**Digital Signal Processing**

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**Lecture - 34**

**IIR Filter Design**

This is 34th lecture on DSP and our topic for today's discussion is IIR filter design.

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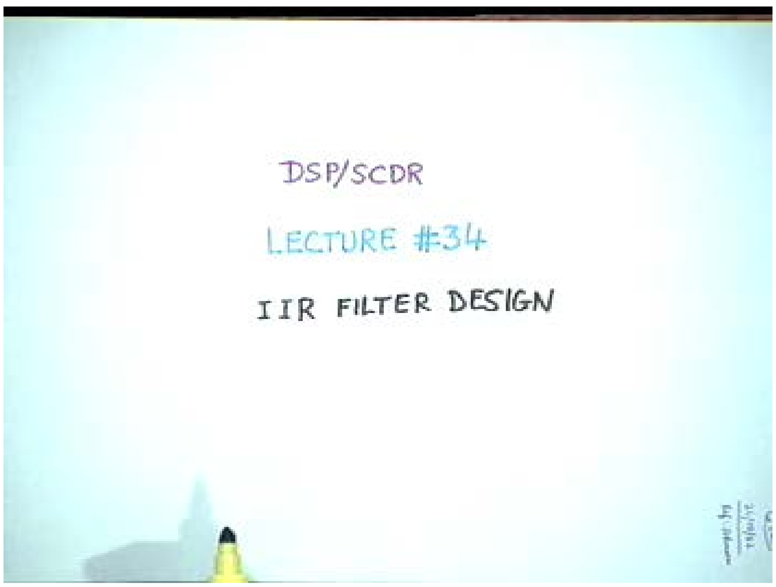


We already started this topic in the last lecture (33rd). In the last lecture, we started with FIR lattice design for various situations of linear phase filters and then we went to non linear phase filters with highest power co-efficient = + 1 or – 1. We saw that they could be synthesized with advantage by using the linear phase concepts. If you decompose a nonlinear phase transfer function into a linear phase and a nonlinear phase filter, you can realize by using taps on the linear phase structure. Next we started discussion on the impulse invariant technique for IIR filter design.

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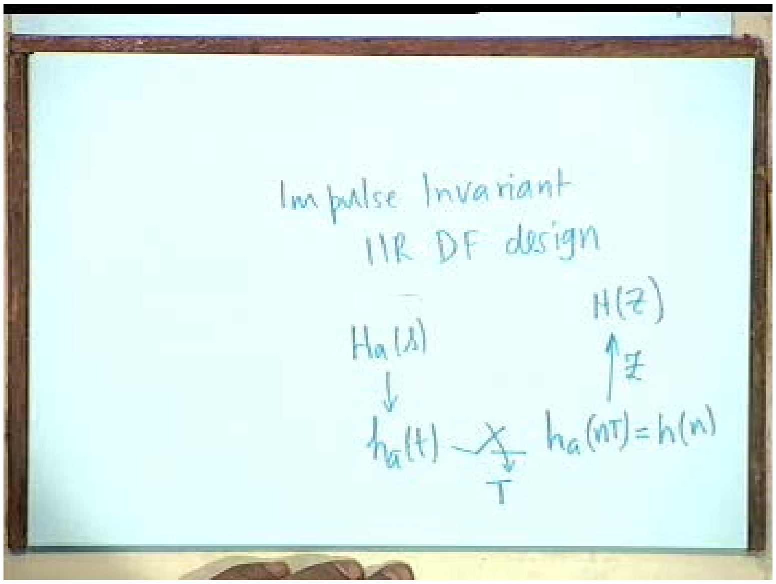


The philosophy was very simple, that is, we take an analog filter $H_a(s)$, derive $h_a(t)$ by inverse Laplace transform, and then we sample $h_a(t)$ to get $h_a(nT)$ which we accept as the impulse response h(n) of the digital filter. And therefore if we take z transform, we get the transfer function H(z). This is the route we follow. So the impulse response is invariant. In other words, if the sampling is done properly, then you can go from $h_a(t)$ to h(n) or h(n) to $h_a(t)$. But then because $H_a(s)$ is hardly ever band limited, i.e. T is never small enough to avoid aliasing, aliasing distortion will occur. What we have to do is to contain it to a tolerable value.

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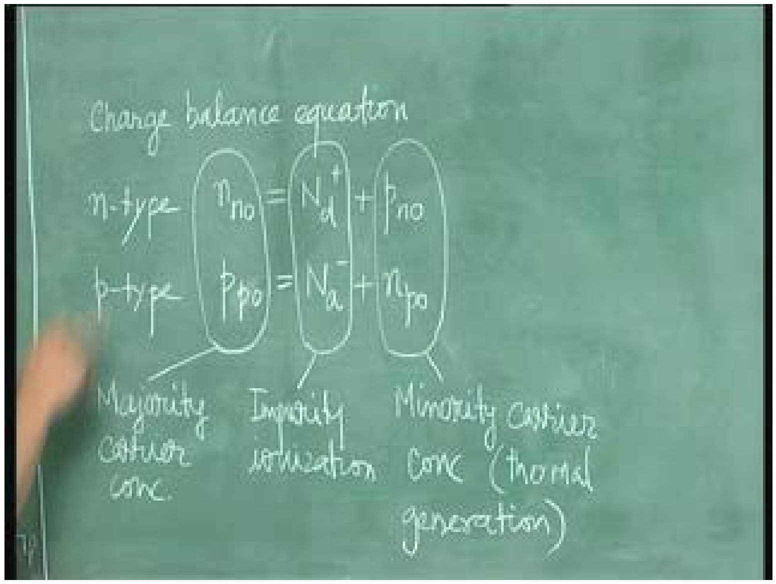


We expand $H_a(s)$ in partial fractions. If there is a pole at $s = p_i$, then in the z domain, this becomes a pole at $e^{-piT}$. We take care of the poles but we cannot take care of the zeros. The transformation obviously is $z = e^{sT}$; we showed that this is a stable transformation. In other words, a stable $H_a(s)$ leads to a stable H(z), because the left half of the s plane maps onto inside of the unit circle. The jω axis maps onto the unit circle and the right half of the s plane goes to outside the unit circle; so it is indeed a stable transformation. In other words, H(z) should be an approximation to the desired digital filter. If impulse response is kept invariant, does it guarantee that other responses are also invariant? The answer is no. To be able to demonstrate this, let us compare the step response of two filters.

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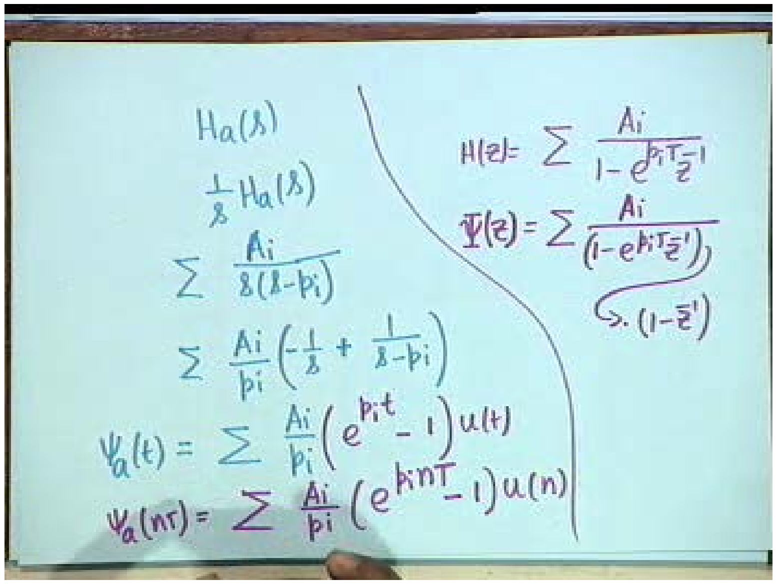


One is derived in the usual manner. For the analog filter, the step response is (1/s) $H_a(s)$. Unit impulse response in the frequency domain is $H_a(s)$ and the Laplace transform of a unit step is (1/s). Then we expand $H_a(s)$ into partial fraction, and get $H_a(s)/s$ as $\sum A_i/[s\ (s - p_i)]$.

We are trying to find out the unit step response of this transfer function and then sample that to get the step response of the digital filter. Now, $A_i/[s\ (s - p_i)] = (A_i/p_i)[(-\ 1/s) + (1/(s - p_i))]$. Therefore the unit step response, which I call as $\psi_a(t)$, shall be = $\sum (A_i/p_i)\ (e^{p_i T} - 1)u(t)$.

Now I want to see if we sample this, shall we get the step response of the digital filter designed by IIT? Now, $\psi_a(nT)$ shall be = $\sum (A_i/p_i)(e^{p_i\ nT} - 1)u(n)$. We want to see whether this matches the unit step response of the digital filter that we derived from impulse invariance. From impulse invariance what was our digital filter? Our digital filter was $H(z) = \sum A_i/(1 - e^{p_i} T z^{-1})$. If you want to find out the unit step response of this digital filter, then you multiply H(z) by the z transform of the unit step i.e. $1/(1 - z^{-1})$. Therefore if we call that as $\Psi(z)$, that is the z transform of unit step response, then this would be = $\sum[A_i /(1 - e^{p_i T}\ z^{-1})] \times (1 - z^{-1})^{-1}$. Taking the inverse z-transform of $\Psi(z)$, we shall get the step response of the filter designed by the impulse invariance technique.

We want to compare the two and if they are identical then step invariance is implied by impulse invariance; if they are not, then the impulse invariance does not ensure step invariance.

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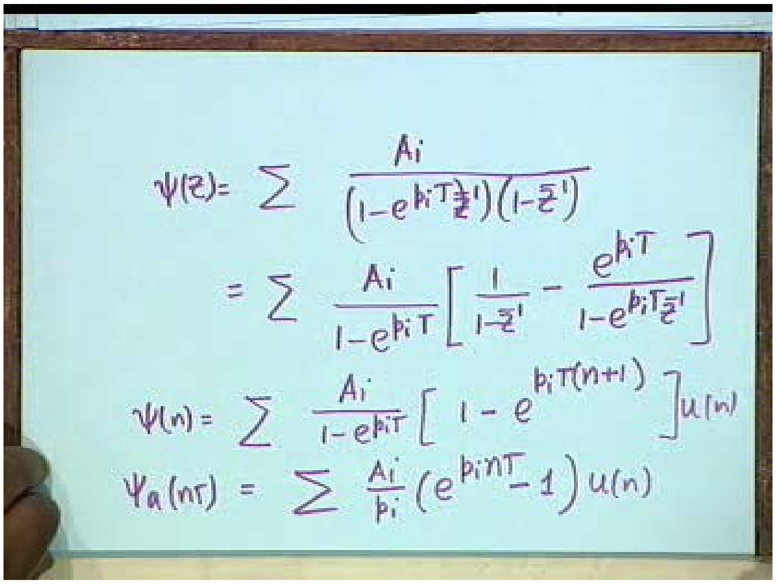


We have $\Psi(z) = \sum A_i/[(1 - e^{p_iT}\ z^{-1})\ (1 - z^{-1})]$ which can be decomposed into partial fractions as $\Sigma[A_i/(1-e^{p_iT})]([1/(1-z^{-1})]-[e^{p_iT}/(1-e^{p_iT}z^{-1})])$. Thus the step response of the impulse invariant design is given by $\Sigma[A_i/(1-e^{p_iT})][1-e^{p_iT(n+1)}]u(n)$. I compare this with $\psi_a(nT)$ which is $\sum(A_i/p_i)(e^{p_inT} - 1)u(n)$. Is it not obvious that the two are not the same? Therefore impulse invariance does not imply step invariance. To be a little more confident about this, let us take an example.

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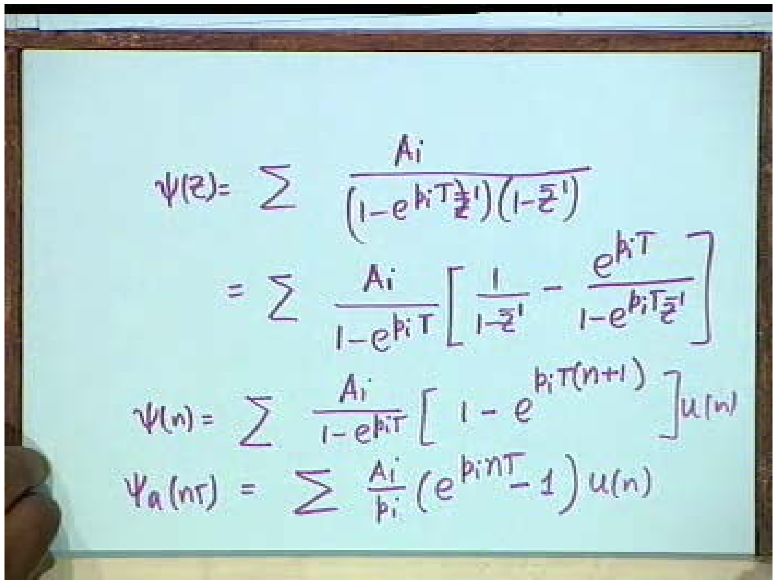


If $H_a(s) = 1/(s + 1)$, a first order filter, then its $\psi_a(nT)$ is $(1 - e^{-nT})$. On the other hand, $\psi(n) = [1/(1 - e^{-T})]\,[1 - e^{-T(n+1)}]$. Obviously these two are quite different. Let us list the values of $\psi_a(nT)$ and $\psi(n)$. If n is 0 then obviously $\psi_a$ is 0, but what is $\psi(n)$? It is = 1. If n = 1 then $\psi_a(nT)$ is $1 - e^{-T}$ whereas $\psi(n)$ is $1 + e^{T}$; therefore $\psi(n)$ is not equal to $\psi_a(nT)$. In other words, impulse invariance does not imply step invariance. If we wanted a step invariant filter, then obviously what we have to do is to take $\psi_a(nT)$, derive its z-transform and then equate that to $\frac{H(z)}{1-z^{-1}}$ to get H(z).

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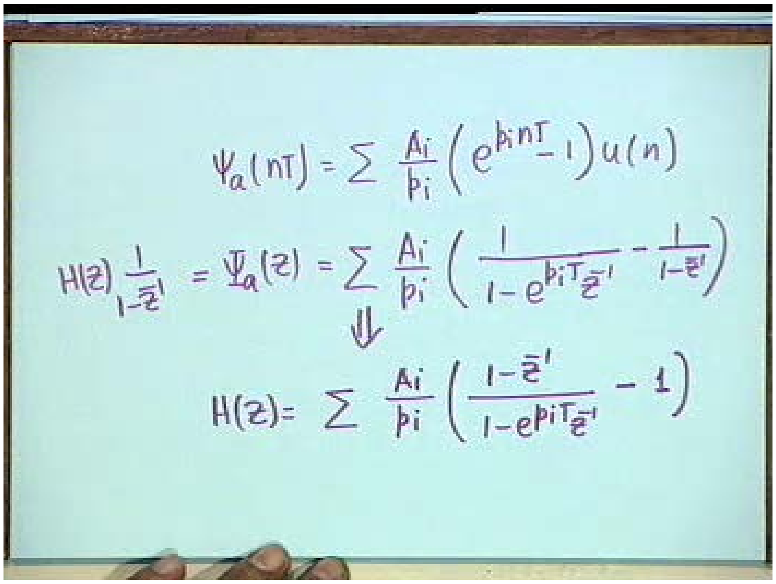


$\Psi_a(z)$ is already known. It is $= \Sigma(A_i / p_i)\{[1/(1-e^{p_iT} z^{-1})]-[1/(1-z^{-1})]\}$ where the former is the z-transform for the first term and latter is the z-transform for the second term in $\Psi_a(nT)$. Now this must be the transfer function H(z) multiplied by $1/(1 - z^{-1})$; therefore the step invariance transformation gives me a transfer function H(z) which is $= \sum (A_i/p_i)/[((1 - z^{-1})/(1 - e^{p_iT} z^{-1})) - 1]$.

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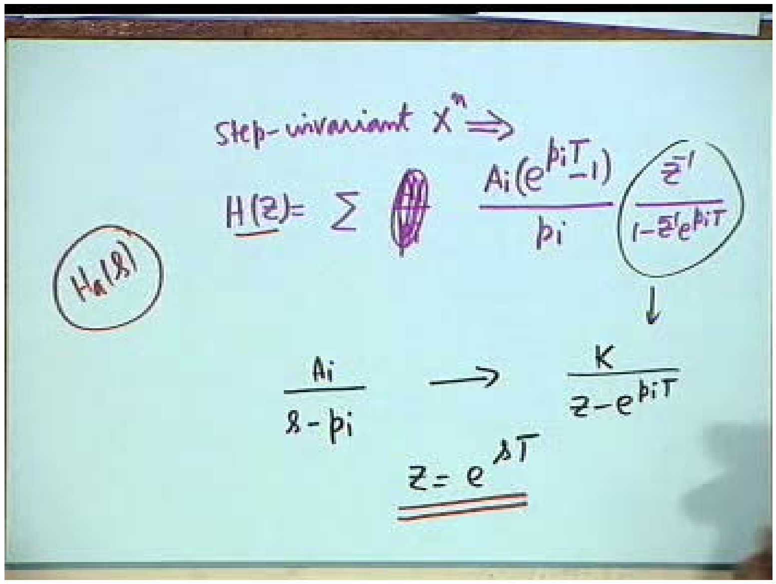


Note that the step invariant transformation leads to a transfer function whose poles are the same as in the impulse invariant transformation; only the zeros change. In addition, if you simplify H(z), it can be written as H(z) = $\Sigma[A_i(e^{p_iT}-1)/p_i]z^{-1}/(1-e^{p_iT}z^{-1})$. Comparing this with the impulse invariant transfer function $\Sigma[A_i(1-e^{p_iT}z^{-1})]$, we see that the residues change and in addition there is a zero at z $=\infty$.

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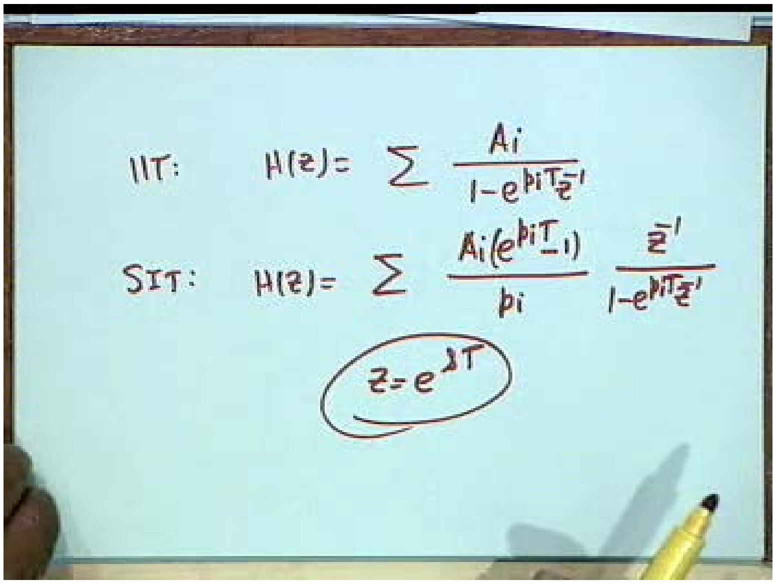


Similarly you could get a ramp invariant transformation or any other performance invariant transformations. But Impulse Invariant transformation is generally in practice.

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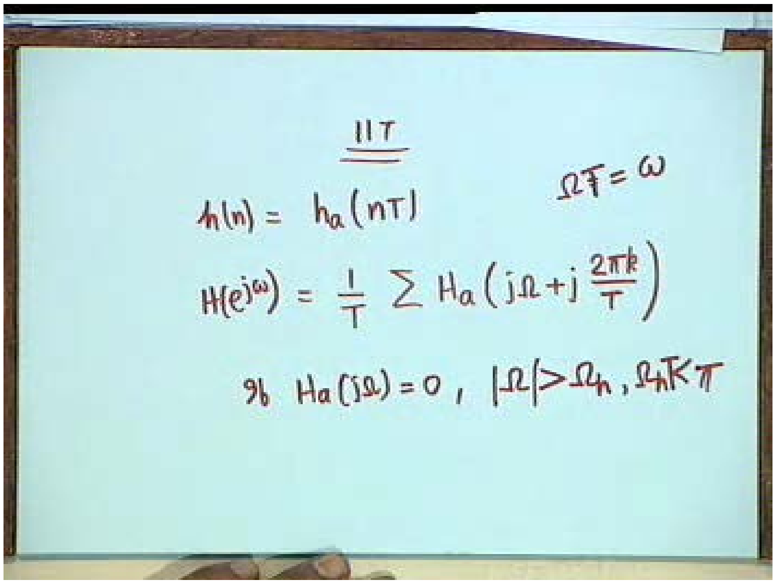

The implication in terms of frequency response is difficult to say at this point, but for the Impulse Invariant Transformation, since $h_a(nT) = h(n)$, the frequency response of the digital filter shall be simply summation of an infinite number of frequency responses of the analog filter shifted by the sampling frequency, on both sides of the frequency axis.

We have already derived that $H(e^{j\omega})$ would be $= (1/T) \sum H_a(j\Omega + j2\pi k/T)$. $k = 0$ gives the basic spectrum; $k = 1$ gives the same spectrum shifted on the left by $(2\pi/T)$ and $k = -1$ gives the same spectrum shifted on the right by $(2\pi/T)$ and so on. In terms of $\omega$, they repeat after $\omega = 2\pi$. Recall that the relationship between $\Omega$ and $\omega$ is $\Omega T = \omega$. This is how aliasing is obtained if $H_a(j\Omega)$ is not band limited. Let us take this ideal case; if $H_a(j\Omega) = 0$ for $|\Omega| > \Omega_h$, then the spectra do not overlap and we have no aliasing distortion.

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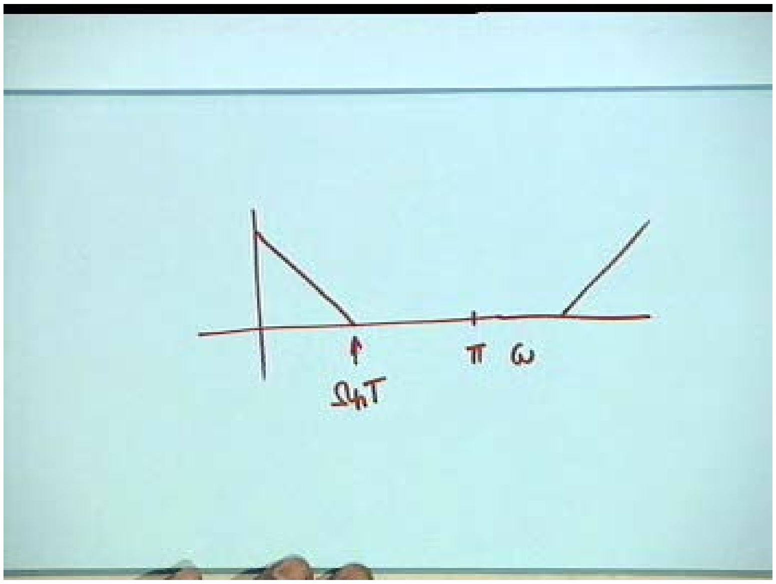


If $\Omega_h T < \pi$, then there is absolutely no problem with Impulse Invariant Transformation.

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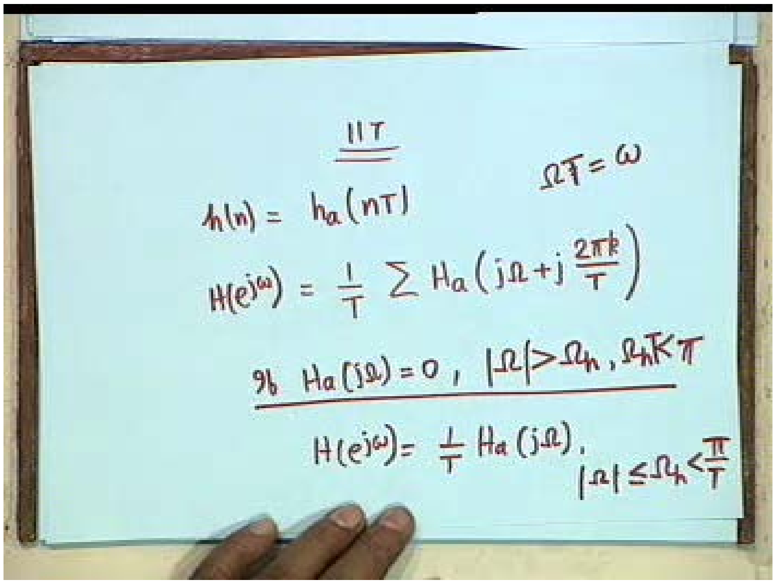


Under this condition, $H(e^{j\omega}) = (1/T)\ H_a(j\Omega)$ where $\Omega$ < or = $\Omega_h$ and $\Omega_h < \pi/T$. If the analog transfer function is band limited, then our Impulse Invariant Transformation will work beautifully without any problem because in $H(e^{j\omega})$, we are only concerned with base band $-\pi \leq \omega \leq \pi$. This is an ideal situation never obtained in practice, and therefore we have to take care. But even in the ideal case we have a problem in practice.

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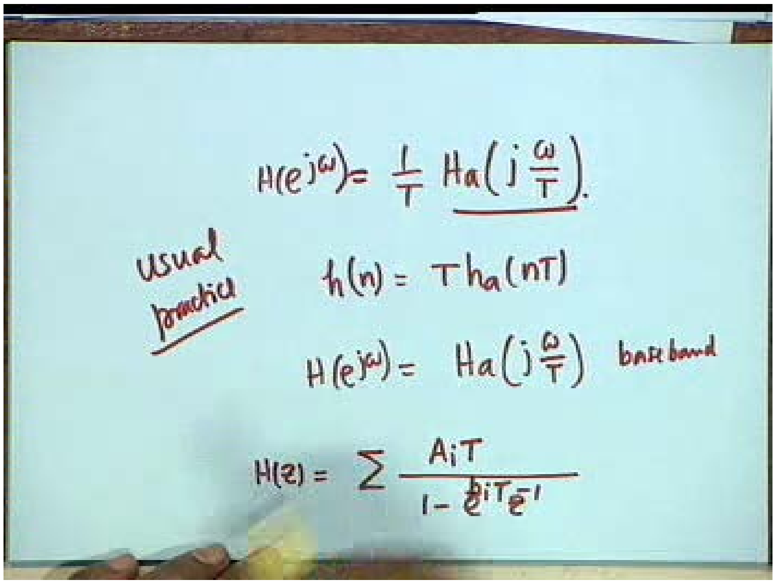


The problem arises because of the factor 1/T. Even if the original spectrum was band limited and sampling was done according to the sampling theorem, when T is small the maximum amplitude of the digital filter transfer function shall be large. That means that there shall be definitely problems of overflow. We cannot allow the signal to grow indefinitely. You would like T to be as small as possible that is sampling frequency as high as possible to avoid all aliasing, but it may create a problem of overflow. To avoid this, what is done in practice is to multiply the impulse response of the analog filters by T, and if I do that then $H(e^{j\omega}) = H_a(j\ \omega/T)$ in the base band and there shall be no problem of overflow. So you take the analog filter, obtain its impulse response; sample the impulse response, then multiply by T and then take the z transform. In other words H(z) for our digital filter would be $\Sigma\ A_iT/(1 - e^{p_iT} z^{-1})$. This comes from practical consideration.

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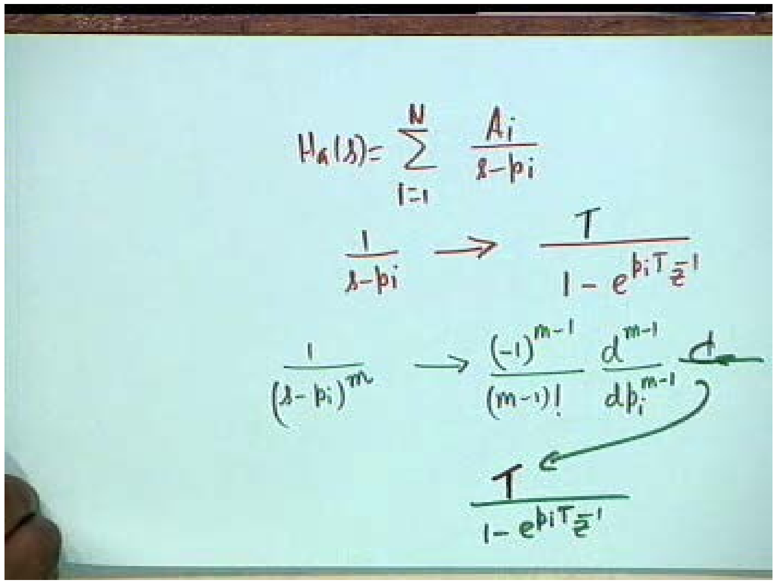

Our derivation so far has been for distinct poles that is $H_a(s) = \Sigma A_i/(s - p_i)$, i = 1 to N. Now what happens if the pole is repeated m times? In the partial fraction expansion, it requires m number of components corresponding to this pole only; you shall have $\Sigma \frac{A_k}{(s-p_i)^k}$ , where k goes from 1 to m. One simplification we have not mentioned is that if you have complex poles, then you combine each pole with its conjugate to get real coefficients.

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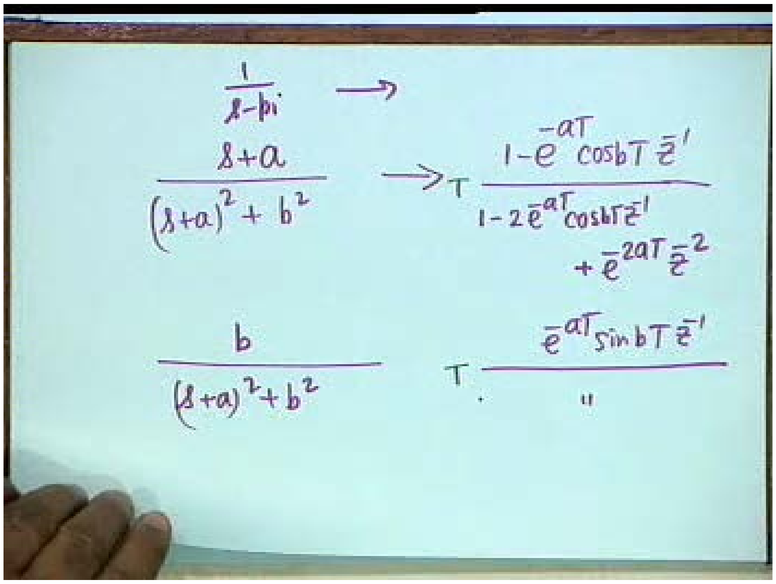

If you have a function with pair of complex conjugate poles at $-a \pm jb$ and the numerator is $s + a$ then this goes in the z-domain as $(1 - e^{-aT} \cos bTz^{-1})/(1 - 2e^{-aT} \cos bTz^{-1} + e^{-2aT}z^{-2})$. If the numerator in the z-domain is a constant and the function of s is of the form $b/[(s + a)^2 + b^2]$, then this goes in the z-domain as $(e^{-aT} \sin bTz^{-1})$ divided by the same denominator. You have to multiply each such factor by T because you took h(n) as $T \times h_a(n)$ and then you take the z transform. We will now take a fairly illustrative example.

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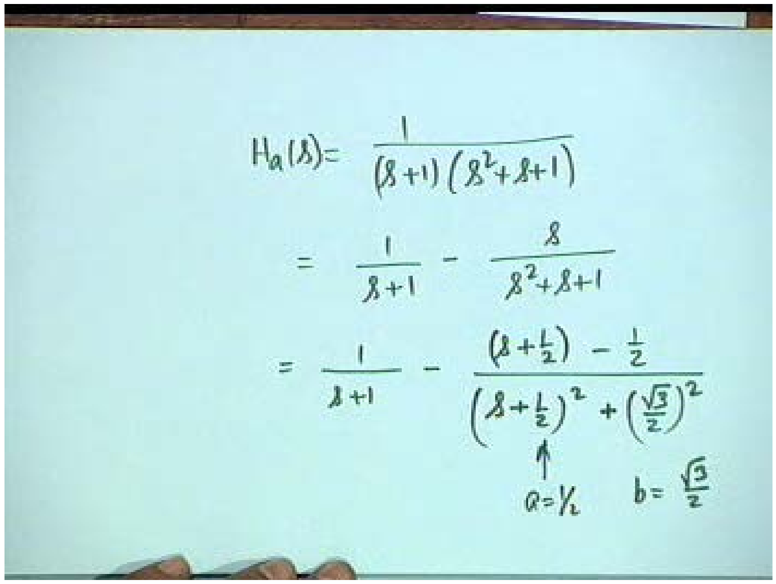


Our example is a third order low pass Butterworth function. You know that the transfer function $H_a(s)$ is $1/[(s + 1)(s^2 + s + 1)]$ and the partial fraction expansion is $[1/s + 1] – [s/(s^2 + s + 1)]$. I can write the second term as $[(s+\frac{1}{2})-\frac{1}{2}]/[(s+\frac{1}{2})^2+(\sqrt{3}/2)^2]$. In order to use the last two transforms, I identify a as ½ and b as $\sqrt{3}/2$.

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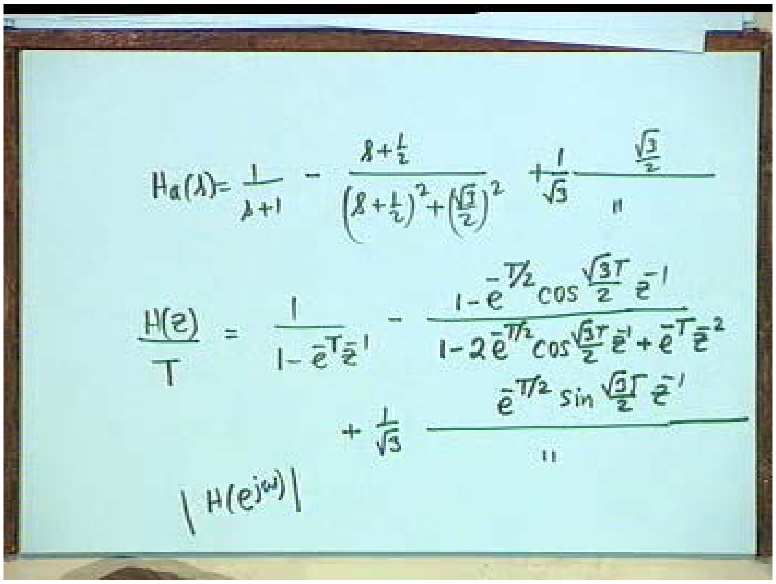

The second term in the numerator, i.e. $\frac{1}{2}$ can be written as $(1/\sqrt{3}) \times (\sqrt{3}/2)$. My scaled transfer function H(z)/T would therefore be equal to $[1/(1 - e^{-T} z^{-1})] - [(1 - e^{-T/2} \cos(\sqrt{3}T/2)z^{-1})/(1 - 2 e^{-T/2}\cos(\sqrt{3}T/2)z^{-1} + e^{-T} z^{-2})] + (1/\sqrt{3}) e^{-T/2} \sin(\sqrt{3}T/2)z^{-1}$ divided by the same denominator. I have obtained the transfer function. Obviously you can bring these last two terms under the same denominator, then you can put $z = e^{j\omega}$ and find out the frequency response. The frequency response that I want is a third order Butterworth low pass response. And if it is identical with the analog response then my job is done, but if it is not, then I have to do something else. Now look at the frequency responses I plotted for this filter.

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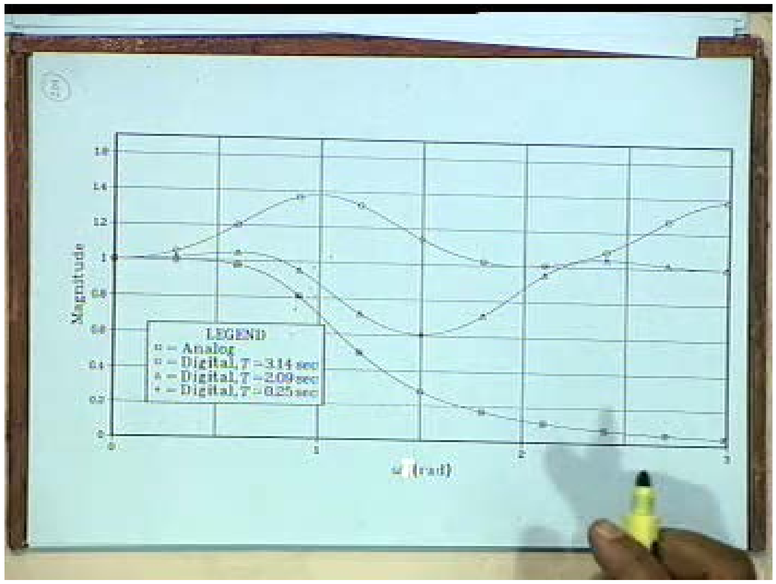


What I have plotted here is versus $\omega = \Omega T$. Now obviously the transfer function depends on T. If you sample at a very low frequency you are asking for trouble. You have to choose a good enough high frequency sampling. If T is large, then what you get may be is a band pass filter. T has to be small enough to get a low-pass response. As shown in the figure, with T = 0.25sec, corresponding to a sampling frequency 4Hz (4Hz is not a practical value; it comes because we took a normalized analog filter whose cutoff was at 1rps) the response is almost indistinguishable from that of the analog filter. You shall have to use MATLAB to draw or watch on the monitor the response, and you have to adjust your T in such a manner that you get a fairly good match with the given analog filter. What will happen beyond $\pi$? The response will rise because it is periodic but we are not concerned about it because we are concerned with a match only in the base band. We will stop here.