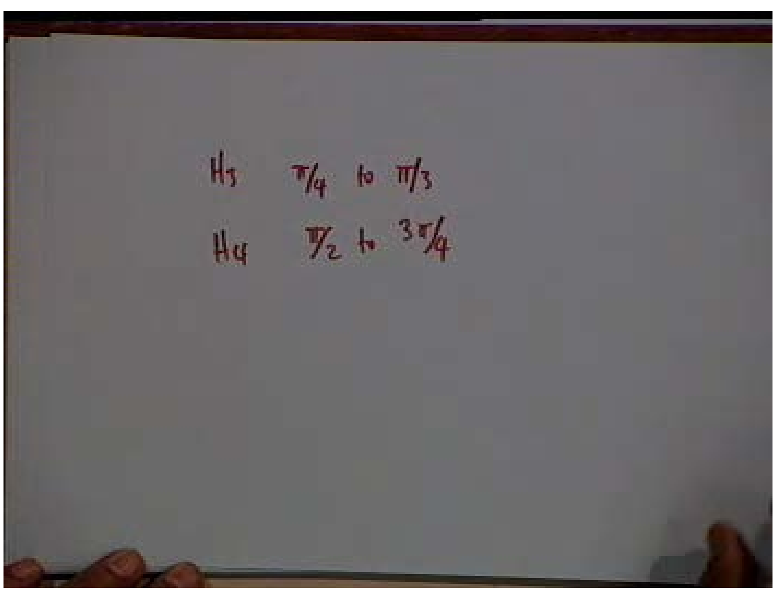

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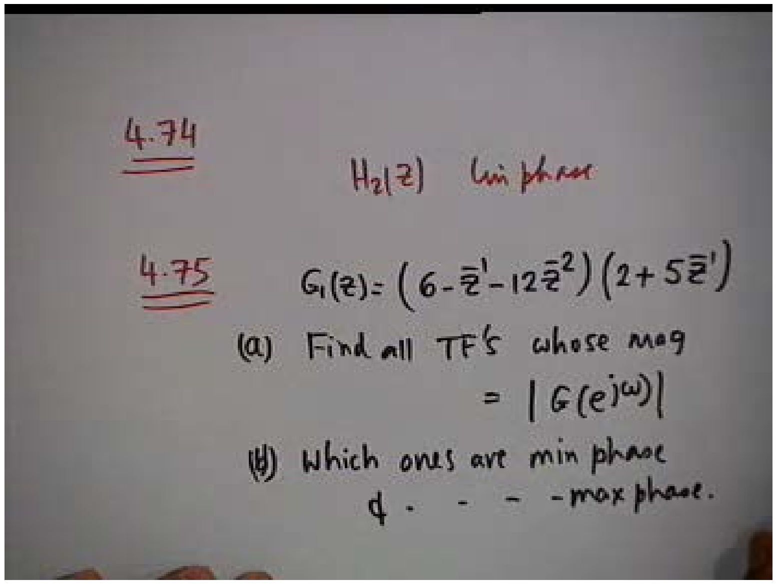

Next, we consider problem 4.74, where five transfer functions are given: H_1, H_2, H_3, H_4 and H_5. These are: $H_1 = 1 - 0.52 z^{-1} + 0.92 z^{-2} + 0.87 z^{-3} + 0.92 z^{-4} - 0.52 z^{-5} + z^{-6}$; $H_2 = 0.36 + 0.384 z^{-1} + 0.1608 z^{-2} + 0.9712 z^{-3} + 0.352 z^{-4} + 0.18 z^{-5} - 0.2 z^{-6}$; $H_3 = 0.6 + 0.652 z^{-2} + 0.6928 z^{-3} + 0.4032 z^{-4} - 0.2 z^{-5} - 0.12 z^{-6}$; $H_4 = 1 - 4.2 z^{-1} + 5.29 z^{-2} + 1.508 z^{-3} - 9.57 z^{-4} + 6.683 z^{-5}$; and $H_5 = 3.12 - 2.5 z^{-1} + 0.5 z^{-3} + 0.06 z^{-4} + z^{-5}$. You have to determine the zero locations of each. Obviously you cannot do it analytically. The order is six and you have to use a Matlab program or a root finding program to determine the roots and then answer the question: does any of them have linear phase? If it is the linear phase, then the coefficients should be symmetrical or anti-symmetrical. The first transfer function $H_1(z)$, if you notice carefully, has symmetrical coefficients and therefore it is a linear phase one. For the others, which one is minimum phase and which one is maximum phase cannot be answered without finding the roots.

In problem 4.75, the statement of the problem is: a third order FIR filter has a transfer function: $G_1(z) = (6 - z^{-1} - 12 z^{-2}) \times (2 + 5 z^{-1})$. You have to answer the following questions: Determine the transfer functions of all other FIR filters which have the same magnitude response as that of $G_1(z)$.

The second part is: which ones are minimum phase and which ones are maximum phase?
The third part relates to the impulse response and partial energy. This part is simply a matter of computation. Let us first find out all transfer functions whose magnitude is the same as that of G(z).

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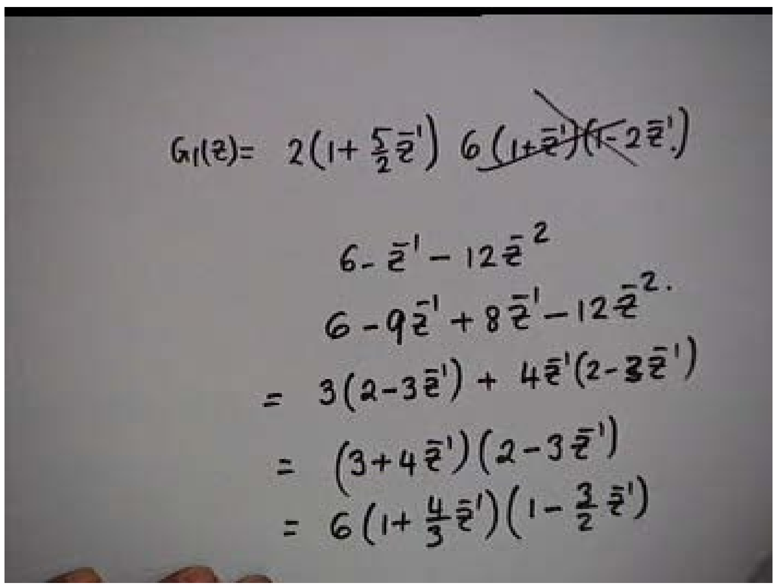

Note that since it is a third order transfer function, it has a real zero, indicated by the factor as $[1 + (5/2) z^{-1}]$. The zero is at z = – 2.5. Then the other factor is $6 - z^{-1} - 12 z^{-2}$. A little bit of thought will show that you can write this as $6[1 + (4/3) z^{-1}] [1 - (3/2) z^{-1}]$.

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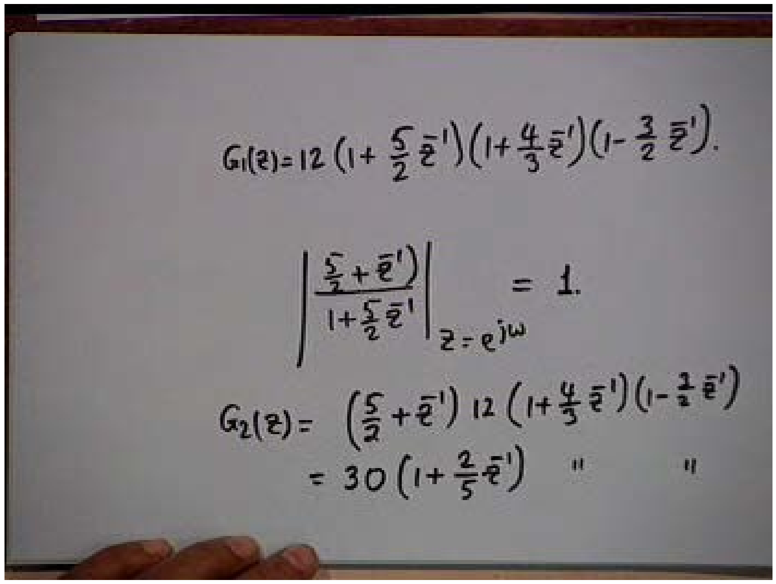

So my given transfer function is $G_1(z) = 12\,[1 + (5/2)\,z^{-1}]\,[1 + (4/3)\,z^{-1}]\,[1 - (3/2)\,z^{-1}]$. Before I find the alternative transfer functions notice that this is the maximum phase, because all zeros are outside the unit circle. What you should do now is to replace each of these factors by another factor such that the magnitude is the same. The magnitude of $[1 + (5/2)\,z^{-1}]/[(5/2) + z^{-1}]$ for $z = e^{j\omega}$ is unity. Therefore, if you write $G_2(z) = [(5/2) + z^{-1}] \times 12\,[1 + (4/3)\,z^{-1}]\,[1 - (3/2)\,z^{-1}]$, its magnitude shall be the same as that of $G_1(z)$. We have to repeat this procedure for the other two factors taken one at a time; so we have G_1, G_2, G_3, G_4. Then take two at a time so we have G_5, G_6 and finally take three at a time to get G_7. So in all, there are seven transfer functions. And the last one shall be a minimum phase. In between you shall have mixed phase transfer functions. Now, the third part. The impulse response can be easily found by taking the inverse z-transforms of the transfer function expressed as a polynomial in z^{-1}. For example, $G_1(z) = 12[1 + (7/3)\,z^{-1} - (29/12)\,z^{-2} - 5\,z^{-3}]$ so that $g_1(n) = \{12, 28, -29, -60\}$. The partial energies refer to $E_1(n) = \sum_{m=0}^{n} g_1(m)$, $0 \le n \le 3$, which can now easily be calculated. Similarly, for the other transfer functions.

If you take all the transfer functions and calculate the partial energies for all, then partial energies for the minimum phase shall be the highest of all of them. The maximum phase one shall have the minimum partial energies.

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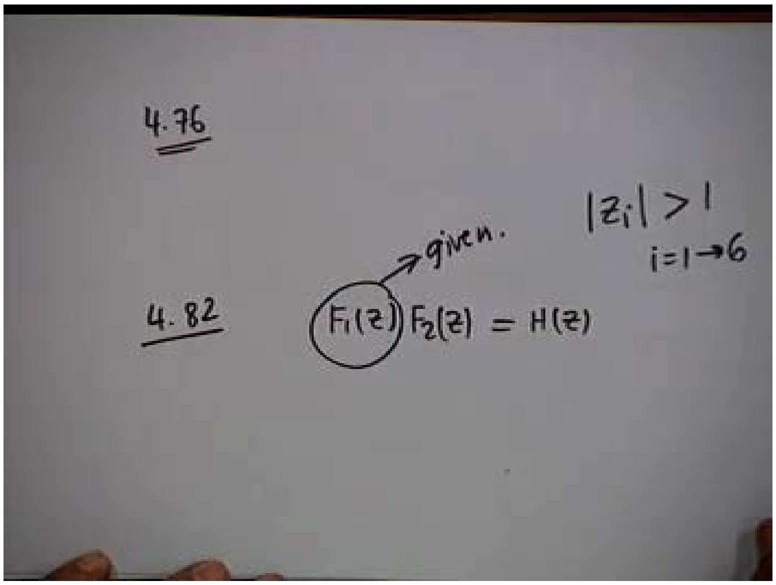

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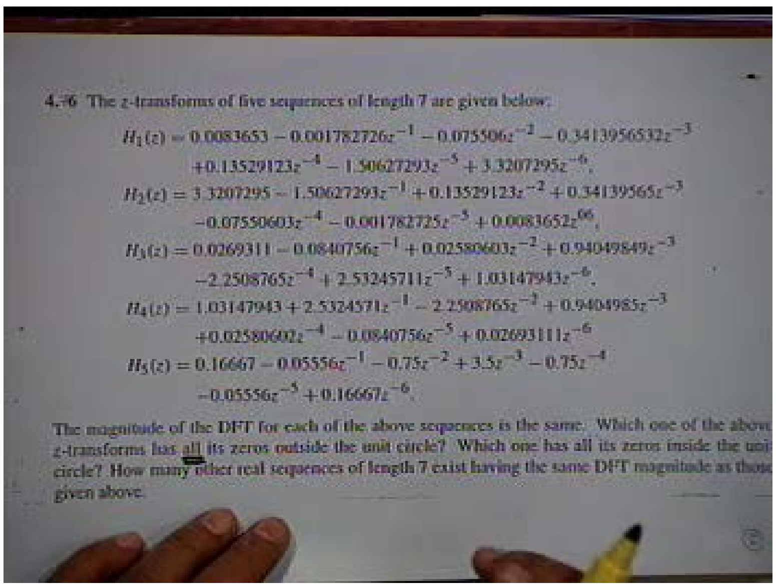

Next we take example 4.76. The z transforms of five sequences of length 7 are given in the Figure. Magnitude of the DFT for each of the above sequences is the same. Which one of the above z transforms will have all its zeros outside the unit circle? All its zeros outside the unit circle means that the product of the roots shall be greater than 1. Therefore the magnitude of the coefficient of z^{-6} > 1 and only H_1 and H_3 are qualified. If all the roots are outside the unit circle then $|z_i| > 1$, i = 1 to 6. The question is which one of the above z transforms will have all its zeros outside the unit circle? Magnitude of the coefficient of z^{-6}>1 is, however, a necessary condition, but not sufficient. We cannot answer this question without actually finding out the roots. We can only say which are the possible candidates.

Calculating partial energies may provide an answer because the maximum phase one will have minimum partial energies. The candidates with all zeros inside the unit circle will have the highest power coefficient magnitude less than 1. So H_5 goes out of the picture; H_5 is neither minimum phase nor maximum phase, it is mixed phase. Amongst the others, you cannot say definitely which one(s) is(are) minimum phase without finding the roots. Therefore the solution to this problem requires finding the roots to arrive at a definite answer. Partial energy is an indication but not a confirmatory test.

The last part of the question is: how many other real sequences of length 7 exist having the same DFT magnitude as those given above? DFT magnitude means magnitudes of X(k) and this number has also to be determined by actual calculation. It turns out to be 26 or 27. This question cannot be answered without using Matlab.

The next problem is 4.82 which states as follows. Let $F_1(z)$ denote one of the factors of a linear phase FIR transfer function H(z). So H(z) = $F_1(z)$ multiplied by some other factor $F_2(z)$. And the question is: determine at least another factor $F_2(z)$ for the following choices of F_1. It is given that a) $F_1(z) = (1 + 2z^{-1} + 3z^{-2})$; and b) $F_1(z) = 3 + 5z^{-1} – 4z^{-2}– 2z^{3}$. One of the approaches can be to find out the roots. We can find the roots for a quadratic. For a cubic we have to search for the real root and then find the roots of the quadratic. But there is a simpler approach, again based on all pass functions. In all pass functions the poles and zeros are reciprocal pairs. The magnitude of the numerator is the same as the magnitude of the denominator in an all pass function.

The second important fact is that for a linear phase function, zeros occur in reciprocal pairs, that is if z_0 is a 0 then $1/z_0$ is also a 0. Now if I want to make an all pass function out of $(1 + 2z^{-1} + 3z^{-2})$ in the numerator, then the denominator obviously shall be $3 + 2z^{-1} + z^{-2}$. Therefore my choice would be $F_2(z) = (3 + 2z^{-1} + z^{-2})$. For the b) part, we shall have $F_2(z) = -2 - 4z^{-1} + 5z^{-2} + 3z^{-3}$. Here we have used two facts: one is that the numerator of an all pass function can be written as $z^{-N} D(z^{-1})$, so that the zeros and poles are in reciprocal pairs and for a linear phase function the zeros occur in reciprocal pairs. I can of course have more terms, which are themselves linear phase.

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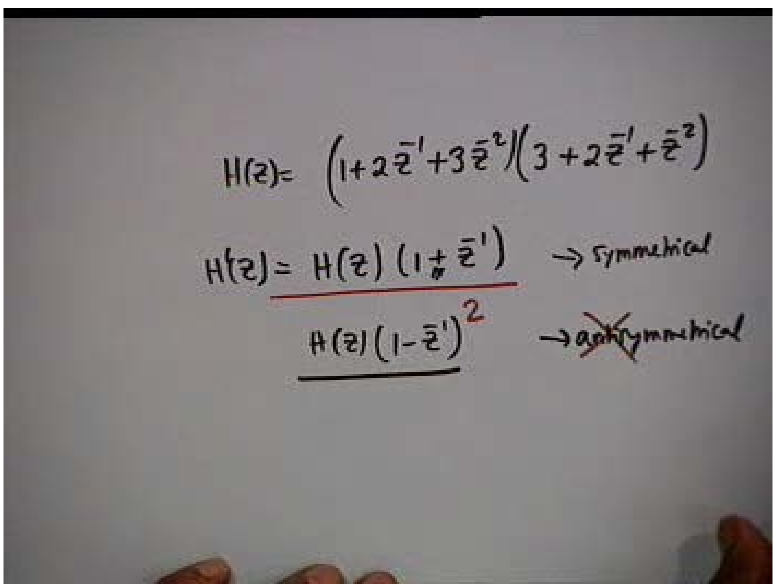

Suppose the question is to find a fifth order linear phase polynomial by using the simplest possible modification of H(z) of part a) of the previous problem. Multiply by z^{-1} and that will become a fifth order. Is it linear phase? Yes, it is. Suppose that is not permitted; what should we do? We can use $1 \pm z^{-1}$. If I use a plus sign, shall the impulse response be symmetrical or anti-symmetrical? If I use the plus sign the impulse response shall be symmetric and if I use the minus sign it becomes anti-symmetric. Suppose you take the minus sign and you want to make the transfer function symmetrical impulse response; then you have to add one or more terms. The degree is no restriction so that you get a symmetrical impulse response. How do you do that? Just square $(1 - z^{-1})$ then this antisymmetry shall go. If you square it, this becomes a case of symmetrical impulse response. So symmetrical multiplied by symmetrical shall also remain symmetrical. On the other hand, if the given transfer function is H(z) $(1 - z^{-1})$ and we want to make asymmetrical, then you raise $(1 - z^{-1})$ to the power 3. So $(1 - z^{-1})$ raised to even powers gives you symmetrical impulse response and odd powers gives rise to anti-symmetric impulse response. If on the other hand, we raise $(1 + z^{-1})$ to any power, even or odd, then does the symmetry or anti-symmetry change? No, it does not.

The next problem is 4.85 that says: show that the phase delay $\tau_p(\omega)$, which is defined as – $\phi(\omega)/\varpi$, of the first order all pass function $A_1(z) = (d_1 + z^{-1})\,(1 + d_1 z_1^{-1})$ is given approximately by $(1 - d_1)/(1 + d_1) = \delta$. The second part says: design a first order all pass filter given that $\delta = 0.5$. The sampling frequency is given as 20 kHz. Then determine the error in samples at 1 kHz in the phase delay from its design value of 0.5 samples. You have to find the error, that is deviation from 0.5 at 1 kHz. Why is the sampling frequency given? The sampling frequency is given so that you can find the normalized digital frequencies ω. Let us do the first part, then the second part would be very simple.

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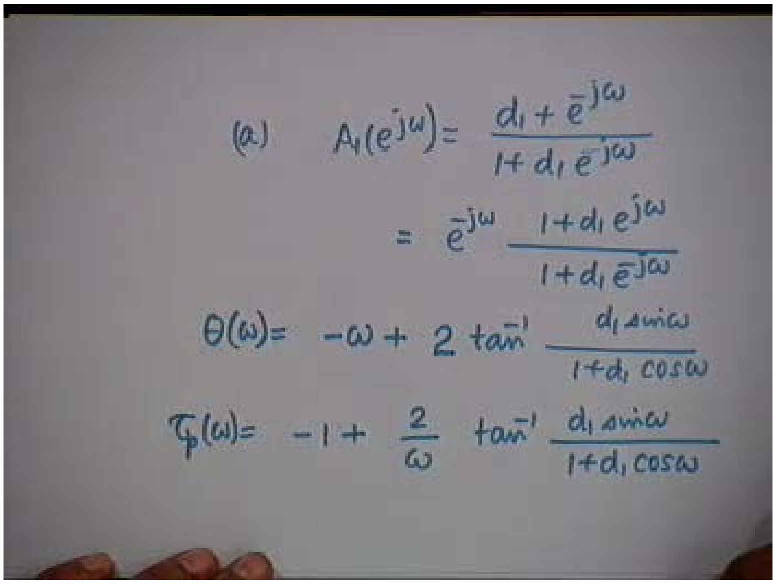

You can write A_1 $(e^{j\omega}) = (d_1 + e^{-j\omega})/(1 + d_1 e^{-j\omega})$. In finding the phase, it is convenient it is convenient to take $e^{-j\omega}$ out from the numerator. Then the phase becomes $-\omega + 2\tan^{-1}(d_1 \sin\omega)/(1 + d_1 \cos\omega)$ so that $\tau_p(\omega) = -1 + (2/\omega) \tan^{-1} (d_1 \sin\omega)/(1 + d_1 \cos\omega)$.

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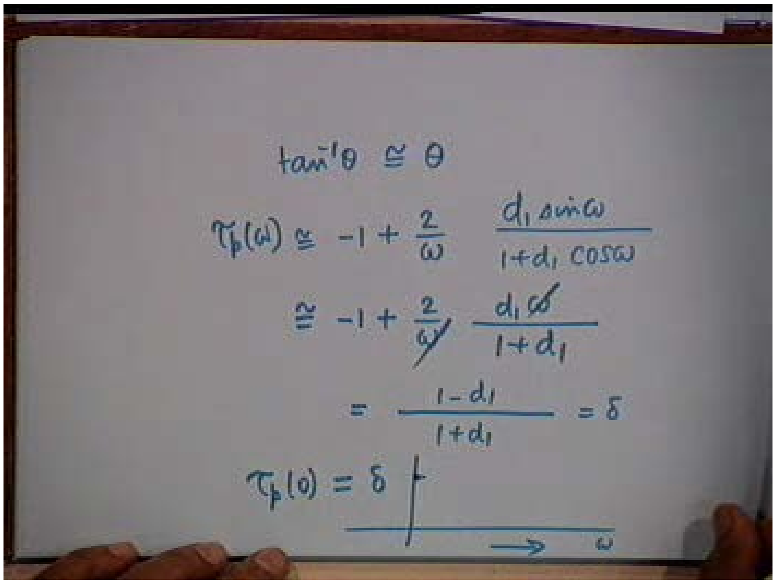

When θ is small, $\tan^{-1}\theta$ is approximately = θ, $\sin\theta$ is approximately = θ and $\cos\theta \cong 1$. Therefore $\tau_p(\omega)$ is approximately = $-1 + (2/\omega)\, d_1\omega/(1 + d_1) = (1 - d_1)/(1 + d_1)$, which has been defined as δ. I have to assume ω small to be able to convert it into a function independent of ω. In other words, this $\tau_p(\omega)$ will be exact when $\omega = 0$. When I increase ω there will be an error. I do not know whether the error will be positive or negative.

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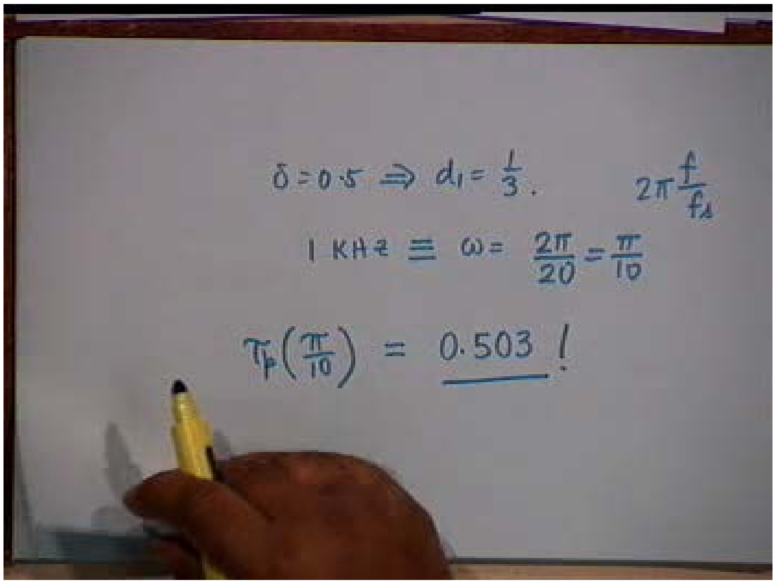

So in the second part $\delta = 0.5$ gives rise to $d_1 = 1/3$ and 1 kHz corresponds to $\omega = \pi/10$. Now you have to calculate τ_p at $\pi/10$ from the exact expression and my calculation comes out as 0.503. This is not much of a deviation from the d.c. value of 0.5.