Digital Signal Processing

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FIR Lattice Synthesis

Lecture - 32

This is the 32nd lecture and our topic for today is FIR Lattice Synthesis. In the last lecture, we took an example of IIR all pass realization by lattice. We talked about tunable filters, low pass-high pass combination and band pass-band stop combination. We also looked into the realization of an arbitrary IIR filter transfer function by tapped lattice. That is, the lattice realizes the all pass constructed with the same denominator, tap some signals, weigh them and then add them together and get the desired transfer function. We showed that this is possible in all cases and then we concluded with an example. We also started our discussion on FIR lattice. The procedure for FIR lattice is synthesis by analysis which is the traditional approach. One can also think of a synthetic approach, but the procedure seems to be quite complicated, involving matrix decomposition.

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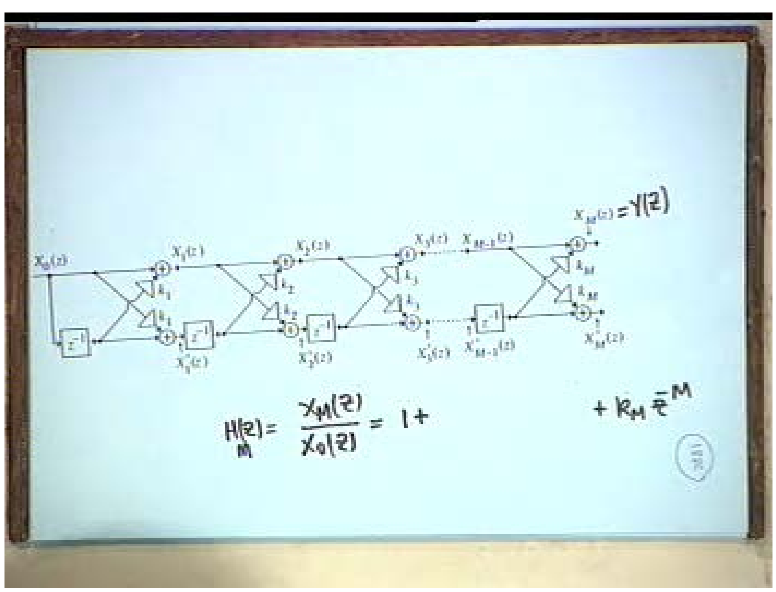

Our structure is shown in the figure (reproduced from Mitra), where instead of Σ for summation Mitra uses the plus sign. My notation for a multiplier is a circle, but Mitra uses a triangle. I require this total diagram to be able to illustrate some points. First, you notice that in this diagram the total number of delays is M, the degree of the transfer function. Since there is no feedback, it should be a FIR transfer function; it is all feedforward. The constant term in the transfer function is equal to 1. This is reflected in the straight path from the input to the output. The desired output Y(z) appears at the end point of the upper line. The signal at the end point of the lower line is not important at this point but you may be able to utilize it later. You notice that if fabricated on a chip, this will also be a multifunction chip. Not only do we get the FIR function here but we also get other FIR functions at the various nodes.

And finally H(z), the transfer function, is $X_M(z)/X_0(z)$. The signal with the highest amount of delay goes totally in the lower path and finally comes through k_m multiplier to the output. So the coefficient of z^{-M} must be k_m. In between we have other impulse response coefficients which can also be calculated from the diagram. For example, if you want the coefficient of z^{-1}, it would be simply $k_1 + k_1 k_2$. If I want the coefficient of z^{-2}, then you have to find all signals which appears at Y(z) after two delays. So far you can do this, but it is not helpful for synthesis. In the synthesis by this procedure, you shall have to solve a set of nonlinear equations which is very cumbersome. So we follow an alternative procedure. And to start with, let our desired transfer function be $H_M(z) = 1+ \Sigma\, h_n z^{-n}$, n = 1 to M.

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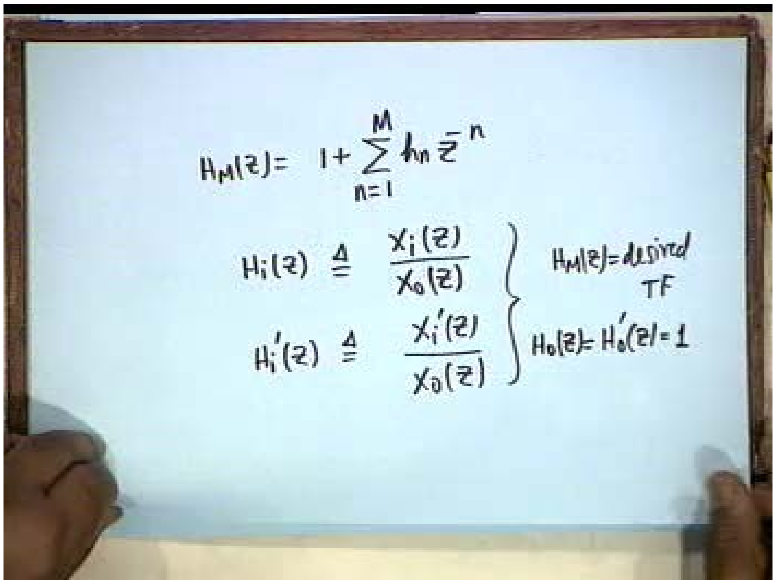

We also adopt the notation $H_i(z) = X_i(z)/X_0(z)$, that is $H_i(z)$ is the transfer function from the input to X_1, X_2 etc so that $H_M(z)$ is the desired transfer function, but in between you get a lot of transfer functions. Precisely, you get 2M number of different transfer functions; it is a multifunction device. We also define the lower signal transfer functions with a prime. That is, we say $H_i'(z) = X_i'(z)/X_0(z)$. Obviously, $H_0(z) = H_0'(z) = 1$. Now, we look at this diagram and try to construct the first few signals; X_1, for example, is $X_0 + z^{-1} k_1 X_0$.

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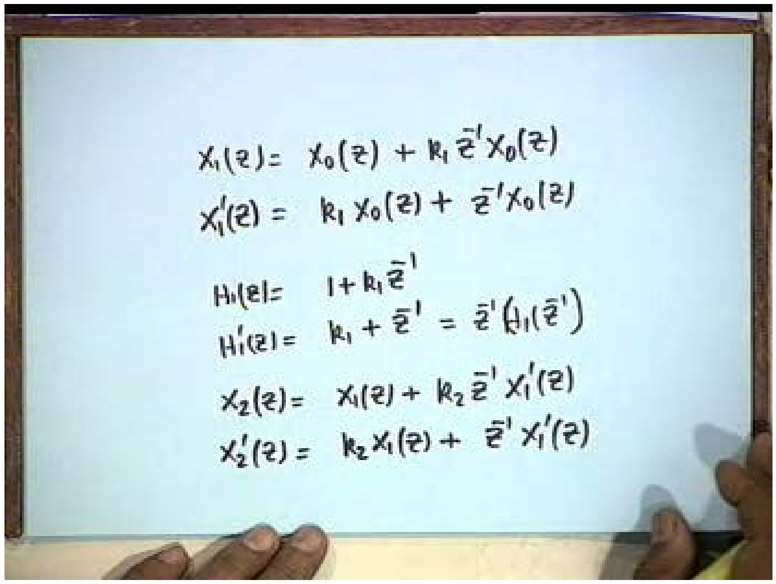

And similarly, $X_1'(z)$ from the diagram comes out as $k_1 X_0(z) + z^{-1} X_0(z)$. In other words, $H_1(z) = 1 + k_1 z^{-1}$ and $H_1'(z) = k_1 + z^{-1}$. And do you see that the ratio of the two is an all pass function? The structure indeed is very intimately related to all pass. Since I have said that the ratio of the two is an all pass function, obviously $H_1'(z)$ can be written as $z^{-1}H_1(z^{-1})$. This is the relationship between H_1' and H_1. In a similar manner, if I want to construct $X_2(z)$, it is $X_1(z) + k_2(z^{-1}) X_1'(z)$. Similarly, $X_2'(z) = k_2X_1(z) + z^{-1}X_1'(z)$. Now I can write two transfer functions $H_2(z)$ and $H_2'(z)$.

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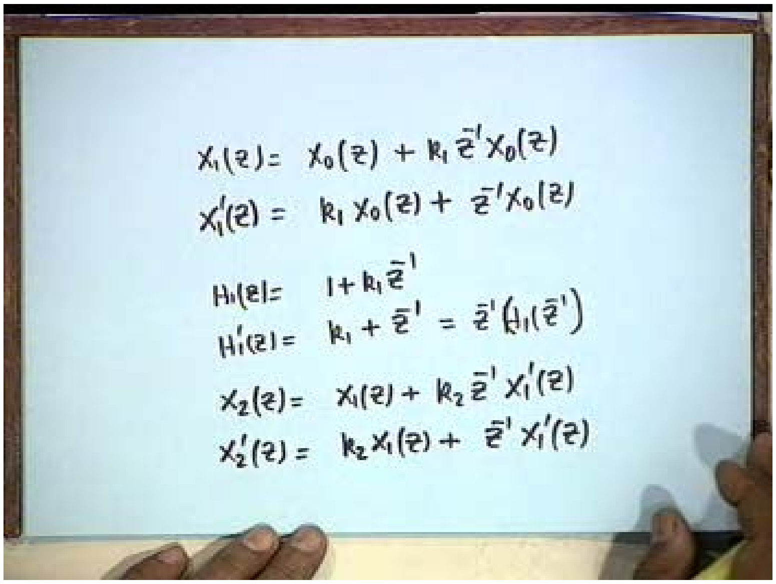

So $H_2(z) = H_1(z) + k_2 z^{-1} H_1'(z)$ and $H_2'(z)$ shall be equal to $k_2 H_1(z) + z^{-1} H_1'(z)$. Here also you see there is an all pass kind of relationship between the two. And if you combine this with the fact that $H_1'(z) = z^{-1} H_1(z^{-1})$ then you can you can get rid of the primes; you get $H_2(z)$ as $H_1(z) + k_2 z^{-2} H_1(z^{-1})$ and $H_2'(z) = k_2 H_1(z) + z^{-2} H_1(z^{-1})$. Don't you see that $H'_2(z)$ is the same as $z^{-2} H_2(z^{-1})$? And, by induction, what has been shown for i = 1 and 2 can be shown to be true for a general i, i.e. $H_i'(z) = z^{-i} H_i(z^{-1})$. Also, from the diagram, we can write $H_i(z) = H_{i-1}(z) + k_i z^{-1} H_{i-1}(z)$ and $H_i(z) = ki\ H_{i-1}(z) + z^{-1} H'_{i-1}(z)$. Combining the first equation with $H_1'(z) = z^{-i} H_i(z^{-1})$, we get $H_i(z) = H_{i-1}(z) + k_i z^{-1} H_{i-1}(z^{-1})$.

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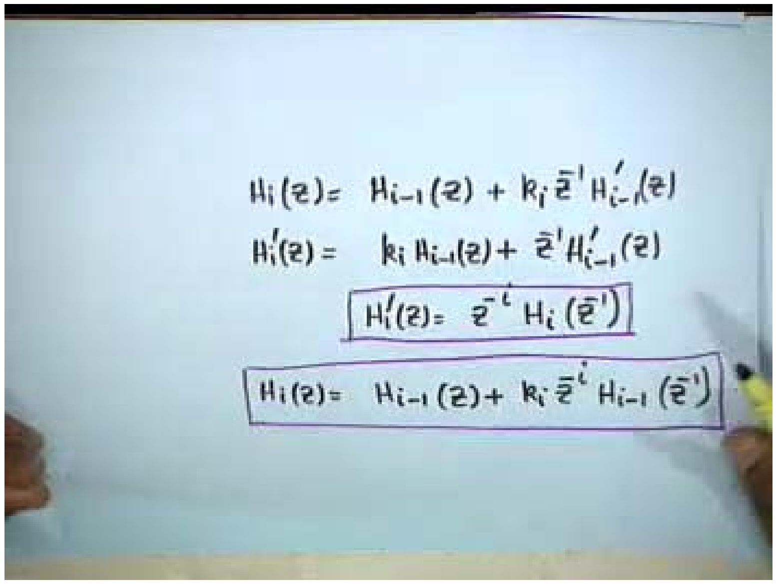

So far what we have derived is by analysis. If we know H_{i-1} then we can find out H_i but that is not what we want to do. What we want to do is given H_M, we want to find out k_M and that is the synthesis problem. So we should proceed in the reverse fashion that is starting from H_M, we should be able to derive H_{M-1}, H_{M-2}, etc. If you find $H_{M-1,}$ then you can find k_{M-1} and so on. We should proceed in the reverse direction, we must start from H_M and then by an iterative method derive the values of k_M, k_{M-1} ,...... upto k_1 because k_0 has already been made a short circuit and X_0 and X_0' are the same. Now, to that end in view, we put i = M to be able to derive H_{M-1} in terms of H_M.

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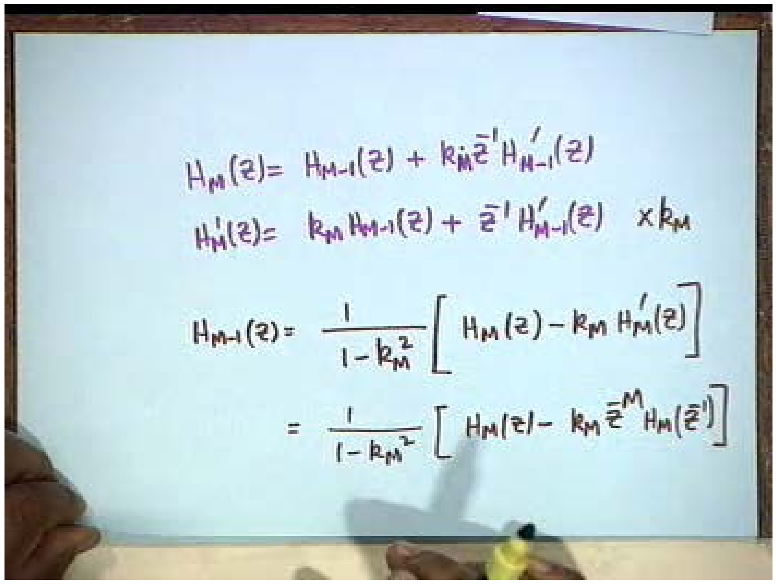

We shall get $H_M(z) = H_{M-1}(z) + k_M z^{-1} H_{M-1}'(z)$. I am using the first relation still. And then $H_M'(z) = k_M H_{M-1}(z) + z^{-1} H_{M-1}'(z)$. We multiply the second relation by k_M and then you subtract from the first relation to get $H_{M-1}(z)$ with $[1/(1 - k_M^2)]$ as a factor if $k_M = 1$ then you are nowhere. What happens if the last coefficient is ± 1? The other factor in $H_{M-1}(z)$ is $[H_M(z) - k_M H_M'(z)]$. We shall consider the case $k_M = \pm 1$ later. Returning to the present situation and assuming $k_M \neq \pm 1$, recognize that we have not yet got rid of H_M' but it is at this point that we get rid of it. That is, we write $H_{M-1}(z) = (1 - k_M^2)^{-1} [H_M(z) - k_M z^{-M} H_M(z^{-1})]$, using that the relation $H_i'(z) = z^{-i} H_i(z^{-1})$. Now I replace $H_M(z)$ by its expanded form.

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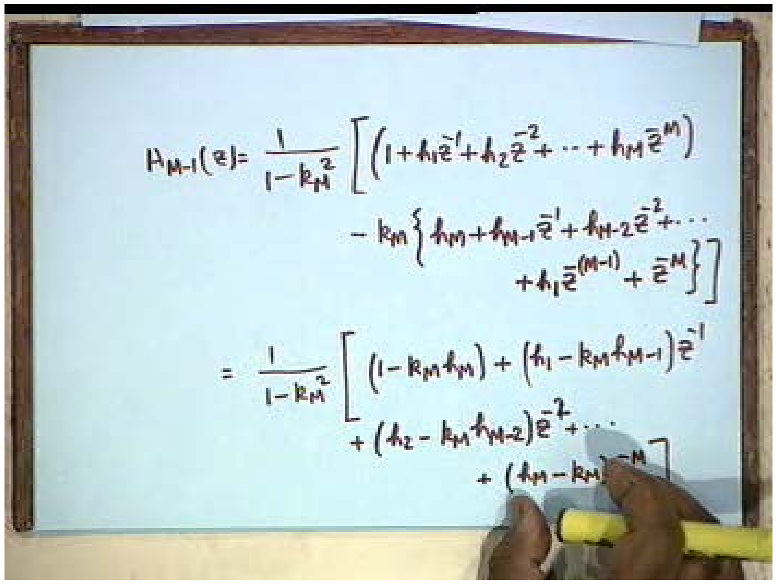

That is, I write $H_{M-1}(z) = [1/(1 - k_M^2)]\ [(1 + h_1z^{-1} + h_2z^{-2} + \ldots + h_Mz^{-M}) - k_M\ \{h_M + h_{M-1}z^{-1} + \ldots + h_1z^{-(M-1)} + z^{-M}\}]$. I collect the coefficients now to get $H_{M-1}(z) = (1 - k_M^2)^{-1}\ [(1 - k_Mh_M) + (h_1 - k_Mh_{M-1})z^{-1} + (h_2 - k_Mh_{M-2})z^{-2} + \ldots (h_M - k_M)z^{-M}]$. This shows that if H_{M-1} is to be of the order M – 1, then the last term should vanish, that is k_M should be equal to h_M. This result was also available from analysis.

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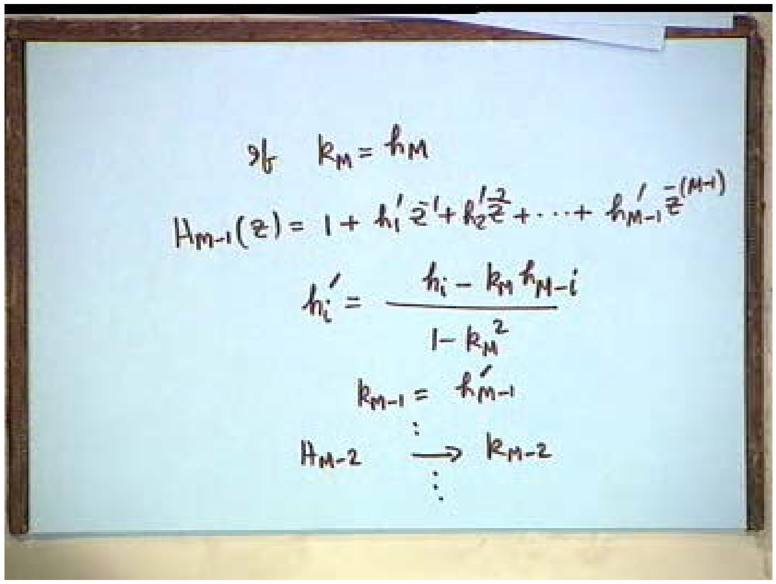

So our major conclusion now is that $k_M = h_M$ the coefficient of the highest power term. Under this condition, $k_M = h_M$, $H_{M-1}(z)$ is of order $M - 1$ and the constant term becomes 1, as is desired. So H_{M-1} is of the same form as H_M and therefore iteration can be used. Let us call the coefficients of $H_{M-1}(z)$ by primes, i.e. $H_{M-1}(z) = 1 + h_1' z^{-1} + h_2' z^{-2} \ldots + h_{M-1}' z^{-(M-1)}$. This is the form where, very interestingly, the coefficient h_i' is $(h_i - k_M h_{M-i})/1 - k_M^2)$. It is the same relationship as in IIR all pass designs. In IIR, it was an all pass function but in FIR, it is an arbitrary function, except for $k_M = 1$. And if I have obtained H_{M-1}; then $k_{M-1} = h_{M-1}'$. Next, you derive H_{M-2} from which you get k_{M-2}, and repeat till you exhaust the function and you will obtain all the lattice coefficients. Now let us look at an example.

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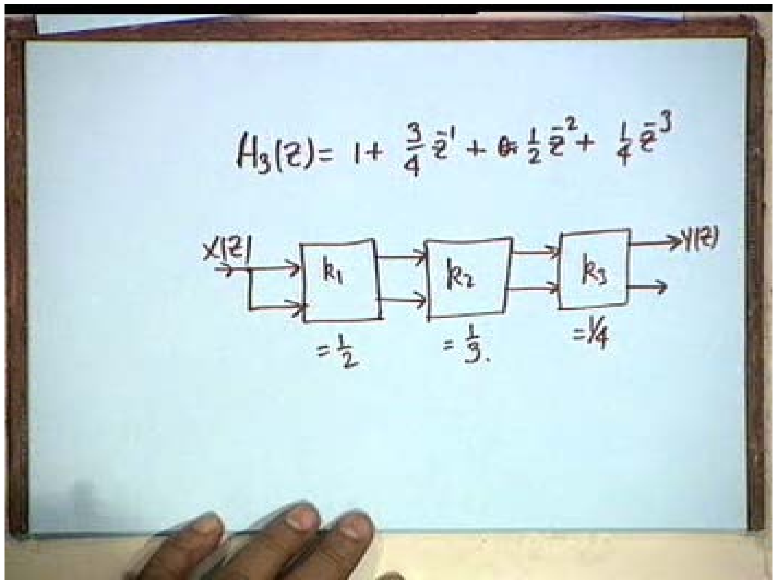

Consider the third order function $H_3(z) = 1 + (3/4) z^{-1} + (1/2)z^{-2} + (1/4)z^{-3}$. If you recall, it was the denominator of an example you worked out in the last lecture. And since recursion relationship is the same except that d's have been replaced by h's, our lattice parameters should be the same. We start from k_3, which would be equal to ¼, and go backwards to k_2 and k_1, which were obtained earlier as 1/3 and ½ respectively. We can now draw this structure. There is no short circuit at the end. Instead, the inputs to k_1 block are shorted. Now what happens if $k_M = \pm 1$? Not only $k_M = \pm 1$, but if at any intermediate stage if k_i becomes ± 1, then you cannot proceed to the next stage.

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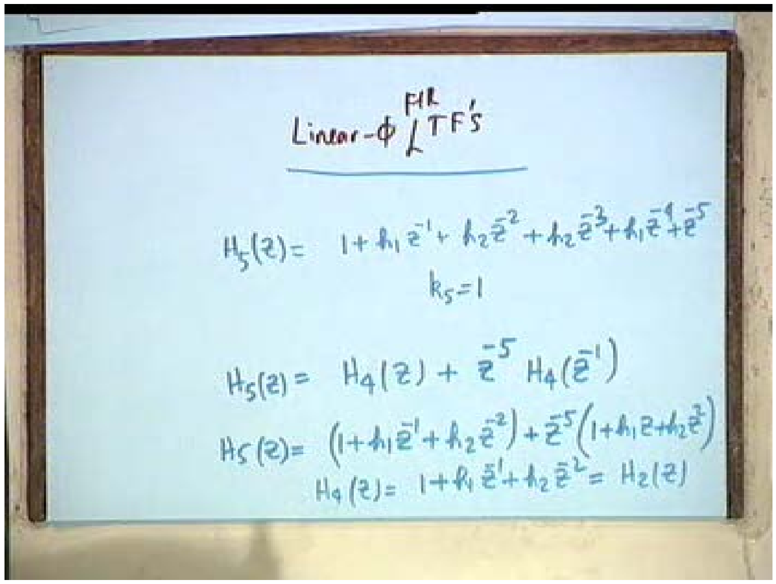

One of the situations where $k_M = \pm 1$ arises is the linear phase FIR transfer function. This has not been considered in any textbook. As you know, in a linear phase transfer function, if the first coefficient is 1, then the last coefficient has to be + 1 or – 1 and $1 - k_M^2 = 0$. Instead of a general procedure, we will take some examples. You know there are four cases of linear phase FIR which are even length symmetrical impulse response, odd length symmetrical impulse response, even length asymmetrical impulse response and odd length asymmetrical impulse response.

We will take an example of each case because each case has a characteristic of its own. We cannot generalize by doing only one case. We will first start with an even length and symmetrical response $H_5(z) = 1 + h_1z^{-1} + h_2z^{-2} + h_2z^{-3} + h_1z^{-4} + z^{-5}$. Since we cannot use the recursion relation, we will do something else. We shall use the relationship $H_i(z) = H_{i-1}(z) + k_iz^{-1} H_{i-1}(z^{-1})$ to derive the lower order transfer function. In our case, this gives $H_5(z) = H_4(z) + z^{-5} H_4(z^{-1})$.

Now, I decompose the given $H_5(z)$ like this: $H_5(z) = (1 + h_1z^{-1} + h_2z^{-2}) + z^{-5} (1 + h_1z + h_2z^2)$. Is my $H_4(z)$ obvious now? If you compare these two you get $H_4(z) = 1 + h_1z^{-1} + h_2z^{-2}$; it is a fourth order transfer function with $h_3 = h_4 = 0$. That is, it is only of second order. It happens because of symmetrical coefficients; we have seen earlier that in linear phase direct form realization, the number of multipliers can be reduced approximately by a factor of ½. This is a reflection of the same fact. Thus $H_4(z) = 1 + h_1z^{-1} + h_2z^{-2} = H_2(z)$. What does this mean in terms of lattice coefficients and what is k_4? From $H_2(z)$ we will be able to obtain k_2 and k_1 but what are k_3 and k_4? They are 0, which means that instead of these two blocks we shall simply have the delays z^{-1} and z^{-1}.

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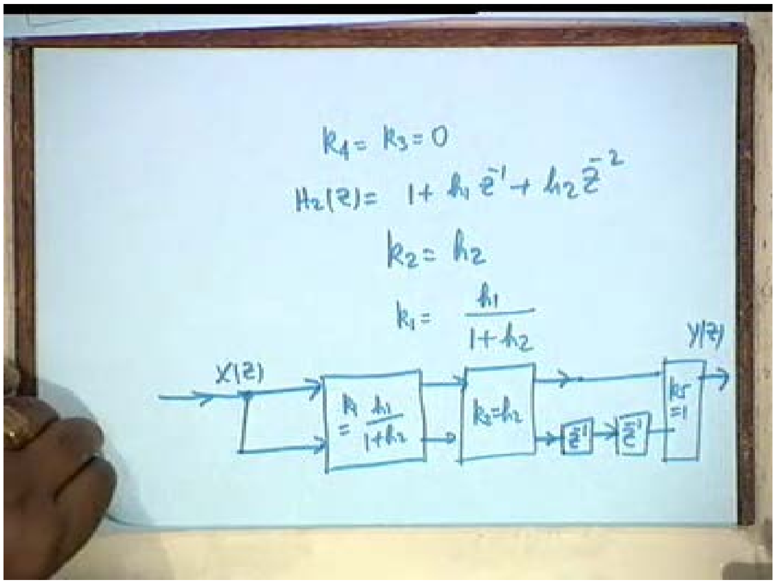

My conclusion is $k_4 = k_3 = 0$, and since $H_2(z) = 1 + h_1\,z^{-1} + h_2z^{-2}$, $k_2 = h_2$. k_1 is also obvious. If you remember, d_1' should be equal to $d_1/(1 + d_2)$ so $k_1 = h_1/(1 + h_2)$ and the synthesis is complete. $H_4(z)$ is a derived transfer function from $H_5(z) = H_4(z) + z^{-5}\,H_4(z^{-1})$ which, in one step, is reduced to a second order transfer function and that is how $k_3 = k_4 = 0$. So my diagram becomes that shown in the figure. It is lattice except that you do not require lattices for k_3 and k_4. Now the number of multipliers is 4 which should have been 2 in canonic realization. Can you have a single multiplier realization of FIR lattice block? We did it for IIR and for FIR lattice, it is an open problem till now.

Now let us take the case of symmetrical impulse response and odd length.

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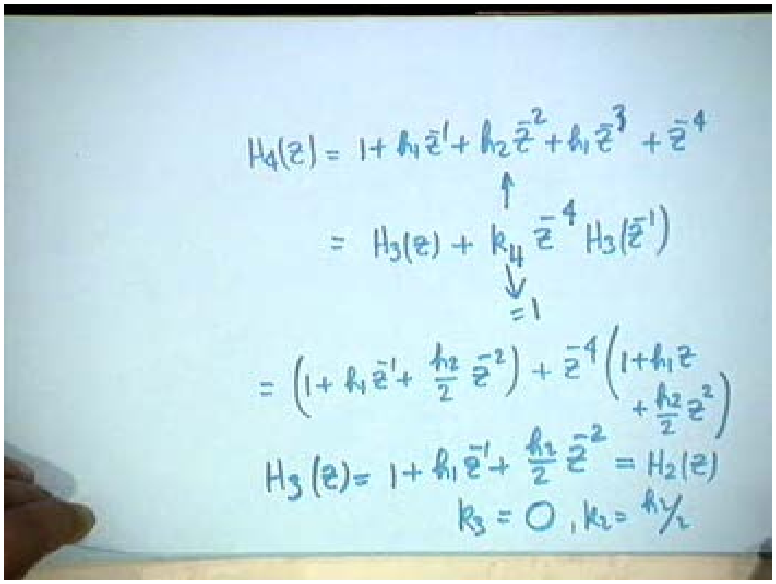

Let the length be 5 so that $H_4(z) = 1 + h_1z^{-1} + h_2z^{-2} + h_1z^{-3} + z^{-4}$. Now how to decompose? Since I want to write this as $H_3(z) + k_4z^{-4}\ H_3(z^{-1})$ with $k_4 = 1$, I split the central term into two equal parts (1/2 is not a multiplier). So you write this as $[1 + h_1z^{-1} + (h_2/2)z^{-2}] + z^{-4}[1 + h_1z + (h_2/2)z^2]$. The identification of H_3 is obvious now: $H_3(z) = 1 + h_1z^{-1} + (h_2/2)\ z^{-2}$, which incidentally is a second order transfer function. So I was able to reduce the order by 1. The conclusion is that $k_3 = 0$; k_2 is also obvious, equal to $h_2/2$. k_1 is also obvious, equal to $h_1/[1 + (h_2/2)]$ and the synthesis is complete.

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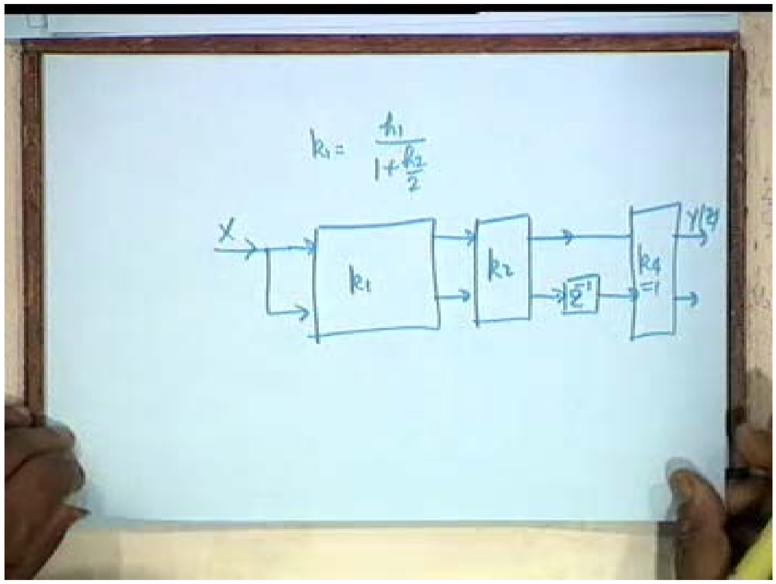

So the diagram would be as shown in the figure. There is no reason why you cannot extend it to any order. Let us look at the other two cases.

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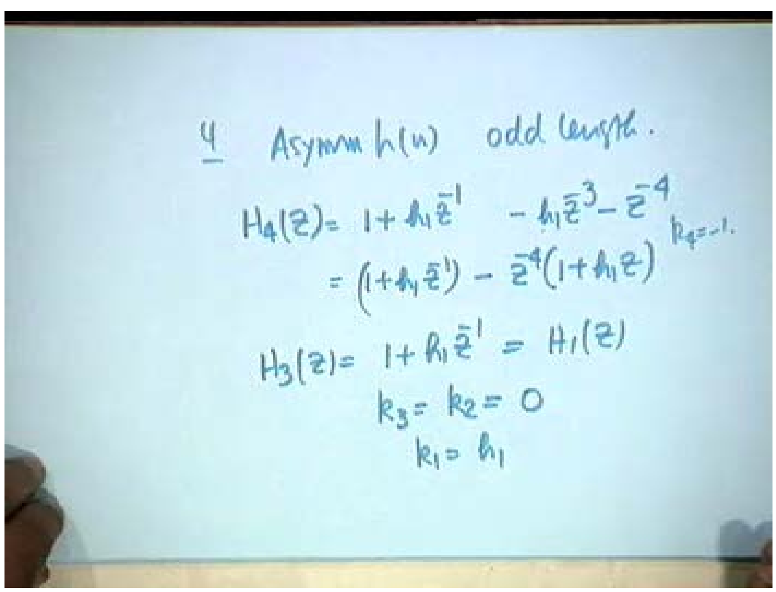

In type 3, we have asymmetrical impulse response of even length. Consider $H_5(z) = 1 + h_1z^{-1} + h_2z^{-2} - h_2z^{-3} - h_1z^{-4} - z^{-5}$ so that $k_5 = -1$. I want to write this as $= H_4(z) + k_5z^{-5} H_4(z^{-1}) = H_4(z) - z^{-5} H_4(z^{-1})$. Therefore, $H_4(z) = 1 + h_1z^{-1} + h_2z^{-2} = H_2(z)$. It means that the realization is identical to the previous example of type 1 except that the last lattice coefficient shall be – 1 instead of + 1. So if you know how to do it for symmetrical case, then you know how to do it for the asymmetrical case also.

The situation is slightly different if the impulse response is asymmetrical and of odd length.

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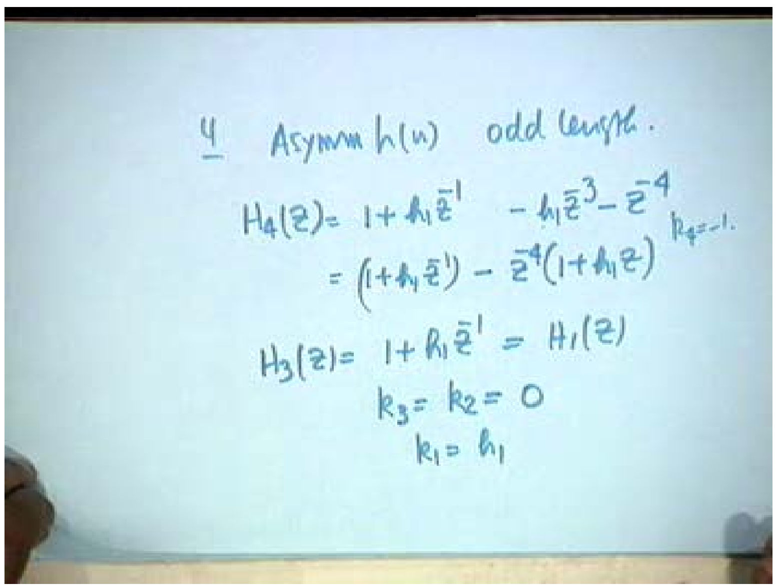

Here the middle coefficient is 0. We again take $H_4(z) = 1 + h_1z^{-1} - h_1z^{-3} - z^{-4}$, and obviously I can write this as $(1 + h_1z^{-1}) - z^{-4} (1 + h_1z)$. Therefore $H_3(z) = 1 + h_1z^{-1}$ which is $H_1(z)$, so that $k_3 = k_2 = 0$ and $k_1 = h_1$. No calculation is required; it is just observation. And the diagram for this case is very simple to draw. We shall have only one complete lattice with $k_1 = h_1$; then k_2 and k_3 both are 0 so we shall have two delays, and finally $k_4 = -1$.

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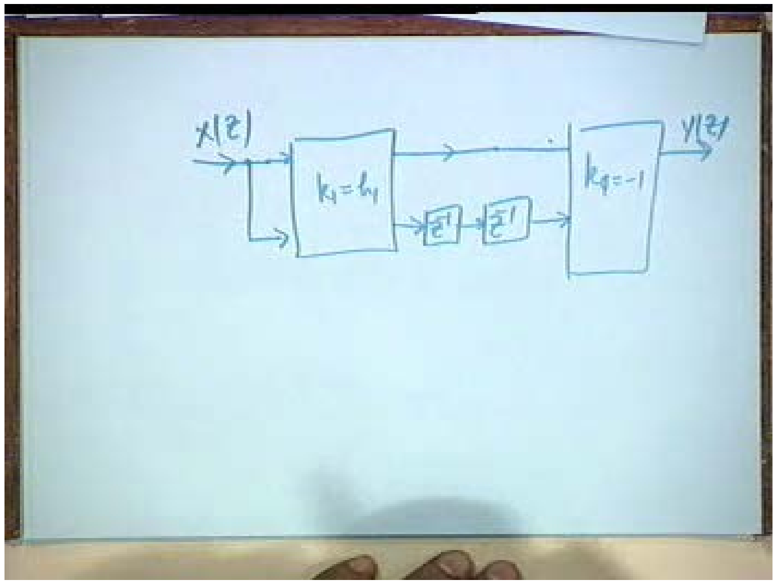

Therefore we have been able to complete all the four cases. Linear phase will use the same number of multipliers as direct canonic structure, approximately half the number of multipliers, if a lattice can be realized with a single multiplier. It is now time to go to the general transfer function, not necessarily linear phase, with the last coefficient = + 1 or – 1.

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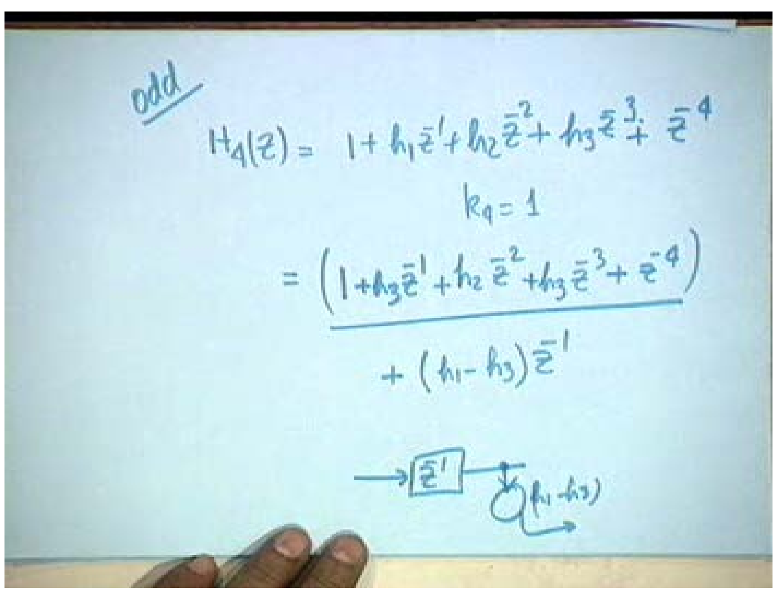

Once again, we have to consider four cases: even length, last coefficient = + 1; odd length, last coefficient = + 1; even length, last coefficient = – 1; and odd length, last coefficient = – 1. Let us start with the first case. Let $H_5(z) = 1 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} + z^{-4}$. The last coefficient is 1 and there is no symmetry or anti–symmetry in the given structures. So what we do is the following: to avoid $k_4 = 1$, we first write this as sum of a linear phase function plus whatever is left. And to minimize the number of delays in the linear phase function we take all the higher powers in the linear phase component. So, I write $H_4(z) = (1 + h_3 z^{-1} + h_2 z^{-2} + h_3 z^{-3} + z^{-4}) + (h_1 – h_3) z^{-1}$.

Now we have already realized the linear phase term with h_1 replacing h_3. This realization is not enough but you have to make a parallel branch in which the transfer function is $(h_1 – h_3) z^{-1}$ which is also linear phase. You have a z^{-1} to start with in the k_1 block; you tap this signal and multiply by $(h_1 – h_3)$ and add it to the main lattice output. So that is how you take care of a general transfer function in which the last coefficient is + 1.

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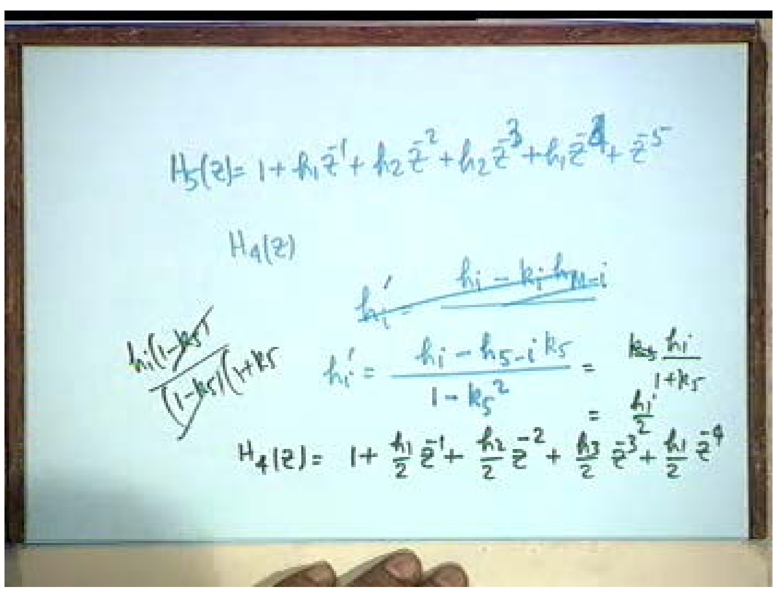

We conclude this lecture with an interesting observation by a student in the context of linear phase Type 2 (i.e. symmetrical impulse response of even length) realization. Let $H_5(z) = 1 + h_1 z^{-1} + h_2 z^{-2} + h_2 z^{-3} + h_1 z^{-4} + z^{-5}$. He pointed out that in finding $H_4(z)$, $h_i' = (h_i – h_{5-i} k_5)/(1 – k_5^2)$.

Let us not put the value of k_5, because you get a 0/0 form, h_i being the same as h_{5-i}: But $h_i = h_{5-i}$ simplifies the formula for h_i' to $h_i' = h_i/(1+k_5)$. Now put the value of $k_{5.}$ If the impulse response would have been anti–symmetrical, then $h_{5-i} = -h_i$ and $k_5 = -1$ would lead to the same relation. So there is no problem you can go from H_5 to $H_{4.}$ h_i would be simply $h_i/2$. So $H_4(z) = 1 + (h_1/2)z^{-1} + (h_2/2)z^{-2}\ h_3/2\ z^{-3} + (h_1/2)\ z^{-4}$. This is a very interesting alternative indeed.