**Digital Signal Processing**

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**Lecture - 30**

**All-pass Realizations**

This is the 30th lecture and our topic is All-pass Realizations.

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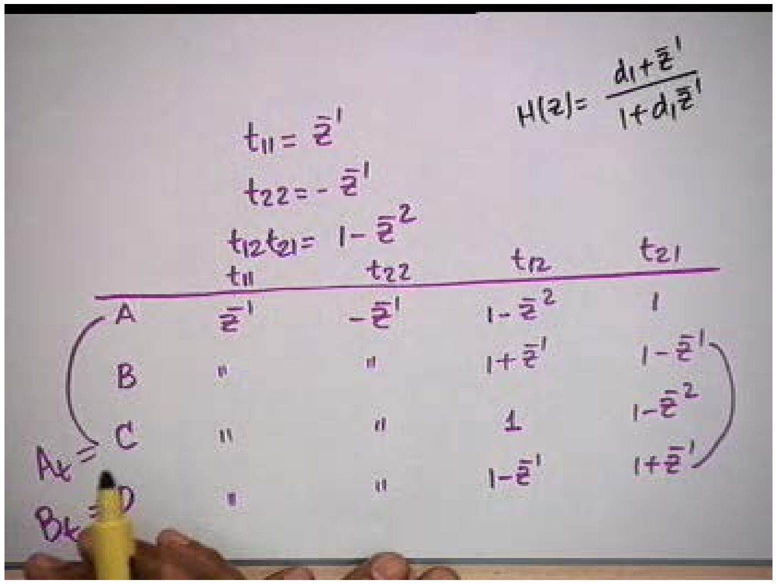


In the last lecture we ended up with the 1st order all-pass filter $H(z) = (d_1 + z^{-1})/(1 + d_1 z^{-1})$. We identified four possible sets of the transmission parameters $t_{11}$, $t_{22}$, $t_{12}$ and $t_{21}$, but there are many other choices. All these realizations are different. We also pointed out that A and C shall have a transposed relationship with respect to each other and similarly B and D are transposes of each other.

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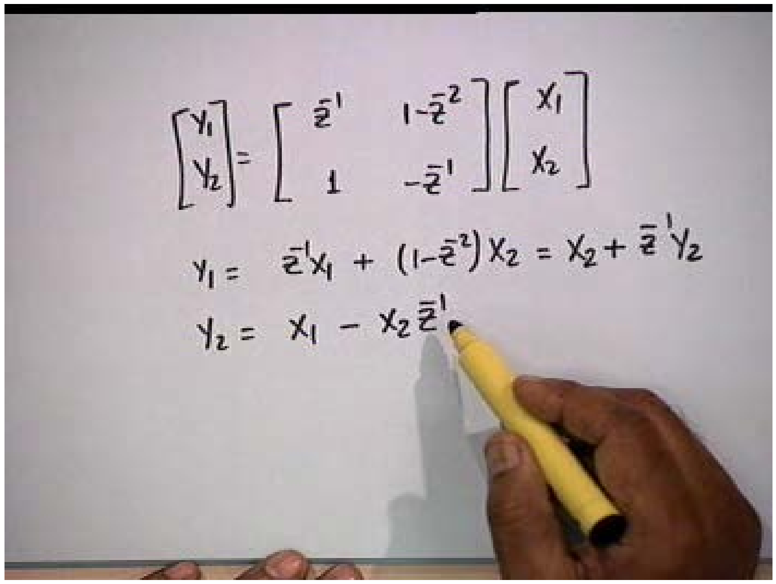


Let us take choice A and look at it closely. Choice A says that $Y_1 = z^{-1} X_1 + (1 - z^{-2}) X_2$ and $Y_2 = X_1 - X_2 z^{-1}$. These are multiplierless as we wanted. But if I realize this directly we shall require two delays whereas we want only one delay. So we have to do something to reduce it to one delay. If I multiply this $Y_2$ by $z^{-1}$, then I get $z^{-1} X_1 - z^{-2} X_2$ on the right hand side; both the terms are present in the first equation. In addition, in the first equation, we have $X_2$, so I can write this as $Y_1 = X_2 + z^{-1} Y_2$. I can realize $Y_2$ by one delay. To construct $Y_1$, I require $X_2$ and $z^{-1} Y_2$. If these two delays can be shared then the job is done. A possible realization of the two equations is shown in the figure, but here I cannot avoid two delays.

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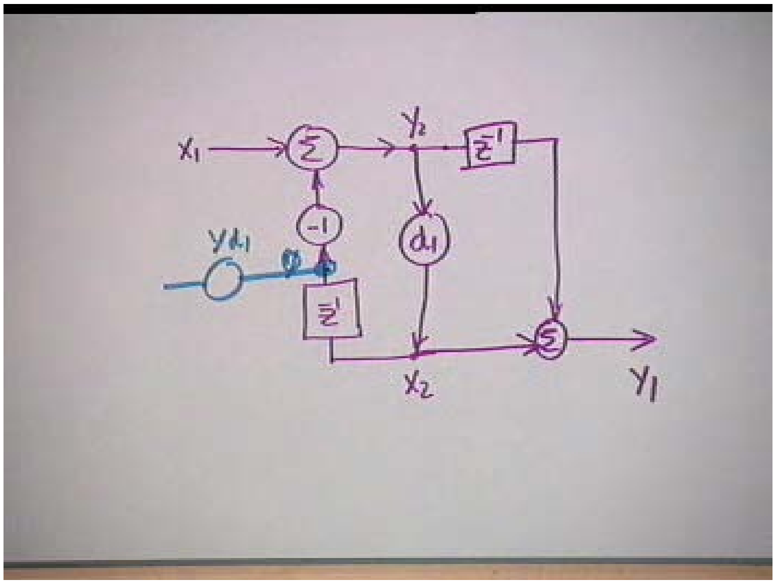


Can you find out a modification of this so as to have one delay? If you can do so, you have done an original work. Let us now consider choice B.

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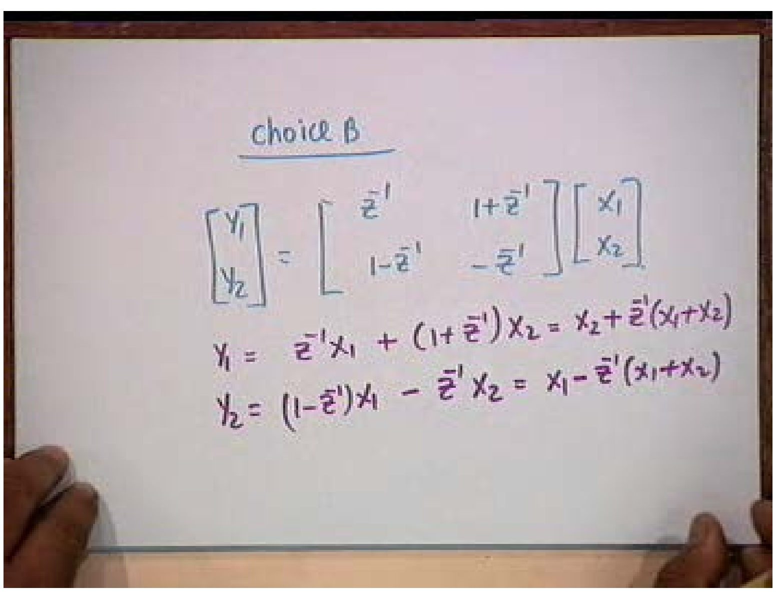

Choice A requires two delays therefore its transpose shall also require two delays. There is no point in investigating a transpose; so we investigate choice B. As in A, one multiplier has been assured by extraction. We have $Y_1 = z^{-1} X_1 + (1 + z^{-1}) X_2$ and $Y_2$ is $(1 - z^{-1}) X_1 - z^{-1} X_2$. And if you notice carefully, I can write $Y_1$ as $X_2 + z^{-1} (X_1 + X_2)$ and $Y_2$ as $X_1 - z^{-1} (X_1 + X_2)$. Now the road to a single delay has opened up. Both of these equations contain $z^{-1} (X_1 + X_2)$, which can be realized with only one delay after adding $X_1$ and $X_2$. The diagram I draw now will be slightly involved.

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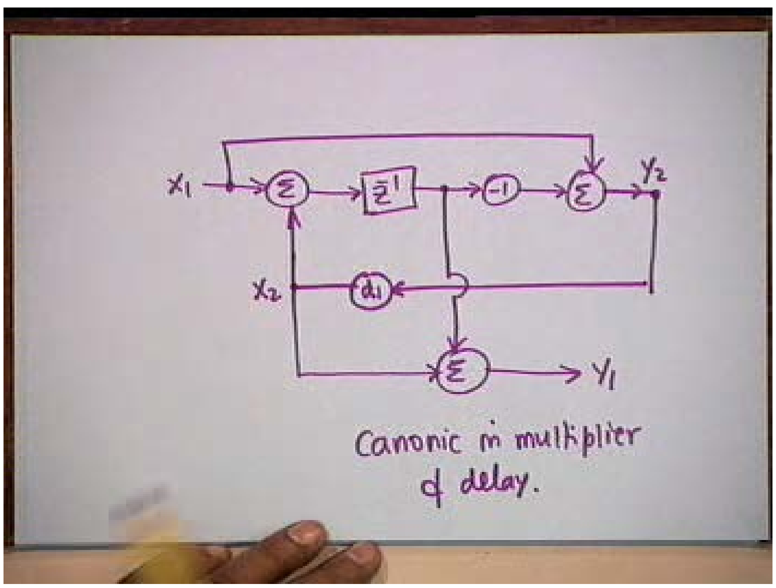


First, we add $X_1$ and $X_2$ and then we delay by one sample. So I have the signal as $z^{-1} (X_1 + X_2)$. Add this to $X_2$ to get $Y_1$. Multiply $z^{-1} (X_1 + X_2)$ by –1 and add to $X_1$. You get $Y_z$. Finally, multiply $Y_2$ by $d_1$ to get $X_2$. This is a single delay single multiplier structure, so it is canonic in multiplier as well as delay. This is a very beautiful first order structure, first derived by Mitra following his concept of digital two pair. Normally, it would not occur to one that one can draw such a structure. Choices B and D are transposes of each other which are equally applicable. The virtue of this is that even if $d_1$ changes due to various reasons, the all-pass property is maintained. The magnitude shall still remain one, the phase shall deviate, but there will be no magnitude distortion.

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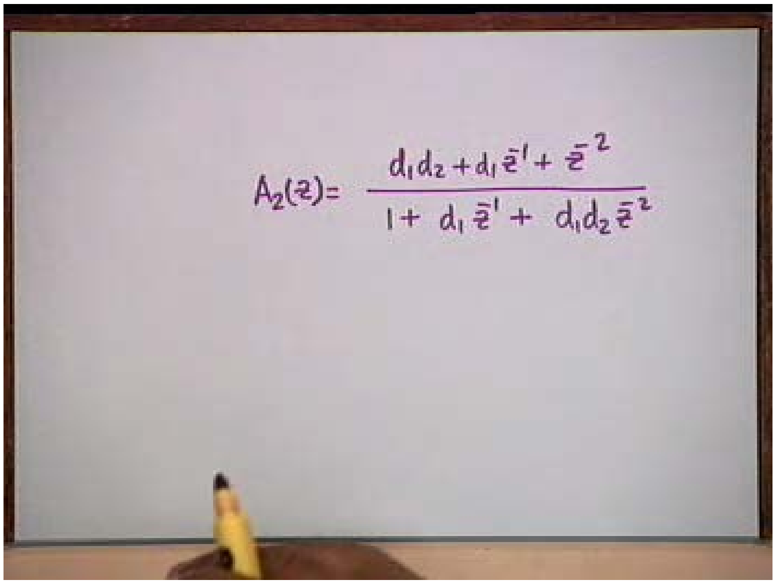


Similarly, let us consider a second order transfer function of the form $A_2(z) = (d_1\, d_2 + d_1\, z^{-1} + z^{-2})/(1 + d_1 z^{-1} + d_1 d_2 z^{-2})$. In other words, I express the coefficient of $z^{-2}$ as a multiplier of $d_1$. There are so many choices and which one shall lead to two delays and two multipliers you do not know beforehand. But if you choose $t_{11,}$ $t_{22}$, $t_{21}$ and $t_{12}$ properly, you should be able to arrive at the structure. One possible structure is shown in the figure.

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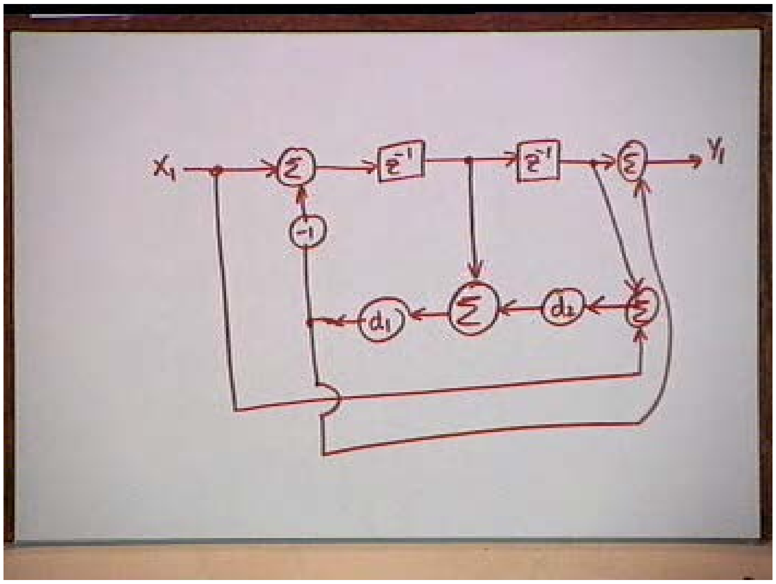


Can you call it a direct structure? Not quite, because the co efficient of $z^{-2}$ is $d_1d_2$, whereas the multiplier is $d_2$ so it is indirect but it does not matter as we are able to do with two multipliers. Now one might ask here why the usual form of the second order transfer function, i.e. $A_2(z) = (d_2 + d_1 z^{-1} + z^{-2})/(1 + d_1 z^{-1} + d_2 z^{-2})$ is not used.

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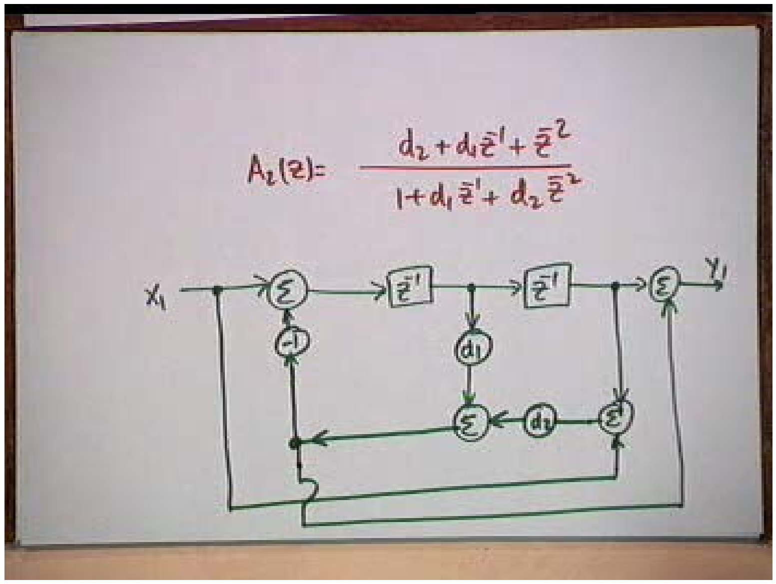


This can also be realized only with two multipliers but a slightly more involved procedure has to be followed to find out the structure. It is shown in the figure.

You can analyze and show that this structure indeed realizes $A_2$'(z). But to get the structure from the given transfer function you shall have to think quite a bit.

Now, the next topic would be the same, that is IIR all-pass realization, but by a different approach. We consider an arbitrary order all-pass function $A_M(z)$ = $(\bar{d}_M + \bar{d}_{M-1} z^{-1} + \ldots.. + z^{-M})/(1 + \bar{d}_1 z^{-1} + \bar{d}_2 z^{-2} + \ldots\ldots. \bar{d}_M z^{-M})$.

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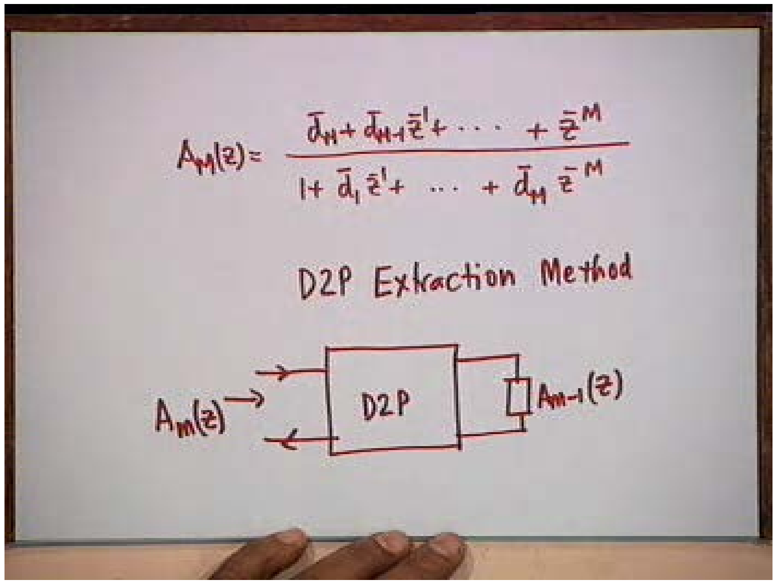


We will find the reason later for using the bar. You can realize this as a cascade of first order and second order all-pass filters but that requires factorization of the denominator. You do not have to factorize the numerator because of the special relationship of the numerator and denominator, viz. $N_M(z) = z^{-M} D_M(z^{-1})$.

Now, in digital filter realization one always wants many structures so that you can choose the one that has no overflow and the lowest quantization error. We shall now show that it is possible to realize an Mth order all-pass filter without factorization. And the approach that is adopted for this is D2P; instead of extracting the multipliers as we did in the first order and second order we extract digital two pairs. In other words, what we do is take a digital two pair and terminate in $A_{M-1}(z)$ to get a transfer function $A_M(z)$.

We start with $A_M(z)$ and then we construct the digital two pair which is terminated in one lower order transfer function. Then I shall repeat the process, i.e. $A_{M-1}(z)$ is realized as a digital two pair terminated in $A_{M-2}(z)$ and so on. It is an iteritative process. This will not require factorization and will give an alternative structure. Now we have to show that at every stage what we get is all-pass.

In IIR stability analysis and testing, we have discussed a procedure using all-pass functions. We apply that procedure to get an alternative realization of the IIR all-pass filter of any order. It turns out that the structure you get is quite well known in statistical signal processing and that structure is a lattice. And for some reason lattice has been found very suitable for VLSI implementation. It was discovered that the same structure can be derived from digital two pair extraction approach and this approach is simpler than the one used in literature on statistical signal processing. For this, we recall the stability testing procedure that we have adopted for an all-pass filter.

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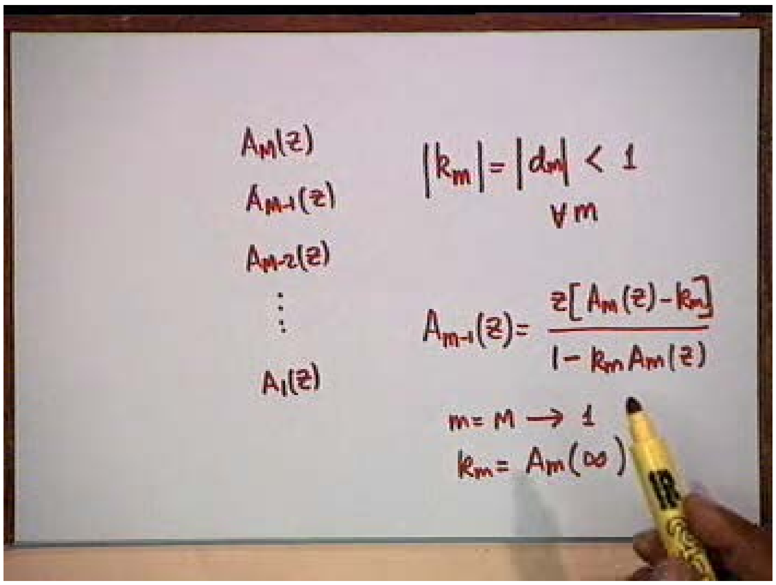


What we did was to start from $A_M(z)$ and derive a set of filters $A_{M-1}(z)$, $A_{M-2}(z)$, ……. upto $A_1(z)$. What is $A_0(z)$? It is 1. We tested the factor $k_m$ which is the same as the coefficient of $z^{-m}$ in the denominator ($k_m$ is the same as $d_m$) of $A_m(z)$, m = M, M –1,………1. The transfer function is stable if $k_m^2$ is less than 1 for all m. How did we derive the lower order from the higher order? The relationship was $A_{m-1}(z) = z\,(A_m(z) - k_m)/(1 - k_m A_m(z))$, where small m goes from M to 1. Note that $k_m = d_m$, which is the same as $A_m(\infty)$. And in addition this relationship gives a relationship between the coefficients of $A_m(z)$ and $A_{m-1}(z)$.

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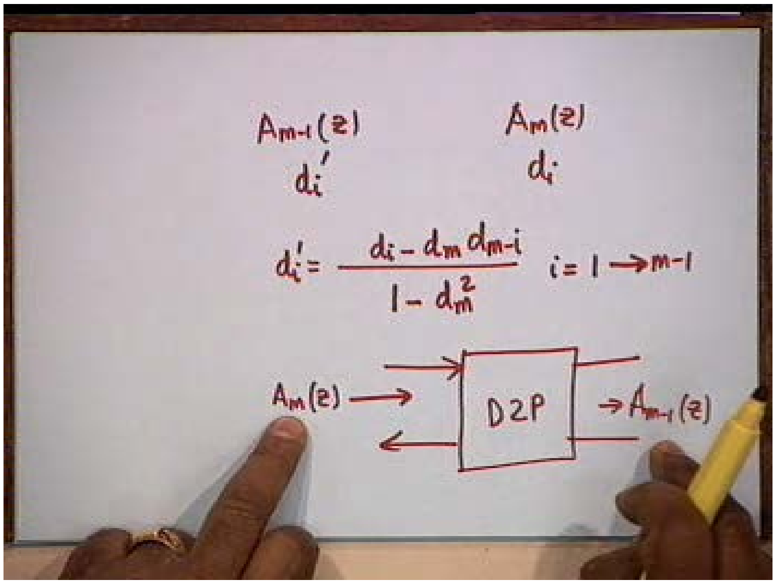


This relationship is $d_i' = (d_i - d_m\, d_{m-i})/(1 - d_m^2)$, to m – 1. This was the stability testing procedure. We shall use this now. Now our aim is to construct $A_M(z)$ as the transfer function of a digital two pair terminated in $A_{M-1}(z)$ which I have not shown as a block, but this block is realized by another digital two pair and another digital two pair and so on. An important point to note is the last all-pass function. The output $Y_2$ should be connected to $X_2$ through a transfer function of $A_0(z) = 1$, that means they are directly connected. So this cascade would be terminated in a straight connection and this point should be obvious. In order to construct this digital two pair in going from mth to (m – 1)th stage I require identifying its transmission parameters. To do so, invert the relationship between $A_m$ and $A_{m-1}$. We have $A_m(z) = (k_m + z^{-1} A_{m-1}(z))/(1 + k_m z^{-1} A_{m-1}(z))$.

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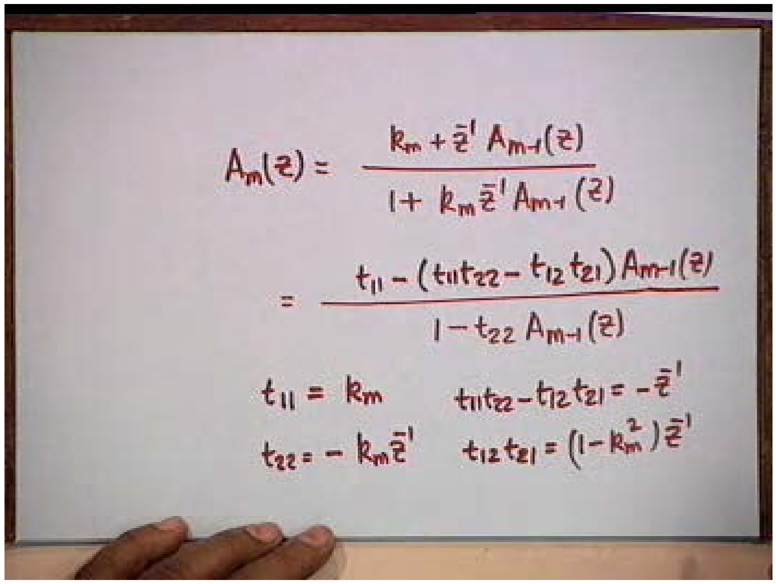


Now if the mth digital two pair has the transmission coefficients $t_{11}$, $t_{22}$ etc then obviously $1 + k_m z^{-1} A_{m-1}(z)$ should be the same as $1 - t_{22} A_{m-1}(z)$. Also, $t_{11} - (t_{11} t_2 - t_{12} t_{21}) A_{m-1}(z)$ should be the same as $k_m + z^{-1} A_{m-1}(z)$. So I compare these two and identify the transmission parameters. Obviously $t_{11}$ should be equal to $k_m$, $t_{22}$ should be equal to $-k_m z^{-1}$ and $t_{11} t_{22} - t_{12} t_{21}$ should be equal to $-z^{-1}$. That is, $t_{12} t_{21}$ should be equal to $(1 - k_m^2) z^{-1}$. So I have got $t_{11}$, $t_{22}$ and the product $t_{21} t_{12}$ and now I can play with these individual factors. Let us list some choices for $t_{12}$ and $t_{21}$ exactly like we did in the first order all-pass case and then investigate some of them.

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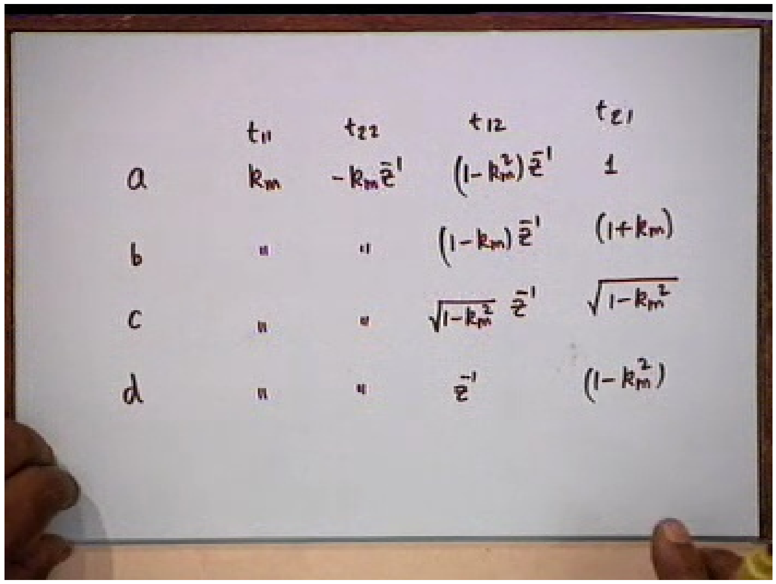


Let us make a table of these choices and denote them by a, b, c and d. So $t_{11}$ and $t_{22}$ are fixed for each of them, $k_m$ and $-k_m z^{-1}$ respectively. Here we will not use transposition. If we can find one, we can find the transposed one also; so let us list unrelated choices. For a, we choose $t_{12} = (1 - k_m) z^{-1}$, then the other one should be simply 1. I can also choose $(1 - k_m) z^{-1}$ for $t_{12}$; then $t_{21} = 1 + k_m$. I can choose $t_{12}$ as $\sqrt{(1 - k_m^2)}\, z^{-1}$; then $t_{21}$ would be $\sqrt{(1 - k_m^2)}$. I can choose $t_{12}$ as $z^{-1}$; then $t_{21}$ shall be $1 - k_m^2$. I can go on doing this. These four choices contain the structures of our interest and therefore we shall explore them. Our aim would be to realize the set ($t_{11}$, $t_{22}$, $t_{12}$, $t_{21}$) in as simple a manner as possible, aiming at a canonic structure. What does it mean? It means we should use one multiplier and one delay only. Consider choice a.

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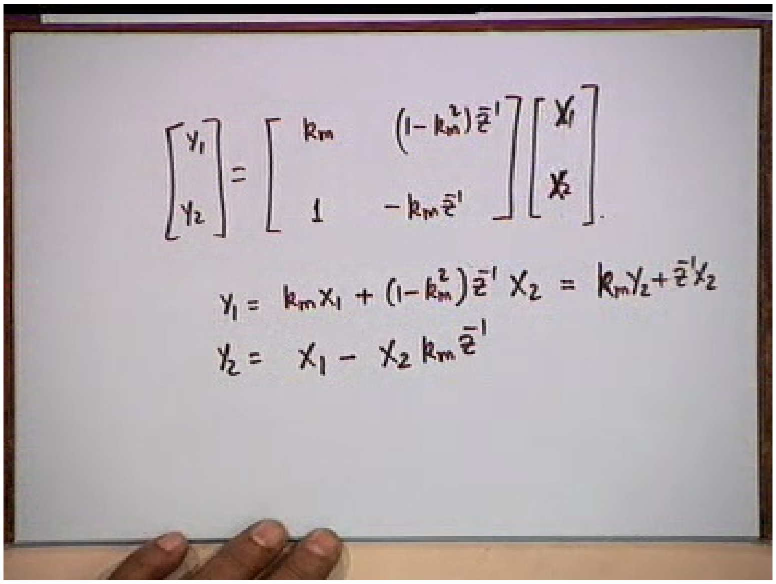


In choice a, we proceed as follows: $Y_1 = k_m X_1 + (1 - k_m^2) z^{-1} X_2$ and $Y_2 = X_1 - X_2 k_m z^{-1}$. If you look at $Y_1$ and $Y_2$, you see that $Y_1$ can be written as $k_m Y_2 + z^{-1} X_2$. Let us look at these relationships and try to construct them: $Y_1 = k_m Y_2 + z^{-1} X_2$ and $Y_2 = X_1 - X_2 k_m z^{-1}$.

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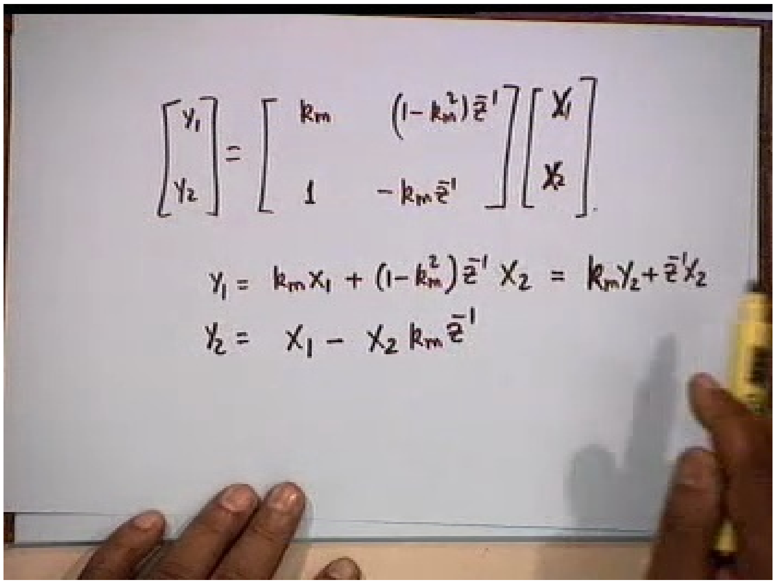


Notice that in both the equations, we require $z^{-1} X_2$, so we delay $X_2$ by one sample, as shown in the figure, with $X_1$, $Y_1$ on the left side and $Y_2$, $X_2$ on the right, as in the convention is D2P.

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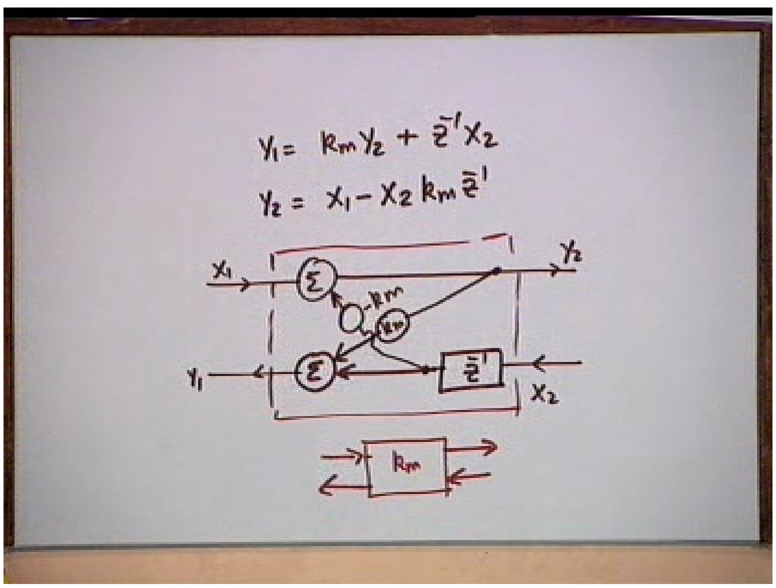

To construct $Y_1$, we require the inputs $z^{-1} X_2$ and $k_m Y_2$. To construct $Y_2$, we add $X_1$ to $-k_m X_2 z^{-1}$. But we have not achieved our purpose, we have used two multipliers $k_m$ and $-k_m$. And it suffers from the same problem of word length or quantization error. The total structure is simply the cascade of such structures ultimately terminated in a straight connection. This block here is simply represented as $k_m$ as shown in the figure.

Now we shall show that it is possible to derive a single multiplier structure. However, the structure just derived has been favored over the single multiple structure, and this is called a lattice. One of the reasons can be that if it is a software program to realize the all-pass structure (the difference between software and hardware is not relevant if you use a host PC to interface with your processor), it does not make a difference. However, in special circumstances, like that of tunable filters, we should not have more than one variable multiplier in a first order filter and we should not have more than two variable multipliers in a second order filter.

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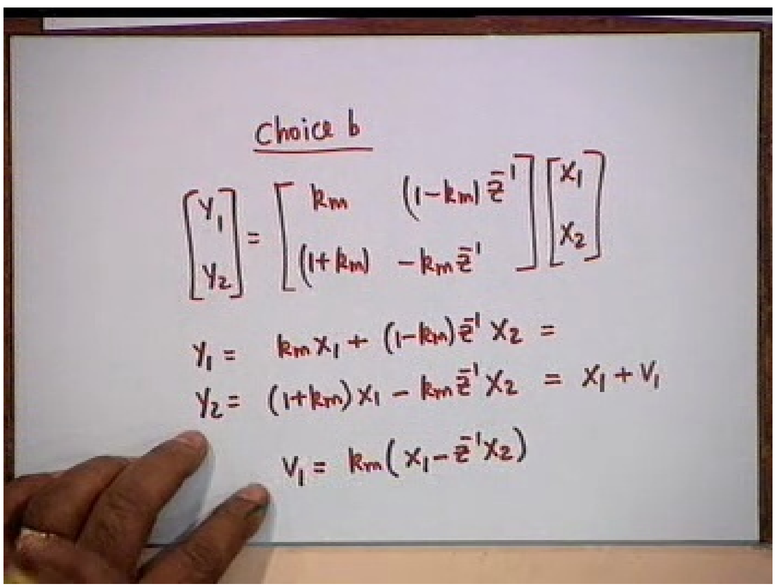


To get a single multiplier structure, we consider choice b. In choice b, $Y_1 = k_m X_1 + (1 - k_m) z^{-1} X_2$ and $Y_2 = (1 + k_m) X_1 - k_m z^{-1} X_2$. If I define a variable $V_1$ as $k_m(X_1 - z^{-1} X_2)$, then I can write both

$Y_1$ and $Y_2$ in terms of $V_1$ and $X_1$ or $X_2$. For example, $Y_2$ is simply $X_1 + V_1$ and no multiplier is required. But a multiplier $k_m$ is needed to construct $V_1$. Also, $Y_1$ is $V_1 + z^{-1} X_2$, so no multiplier is needed here. One multiplier has done the job. Now let us look at the realization.

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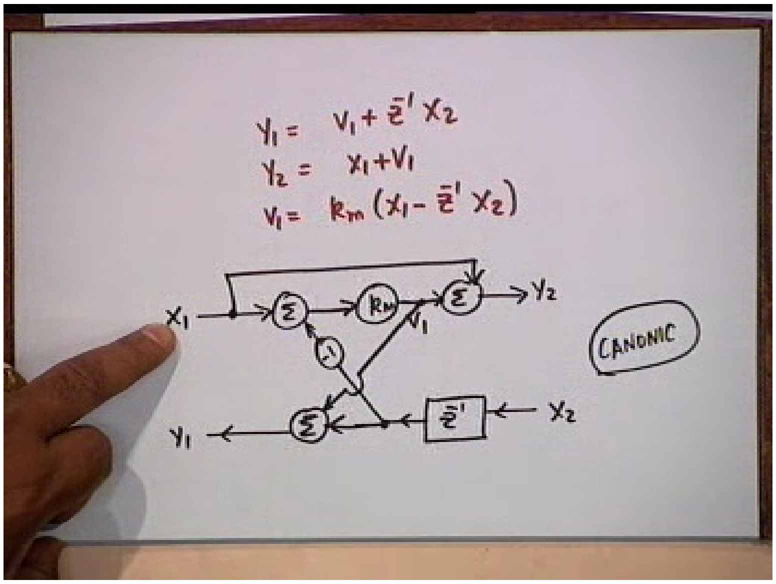


We have $Y_1 = V + z^{-1} X_2$ and $Y_2 = X_1 + V_1 = k_m(X_1 - z^{-1} X_2)$. To draw the realization we first draw the $X_1$ $Y_1$ lines and $Y_2$ $X_2$ lines. Our digital two pair is between the first two terminals. We require $z^{-1} X_2$ so we use a delay $z^{-1}$ with $X_2$ as input. And then we need to construct $V_{1.}$ So I add $X_1$ to $-z^{-1} X_2$ and I multiply the adder output by $k_m$ to get $V_1$. Once I have constructed $V_1$, then the construction of $Y_1$ and $Y_2$ easy, as shown in the figure. It is a single multiplier and single delay structure and is indeed canonic.

One reason I guess why this structure is not popular in comparison to the two multiplier structure is that there is an additional feedforward line, but notice that there is no delay free loop. A second order all-pass would have required two such sections, while a third order all-pass function can be realized by three such lattice sections, as shown in the next figure.

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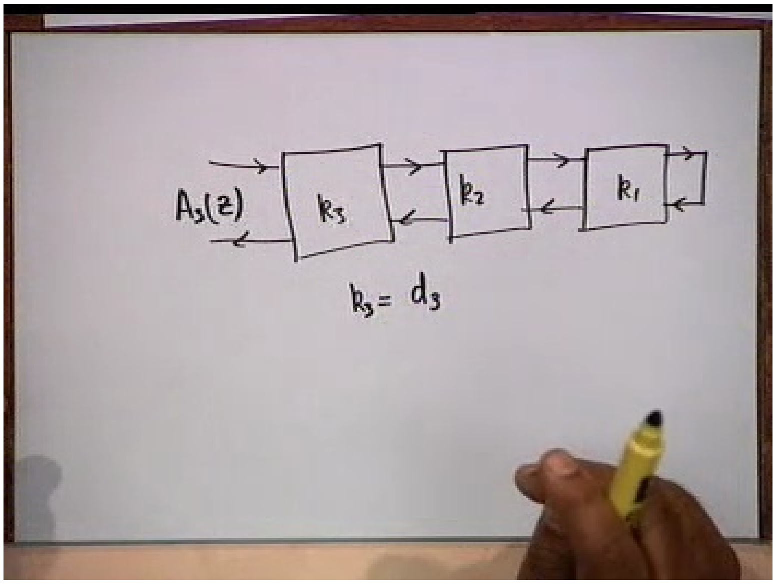


First section will be $k_3$, then $k_2$, then $k_1$, and termination by a direct connection. Obviously $k_{3,}$ $k_2$ and $k_1$ have to be derived by using the relationship $d_i' = (d_i - d_m\, d_{m-i})/(1 - d_m^2)$. $k_3$ is the same as $d_3$; so you have to derive $k_2$ and $k_1$. And if in the process, magnitude of one of these exceeds or is equal to unity, then you need not proceed further because the structure is unstable and you cannot realize the transfer function.